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SOME COMBINATORIAL PROPERTIES OF CONICS IN THE HJELMSLEV PLANE

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ABSTRACT. We prove some combinatorial properties of conics in desarguesian Hjelmslev planes. The results generalize analogous properties of conics in finite projective geometries [4].

1. Introductory notes and definitions

By a *special local ring* we understand a finite commutative local ring R, the ideal I of divisors of zero of which is principal. Let g be a generator of I. The smallest integer $\nu \in \mathbb{N}$ such that $g^\nu = 0$ is called the *index of nilpotency* of R. We assume that R is not a field and that the characteristic of R is odd. The image of R under the canonical homomorphism ψ will be denoted by $\overline{R}$. The coordinatized Hjelmslev plane over R will be denoted by $H(R)$.

A *conic* Q in $H(R)$ is the set of all points whose coordinates x_i satisfy

$$\sum_{i,j=1}^{3} a_{ij}x_ix_j = 0\,. \tag{1}$$

In this paper, we shall assume that the conic Q is regular, i.e., $\det[a_{ij}] \notin I$.

Observe that in a suitable coordinate system the conic given by (1) satisfies the equation

$$ax^2 + by^2 + cz^2 = 0\,. \tag{2}$$

It is known that in the projective plane over the skewfield $\overline{R}$ a conic has exactly $|\overline{R}|+1$ points. On the other hand, it can be shown that a conic in $H(R)$ has exactly $|R| + |I|$ points.

A line t intersecting the given conic Q at more than two points is said to be a *tangent* to Q. If t intersects Q at exactly two points, then t is said to be a *secant*. In all other cases, t is said to be a *nonsecant*.

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Observe that the intersecting points of a tangent are neighbouring.

In this paper, we will use the following algebraic result, which we state without proof.

THEOREM 1.1. *Let $d = d_1 g^\alpha \in R$. Then*

1. *there is a square root of d in R if and only if $\overline{d}_1$ is a square in $\overline{R}$ and $\alpha = 2\delta$;*
2. *if condition 1 is fulfilled, then d has*
   a) $2|\overline{R}|^\delta$ *square roots if $d_1 \neq 0$,*
   b) $|\overline{R}|^{\lceil \frac{\nu}{2} \rceil}$ *square roots if $d_1 = d = 0$.*

2. The line and the conic in $H(R)$

In this section, we prove several combinatorial properties of conics in the Hjelmslev plane $H(R)$.

THEOREM 2.1. *The number of tangents through each point of a conic is exactly $|I|$.*

Proof. It is known that the number of lines passing through each point in $H(R)$ is exactly $|R| + |I|$. Since a conic in $H(R)$ has exactly $|R| + |I|$ points, the number of secants passing through each point of the conic is $|\overline{R}| \cdot |I| = |R|$ (lines connecting the given point with all nonneighbouring points of the conic). Then the number of tangents is $|R| + |I| - |R| = |I|$. □

THEOREM 2.2. *Let $t\colon A_1x_1 + A_2x_2 + A_3x_3 = 0$ be a line and let $Q\colon \sum\limits_{i,j} a_{ij}x_ix_j = 0$ be a conic. Then t is a tangent to Q if and only if*

$$\chi = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} & A_1 \\ a_{21} & a_{22} & a_{23} & A_2 \\ a_{31} & a_{32} & a_{33} & A_3 \\ A_1 & A_2 & A_3 & 0 \end{bmatrix} = n^2, \qquad n \in I. \tag{3}$$

Proof.

1. Assume that the conic Q satisfies, with respect to a given coordinate system, the equation $ax^2 + by^2 + cz^2 = 0$.

a) Let the line $t\colon Ax + By + Cz = 0$ be a tangent to the conic Q. We shall prove that $\chi = n^2$, $n \in I$, holds true. Consider the intersection of the line t and the conic Q. Clearly, at least for one of the coefficients of the line t, say B, we have $B \notin I$. Then

$$f = \frac{-Ax - Cz}{B},$$

and hence

$$x^2(aB^2 + bA^2) + 2ACbxz + z^2(bC^2 + cB^2) = 0. \tag{4}$$

Since the line t is a tangent to the conic Q, the discriminant of (4) is a singular square, i.e.,

$$\Delta = -4B^2(bcA^2 + acB^2 + abC^2) = n'^2 .$$

On the other hand, the determinant χ is equal to $-A^2bc - B^2ac - C^2ab$. Thus

$$bcA^2 + acB^2 + abC^2 = \frac{n'^2}{4B^2} ,$$

and hence χ is a singular square.

b) Let

$$\det \begin{bmatrix} a & 0 & 0 & A \\ 0 & b & 0 & B \\ 0 & 0 & c & C \\ A & B & C & 0 \end{bmatrix} = n^2 , \qquad n \in I . \tag{5}$$

We show that the line t: $Ax + By + Cz = 0$ is a tangent to the conic Q: $ax^2 + by^2 + cz^2 = 0$. Arranging (5) we get

$$-A^2bc - B^2ac - C^2ab = n^2 . \tag{6}$$

From $Ax + By + Cz = 0$, since $B \notin I$, we have

$$y = \frac{-Ax - Cz}{B} ,$$

and substituting into the equation of the conic we obtain

$$x^2(aB^2 + bA^2) + 2ACbxz + z^2(bC^2 + cB^2) = 0 .$$

The discriminant of the last equation is

$$\Delta = (2ACb)^2 - 4(aB^2 + bA^2) \cdot (bC^2 + cB^2) ,$$

and hence, in view of (4),

$$\Delta = 4B^2n'^2 .$$

2. Given Q: $\sum a_{ij}x_ix_j = 0$, where $\mathbf{M} = [a_{ij}]$ is the corresponding matrix, let p: $A_1x_1 + A_2x_2 + A_3x_3 = 0$ be a line. It is known that there is a coordinate system with respect to which the equation of the conic is $ax^2 + by^2 + cz^2 = 0$. Denote by $\mathbf{P}$ the transition matrix between the coordinate systems in question. Then the matrix of the conic Q is $\mathbf{P}'\mathbf{M}\mathbf{P} = \widetilde{\mathbf{M}}$ and the equation of the line p is $\widetilde{A}x + \widetilde{B}y + \widetilde{C}z = 0$. Put $(\widetilde{A}, \widetilde{B}, \widetilde{C}) = \widetilde{\boldsymbol{N}}$, $(A, B, C) = \boldsymbol{N}$. Consider the matrix

$$\mathbf{D} = \begin{bmatrix} \mathbf{M} & \boldsymbol{N}' \\ \boldsymbol{N} & 0 \end{bmatrix} .$$

Clearly, we have

$$\begin{bmatrix} \mathbf{P}' & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} \mathbf{D} \begin{bmatrix} \mathbf{P} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{P}' & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{M} & \boldsymbol{N}' \\ \boldsymbol{N} & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{P} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{P}'\mathbf{M}\mathbf{P} & \mathbf{P}'\boldsymbol{N}' \\ \boldsymbol{N}\mathbf{P} & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{M} & \boldsymbol{N}' \\ \boldsymbol{N} & 0 \end{bmatrix} = \widetilde{\mathbf{D}} . \tag{7}$$

From the first part of the proof it follows that $\det[\mathbf{D}] = n^2$. From (7), we immediately have

$$\det[\widetilde{\mathbf{D}}] = \det[\mathbf{P}]^2 \cdot \det[\mathbf{D}] ,$$

where $\det[\mathbf{P}] \notin I$, and

$$\det[\mathbf{D}] = \frac{1}{\det[\mathbf{P}]^2} \cdot \det[\widetilde{\mathbf{D}}] \in I .$$

□

In what follows, by a zero tangent we will understand the polar of a point which lies on the conic. We shall show that a zero tangent is also a tangent in the sense of §1.

THEOREM 2.3. *The line $Ax + By + Cz = 0$ is a zero tangent to the conic $Q\colon ax^2 + by^2 + cz^2 = 0$ if and only if*

$$\det \begin{bmatrix} a & 0 & 0 & A \\ 0 & b & 0 & B \\ 0 & 0 & c & C \\ A & B & C & 0 \end{bmatrix} = 0 .$$

Proof.

1. Let $ax^2 + by^2 + cz^2 = 0$ be a conic and let $ax_1x + by_1y + cz_1z = 0$ be the equation of the zero tangent at the point $T = [x_1, y_1, z_1]$. Let us calculate the corresponding determinant. Then

$$\det \begin{bmatrix} a & 0 & 0 & A \\ 0 & b & 0 & B \\ 0 & 0 & c & C \\ A & B & C & 0 \end{bmatrix} = \det \begin{bmatrix} a & 0 & 0 & ax_1 \\ 0 & b & 0 & by_1 \\ 0 & 0 & c & cz_1 \\ ax_1 & by_1 & cz_1 & 0 \end{bmatrix}$$
$$= a(-cB^2y^2 - bC^2z_1^2) - ax_1ax_1bc = -abc(ax_1^2 + by_1^2 + cz_1^2) = 0$$

because the point T lies on the conic Q'.

2. Let

$$\det \begin{bmatrix} a & 0 & 0 & A \\ 0 & b & 0 & B \\ 0 & 0 & c & C \\ A & B & C & 0 \end{bmatrix} = A^2bc + B^2ac + C^2Ab = 0 .$$

By Theorem 2.1, if $A^2bc + B^2ac + C^2ab = 0$, then the line $Ax + By + Cz = 0$ is a tangent to the conic Q.

Let G be the set of all triples (A, B, C) of elements of R such that

$$\det \begin{bmatrix} a & 0 & 0 & A \\ 0 & b & 0 & B \\ 0 & 0 & c & C \\ A & B & C & 0 \end{bmatrix} = 0,$$

and let $G(0)$ be the set of all zero tangents to Q. Evidently, $G(0) \subset G$. Thus a triple (A, B, C) belongs to G if and only if we have

$$Q'\colon A^2bc + B^2ac + C^2ab = 0. \tag{8}$$

Let Q' denote the conic determined by (8) in variables A, B, C. Then the number $|Q'|$ of points of Q' (the cardinality of Q') satisfies

$$|G| = |Q'| = \big(|\overline{R}| + 1\big)|I|.$$

Since $G(0) \subset G$, and for the cardinality of $G(0)$ we have

$$G(0) = |Q| = \big(|\overline{R}| + 1\big)|I|,$$

it follows that $G(0) = G$.

COROLLARY 2.1. *The number of all zero tangents to a given conic equals* $|R| + |I|$.

The next theorem gives the number of all points a conic and its tangent have in common.

THEOREM 2.4. *Let t be a tangent to a conic Q. Then:*

1. *if t is a zero tangent, then Q and t have in common $|\overline{R}|^{\lceil \frac{\nu}{2} \rceil}$ points;*
2. *if t is not a zero tangent, then there is a number $\delta(t)$, $0 \le \delta(t) \le \left\lceil \frac{\nu}{2} \right\rceil$, such that Q and t have in common $2|\overline{R}|^{\delta(t)}$ points.*

Proof. Consider a conic $Q\colon ax^2 + by^2 + cz^2 = 0$ and a line $t\colon Ax + By + Cz = 0$. According to Theorem 2.1, t is a tangent to Q if and only if

$$\det \begin{bmatrix} a & 0 & 0 & A \\ 0 & b & 0 & B \\ 0 & 0 & c & C \\ A & B & C & 0 \end{bmatrix} = n^2,$$

i.e., $-A^2bc - B^2ac - C^2ab = n^2$.

One of the coefficients of t, say B, is regular, i.e., $B \notin I$.

Then
$$y = \frac{-Ax - Cz}{B}\,.$$
Substituting y into Q we get
$$x^2(aB^2 + bA^2) + 2ACbxz + z^2(bC^2 + cB^2) = 0\,, \tag{9}$$
and the discriminant of (9) is given by
$$\Delta = 4B^2n^2\,. \tag{10}$$
The assertion now follows directly from Theorem 1.1. □

Remark 2.1. In case $R = \mathbb{Z}_{p^k}$, the number $\delta(t)$ can be calculated directly. It would be interesting to determine $\delta(t)$ in general. Among nonsecants, there are distinguished ones, which we will call *imaginary tangents.*

DEFINITION 2.1. Let $Q\colon \sum a_{ij}x_ix_j = 0$ be a conic, and let $t\colon Ax + By + Cz = 0$ be a line. If the determinant (3) is a singular square, then t is said to be an *imaginary tangent.*

Observe that an imaginary tangent is mapped by the canonical map to a tangent in the projective plane $\Pi(|\overline{R}|)$.

The relationship between the classification of lines with respect to a given conic and the determinant (3) is described by the next theorem.

THEOREM 2.5. *Let $t\colon Ax+By+Cz = 0$ be a line and let $Q\colon ax^2+by^2+cz^2 = 0$ be a conic. Then:*

1. *t is a tangent to Q if and only if the determinant* (3) *is a singular square;*
2. *t is an imaginary tangent to Q if and only if the determinant* (3) *is a singular nonsquare;*
3. *t is a secant to Q if and only if the determinant* (3) *is a regular square;*
4. *t is a nonsecant to Q if and only if the determinant* (3) *is a nonsquare.*

Proof. The intersection of t and Q is given by the following equation:
$$x^2(B^2a + A^2b) + 2ACbxz + z^2(bC^2 + cB^2) = 0\,.$$
The mutual position of t and Q depends on its discriminant
$$D = 4B^2(-abC^2 - acB^2 - bcA^2) = 4B^2\chi\,.$$
The assertion is now a straightforward consequence. □

The proof of the next auxiliary statement can be found e.g. in [3].

LEMMA 2.1. *Let R be a special local ring, and let ν be the index of nilpotency of R. Then the number W of singular squares in R is given by*

$$W = 1 + \frac{|\overline{R}|^{\nu-1} - |\overline{R}|}{2(|\overline{R}|+1)} \qquad \textit{for} \quad \nu = 2k\,, \tag{11}$$

and

$$W = 1 + \frac{|\overline{R}|^{\nu-1} - 1}{2(|\overline{R}|+1)} \qquad \textit{otherwise.} \tag{12}$$

Let $n \in I$. The set of all points $[x,y,z] \in H(R)$ such that

$$ax^2 + by^2 + cz^2 = r^2 n\,, \qquad r \in R - I \text{ for fixed } n \tag{13}$$

is said to be a *quasiconic* $Q(n)$.

Clearly, $Q(0)$ is a conic, and, moreover, $Q(0) = Q$. Let the symbol $[n]$ designate the set $\{z \in R;\ z = r^2 n,\ r \notin I\}$.

LEMMA 2.2. *For all $n \in I$ we have $|Q(n)| = |Q| \cdot |[n]|$.*

P r o o f. Let $P = [x_1, y_1, z_1]$ be a point of $Q(0)$. Then $ax_1^2 + by_1^2 + cz_1^2 = 0$. Assume that, e.g., $x_1 \notin I$. Hence $P = [1, y_1, z_1]$. Under the canonical map, P is mapped onto $\overline{P} = [1, \overline{y}_1, \overline{z}_1]$. We prove that for each $r \in R - I$ there is a unique triple $(1, y_1, \tilde{z}_1)$ such that $a + by_1^2 + c\tilde{z}_1^{\,2} = r^2 n$. Consequently, $[1, y_1, \tilde{z}_1]$ is a point of the quasiconic $Q(n)$, and

$$[1, \overline{y}_1, \overline{\tilde{z}}_1] = [1, \overline{y}_1, \overline{z}_1]\,. \tag{14}$$

Consider the equation

$$a + by_1^2 + cz_1^2 = r^2 n\,. \tag{15}$$

Since the equation $\overline{a} + \overline{b}\overline{y}_1^2 + \overline{c}\overline{z}_1^{\,2} = 0$ has two solutions in $\overline{R}$, the equation (15) has two solutions as well. One of them is clearly $[1, y_1, \tilde{z}_1]$ and it will be mapped into (14). Each triple $(1, y_1, \tilde{z}_1)$ determines a point of $Q(n)$. For different $r^2 n$ the corresponding triples are different, too. Hence to each point of $Q(0)$ there correspond exactly $[n]$ points of the quasiconic $Q(n)$. □

THEOREM 2.6. *To each conic in Hjelmslev plane there are exactly*

$$(|R| + |I|)W$$

tangents, where W is the number of singular squares in R.

P r o o f. According to Theorem 2.1, a line $Ax + By + Cz = 0$ is a tangent of the conic $ax^2 + by^2 + cz^2 = 0$ if and only if

$$A^2bc + B^2ac + C^2ab = -n^2\,, \qquad n \in I\,. \tag{16}$$

Evidently, a triple (A, B, C) satisfies (16) if and only if $[A, B, C]$ is a point of the quasiconic

$$-A^2bc - B^2ac - C^2ab = R^2n^2 . \tag{17}$$

By Lemma 2.2, for a fixed $n \in I$, the quasiconic (17) has exactly $|[n^2]|\big(|R|+|I|\big)$ points. For each $n^2 \in I$, the number of points of (17) is $\big(|R|+|I|\big)\sum|[n^2]|$. But $\sum|[n^2]| = W$, and the assertion follows. □

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