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# QUASIUNIFORM LIMITS OF QUASICONTINUOUS FUNCTIONS[1)]

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ABSTRACT. It is proved that every cliquish function $f\colon \mathbb{R} \to \mathbb{R}$ is a quasiuniform limit of a sequence of quasicontinuous functions.

A real function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be quasicontinuous (cliquish) at $x \in \mathbb{R}$ if for every neighbourhood $U$ of $x$ and every $\varepsilon > 0$ there is a nonempty open set $G \subset U$ such that $|f(x) - f(y)| < \varepsilon$ for each $y \in G$ ($|f(y) - f(z)| < \varepsilon$ for each $y, z \in G$). A function $f$ is quasicontinuous (cliquish) if it is such at each point of its domain [5].

A sequence $(f_n), f_n\colon \mathbb{R} \to \mathbb{R}$ quasiuniformly converges to $f\colon \mathbb{R} \to \mathbb{R}$ [6] if the sequence $(f_n)$ pointwise converges to $f$ and

$$\forall \varepsilon > 0 \quad \forall m \in \mathbb{N} \quad \exists p \in \mathbb{N} \quad \forall x \in \mathbb{R}: \\ \min\{|f_{m+1}(x) - f(x)|, \dots, |f_{m+p}(x) - f(x)|\} < \varepsilon .$$

The letters $\mathbb{R}$ and $\mathbb{N}$ stand for the set of real and natural numbers, respectively. If $A$ is a subset of $\mathbb{R}$ and $x \in \mathbb{R}$, then $d(x, A) = \inf\{|x - a|:\ a \in A\}$. If $f\colon \mathbb{R} \to \mathbb{R}$ is a function , then $C_f$ and $Q_f$ stand for the set of all continuity and quasicontinuity points of $f$, respectively.

If $f\colon \mathbb{R} \to \mathbb{R}$, then the function $\omega_f\colon\ \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, given by the formula $\omega_f(x) = \inf\{\sup\{|f(y) - f(z)|:\ y, z \in U\}: U \text{ is a neighbourhood of } x\}$, is said to be oscillation of the function $f$. It is well known that $\omega_f$ is upper semicontinuous and $\omega_f(x) = 0$ if and only if $f$ is continuous at $x$ [6].

If $\mathcal{K}$ is a family of functions $f\colon \mathbb{R} \to \mathbb{R}$, then $B(\mathcal{K})$, $U(\mathcal{K})$ and $D(\mathcal{K})$ denote the collection of all pointwise, uniform and quasiuniform limits of sequences taken from $\mathcal{K}$, respectively. Further we denote by $\mathcal{C}$, $\mathcal{Q}$ and $\mathcal{P}$ the family of all continuous , quasicontinuous and cliquish functions $f\colon \mathbb{R} \to \mathbb{R}$, respectively.

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It is well known that $U(\mathcal{C}) = D(\mathcal{C}) = \mathcal{C}$ and $B(\mathcal{C})$ is the family of Baire 1 functions. Further $U(\mathcal{Q}) = \mathcal{Q}$ and $U(\mathcal{P}) = \mathcal{P}$ [5]. In [3] it is shown that $B(\mathcal{Q}) = \mathcal{P}$ and $B(\mathcal{P})$ is the family of all functions with Baire property. In [1] it is shown that $D(\mathcal{P}) = \mathcal{P}$ (see also [2]) and that $D(\mathcal{Q}) \neq \mathcal{Q}$. We shall show that $D(\mathcal{Q}) = P$. The inclusion $D(\mathcal{Q}) \subset D(\mathcal{P}) = \mathcal{P}$ is obvious.

**THEOREM.** *Every cliquish function $f\colon \mathbb{R} \to \mathbb{R}$ is the quasiuniform limit of a sequence of quasicontinuous functions.*

Proof. Put $A_n = \{x \in \mathbb{R}\colon \omega_f(x) \geq 2^{-n}\}$. Then $A_n$ are closed sets with regard of the upper semi-continuity of $\omega_f$ and because the set $\mathbb{R} - C_f = \bigcup\limits_{n=1}^{\infty} A_n$ is a set of the first category [2], they are nowhere dense. Moreover, $A_1 \subset A_2 \subset \cdots \subset A_n \subset \ldots$. Since the set $C_f$ is dense in $\mathbb{R}$ we can define a function $g\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} \limsup\limits_{u \in C_f, u \to x} f(u), & \text{for } x \in \mathbb{R} - A_1, \\ f(x), & \text{for } x \in A_1. \end{cases}$$

Since $f$ is bounded on some neighbourhood of $x \in \mathbb{R} - A_1$ we have $g(x) < \infty$ for each $x \in \mathbb{R}$. The function $g$ has the following properties:

(1) $g(x) = f(x)$ for each $x \in C_f$,
(2) if $x \notin A_k$ then $|f(x) - g(x)| \leq 2^{-k}$ and
(3) $\mathbb{R} - A_1 \subset Q_g$.

(1) is obvious.

(2): Let $x \notin A_k$. Then $\omega_f(x) < 2^{-k}$ and there is a neighbourhood $U$ of $x$ such that $|f(x) - f(y)| < 2^{-k}$ for each $y \in U$. There exists a sequence $(u_n)$ of points in $C_f$ such that $(u_n)$ converge to $x$ and $g(x) = \lim\limits_{n\to\infty} f(u_n)$. Then $|f(x) - g(x)| = |f(x) - \lim\limits_{n\to\infty} f(u_n)| \leq 2^{-k}$.

(3): Let $x \in \mathbb{R} - A_1$, $U$ be a neighbourhood of $x$ and $\varepsilon > 0$. Then there is $u \in C_f \cap U$ such that $|f(u) - g(x)| < \frac{\varepsilon}{2}$. Since $u \in C_f$ there is an open neighbourhood $G$ of $u$, $G \subset U$ such that $|g(u) - g(y)| < \frac{\varepsilon}{2}$ for each $y \in G$. Therefore, with respect to (1), for each $y \in G$ we have

$$|g(x) - g(y)| \leq |g(x) - f(u)| + |f(u) - g(u)| + |g(u) - g(y)| < \varepsilon,$$

which yields $x \in Q_g$.

Let $k \in \mathbb{N}$. Since $\mathbb{R} - A_k$ is open, $\mathbb{R} - A_k = \bigcup\limits_{i=1}^{s_k} (a_i^k, b_i^k)$, where $s_k \in \{0, \infty\} \cup \mathbb{N}$ and $(a_i^k, b_i^k) \cap (a_j^k, b_j^k) = \emptyset$ for $i \neq j$.

Let $i \in \mathbb{N}$. Let $a_i^k \neq -\infty$. Then $a_i^k \in A_k$. If $a_i^k \notin A_1$, then $a_i^k \in A_{t+1} - A_t$ for some $t \in \{1, 2, \dots, k-1\}$. Since $a_i^k \notin A_t$ so $\omega_f(a_i^k) < 2^{-t}$. Therefore there is $\alpha_i^k > 0$ such that $|f(a_i^k) - f(y)| < 2^{-t}$ for each $y \in (a_i^k - 2\alpha_i^k, a_i^k + 2\alpha_i^k)$. If $a_i^k \in A_1$, put $\alpha_i^k = \frac{1}{k}$. Now put

$$c_i^k = \begin{cases} \min\{a_i^k + \frac{1}{k},\ a_i^k + \alpha_i^k,\ a_i^k + \frac{1}{3}(b_i^k - a_i^k)\}, & \text{if } b_i^k \neq \infty, \\ \min\{a_i^k + \frac{1}{k},\ a_i^k + \alpha_i^k\}, & \text{if } b_i^k = \infty. \end{cases} \tag{4}$$

Further put

$$q_i^k = \min\left\{ s \in \mathbb{N}\colon d(a_i^k, A_s) < \frac{1}{k} \right\}.$$

Evidently $q_i^k \leq k$. If $a_i^k = -\infty$, we put $c_i^k = -\infty$.

Now we define a function $f_{2k-1} \colon \mathbb{R} \to \mathbb{R}$ as follows

$$f_{2k-1}(x) = \begin{cases} f(x), & \text{for } x \in A_k, \\ g(x), & \text{for } x \in (c_i^k, b_i^k), \\ \frac{1}{x - a_i^k} \sin \frac{1}{x - a_i^k}, & \text{for } x \in (a_i^k, c_i^k] \text{ and } q_i^k = 1, \\ f(a_i^k) + 2^{1-q_i^k} \sin \frac{1}{x - a_i^k}, & \text{for } x \in (a_i^k, c_i^k] \text{ and } q_i^k \in \{2, 3, \dots, k\}. \end{cases}$$

Let $i \in \mathbb{N}$. Let $b_i^k \neq \infty$. Then $b_i^k \in A_k$. If $b_i^k \notin A_1$, then $b_i^k \in A_{t+1} - A_t$ for some $t \in \{1, 2, \dots, k-1\}$ and hence there is $\beta_i^k > 0$ such that $|f(b_i^k) - f(y)| < 2^{-t}$ for each $y \in (b_i^k - 2\beta_i^k, b_i^k + 2\beta_i^k)$. If $b_i^k \in A_1$, put $\beta_i^k = \frac{1}{k}$. Put

$$d_i^k = \begin{cases} \max\{b_i^k - \frac{1}{k},\ b_i^k - \beta_i^k,\ a_i^k + \frac{2}{3}(b_i^k - a_i^k)\}, & \text{if } a_i^k \neq -\infty, \\ \max\{b_i^k - \frac{1}{k},\ b_i^k - \beta_i^k\}, & \text{if } a_i^k = -\infty. \end{cases}$$

Let

$$r_i^k = \min\left\{ s \in \mathbb{N}\colon d(b_i^k, A_s) < \frac{1}{k} \right\}.$$

If $b_i^k = \infty$, we put $d_i^k = \infty$.

Now we define a function $f_{2k} \colon \mathbb{R} \to \mathbb{R}$ as follows:

$$f_{2k}(x) = \begin{cases} f(x), & \text{for } x \in A_k, \\ g(x), & \text{for } x \in (a_i^k, d_i^k), \\ \frac{1}{b_i^k - x} \sin \frac{1}{b_i^k - x}, & \text{for } x \in [d_i^k, b_i^k) \text{ and } r_i^k = 1, \\ f(b_i^k) + 2^{1-r_i^k} \sin \frac{1}{b_i^k - x}, & \text{for } x \in [d_i^k, b_i^k) \text{ and } r_i^k \in \{2, 3, \dots, k\}. \end{cases}$$

We shall show that $f_{2k-1}$ is a quasicontinuous function. Analogically we can prove that $f_{2k}$ is quasicontinuous.

Let $x \in \mathbb{R}$.
If $x \in (a_i^k, c_i^k)$, then $f_{2k-1}$ is continuous at $x$ and hence $x \in Q_{f_{2k-1}}$.
If $x = c_i^k$, then $f_{2k-1}$ is continuous from left at $x$ and hence $x \in Q_{f_{2k-1}}$.
If $x \in (c_i^k, b_i^k)$, then according to (3) the function $g$ is quasicontinuous on the open set $(c_i^k, b_i^k)$ and hence $x \in Q_{f_{2k-1}}$.
Now let $x \in A_k$, let $\delta > 0$ and $\varepsilon > 0$. We may assume that $\delta < \frac{1}{k}$.

If $x \in A_1$, then there is $i \in \mathbb{N}$ such that $(x, x+\delta) \cap (a_i^k, b_i^k) = (v, w) \neq \emptyset$. Then $d(a_i^k, A_1) < \frac{1}{k}$ and $q_i^k = 1$. Since $x \le a_i^k$ so $v = a_i^k$. Since $f_{2k-1}\big((v,w)\big) = \mathbb{R}$, then with respect to the continuity of $f_{2k-1}$ on $(v,w)$ there is $y \in (v,w)$ such that $f_{2k-1}(x) = f_{2k-1}(y)$. Hence there is an open set $G \subset (v,w) \subset (x-\delta, x+\delta)$ such that $|f_{2k-1}(z) - f_{2k-1}(x)| < \varepsilon$ for each $z \in G$. Thus $x \in Q_{f_{2k-1}}$.

Let $x \notin A_1$. Then there is $m \in \{2, 3, \ldots, k\}$ such that $x \in A_m - A_{m-1}$. Since $x \notin A_{m-1}$ and $A_{m-1}$ is closed there is $\beta > 0$ such that $(x-\beta, x+\beta) \cap A_{m-1} = \emptyset$. Since $\omega_f(x) < 2^{1-m}$ there is $\alpha > 0$ such that $|f(x) - f(y)| < 2^{1-m}$ for each $y \in (x-\alpha, x+\alpha)$. Denote

$$\gamma = \min\{\alpha,\ \beta,\ \delta\} > 0\,.$$

Since $x \in A_k$ there is $i \in \mathbb{N}$ such that $(x, x+\gamma) \cap (a_i^k, b_i^k) = (v,w) \neq \emptyset$. Then $v = a_i^k$ and $d(a_i^k, A_m) \le a_i^k - x < \gamma < \frac{1}{k}$. Therefore $q_i^k \le m$.
If $q_i^k = 1$, then quasicontinuity of $f_{2k-1}$ at $x$ we can prove similarly as for $x \in A_1$.

Let $q_i^k \in \{2, 3, \ldots, m\}$. Put $\xi = \min\{\gamma, c_i^k\}$. Then for each $y \in (a_i^k, \xi)$ we have

$$f_{2k-1}(y) = f(a_i^k) + 2^{1-q_i^k} \sin \frac{1}{y - a_i^k}$$

and $f_{2k-1}\big((a_i^k, \xi)\big) = [\,f(a_i^k) - 2^{1-q_i^k}, f(a_i^k) + 2^{1-q_i^k}\,]$.
Since $|x - a_i^k| < \gamma < \alpha$, so $|f(x) - f(a_i^k)| < 2^{1-m}$. Thus

$$f(x) \in \big(f(a_i^k) - 2^{1-m}, f(a_i^k) + 2^{1-m}\big) \subset \big(f(a_i^k) - 2^{1-q_i^k}, f(a_i^k) + 2^{1-q_i^k}\big)$$

and hence there is $u \in (a_i^k, \xi)$ such that $f(x) = f_{2k-1}(u)$. Now there is an open set $G \subset (a_i^k, \xi) \subset (x-\delta, x+\delta)$ such that for each $y \in G$ we have

$$|f_{2k-1}(x) - f_{2k-1}(y)| = |f(x) - f_{2k-1}(y)| = |f_{2k-1}(u) - f_{2k-1}(y)| < \varepsilon\,.$$

Therefore $x \in Q_{f_{2k-1}}$.

Now we will prove that the sequence $(f_n)$ is quasiuniformly convergent to the function $f$. First we will prove pointwise convergence.

If $x \notin C_f$, then there is $k \in \mathbb{N}$ such that $x \in A_k$ and then $f_n(x) = f(x)$ for each $n \geq 2k-1$.

Let $x \in C_f$. Then according to (1) we have $f(x) = g(x)$. Let $\varepsilon > 0$. Let $m \in \mathbb{N}$ be such that $2^{2-m} < \varepsilon$. Since $x \notin A_m$ there is $k > m$ such that $\left(x - \frac{2}{k}, x + \frac{2}{k}\right) \cap A_m = \emptyset$. Therefore

$$d(x, A_m) \geq \frac{2}{k}. \tag{5}$$

Let $n > 2k$ and $n$ be odd. Then $n = 2j-1$, where $j \in \mathbb{N}$ and $j > k$. Since $x \notin A_j$ there is $i \in \mathbb{N}$ such that $x \in (a_i^j, b_i^j)$.

a) If $d(a_i^j, A_m) < \frac{1}{k}$ then $x - a_i^j > \frac{1}{j}$. Indeed, if $x - a_i^j \leq \frac{1}{j}$, then there is $z \in A_m$ such that $|a_i^j - z| < \frac{1}{k}$ and hence $|x-z| \leq |x-a_i^j| + |a_i^j - z| < \frac{1}{j} + \frac{1}{k} < \frac{2}{k}$, a contradiction with (5). However, then $x \in (c_i^j, b_i^j)$ and $f_n(x) = f_{2j-1}(x) = g(x) = f(x)$.

b) Let $d(a_i^j, A_m) \geq \frac{1}{k}$. Then $q_i^j > m$.
If $x \in (c_i^j, b_i^j)$, then $f_n(x) = f(x)$.
If $x \in (a_i^j, c_i^j]$, then $f_n(x) = f(a_i^j) + 2^{1-q_i^j} \sin \frac{1}{x - a_i^j}$. Since $a_i^j \notin A_m$, then for each $y \in (a_i^j, c_i^j]$, with respect to (4), we have $|f(a_i^j) - f(y)| < 2^{-m}$. Therefore $|f_n(x) - f(x)| \leq |f_n(x) - f(a_i^j)| + |f(a_i^j) - f(x)| < 2^{1-q_i^j} + 2^{-m} < 3 \cdot 2^{-m} < \varepsilon$.

Similarly, for $n$ even we can prove that $|f_n(x) - f(x)| < \varepsilon$. Therefore $\lim_{n \to \infty} f_n(x) = f(x)$.

Now let $m \in \mathbb{N}$ and $\varepsilon > 0$. Let $r \in \mathbb{N}$ be such that $2^{-r} < \varepsilon$ and let $p = m + 2r$. Let $x \in \mathbb{R}$.

$\alpha$) If $x \in A_{m+r}$, then $f_{m+p-1}(x) = f_{2(m+r)-1}(x) = f(x)$ and hence

$$|f_{m+p-1}(x) - f(x)| < \varepsilon.$$

$\beta$) Let $x \notin A_{m+r}$. Then there is $i \in \mathbb{N}$ such that $x \in (a_i^{r+m}, b_i^{r+m})$. If $x \in \left[\frac{1}{2}(a_i^{m+r} + b_i^{m+r}), b_i^{m+r}\right) \subset (c_i^{m+r}, b_i^{m+r})$, then $f_{2(m+r)-1}(x) = g(x)$. According to (2) we have

$$|f_{m+p-1}(x) - f(x)| = |g(x) - f(x)| \leq 2^{-(m+r)} < 2^{-r} < \varepsilon.$$

If $x \in \left(a_i^{m+r}, \frac{1}{2}(a_i^{m+r} + b_i^{m+r})\right)$, then similarly $|f_{m+p}(x) - f(x)| < \varepsilon$.

Therefore for each $x \in \mathbb{R}$ we have

$$\min\{|f_{m+1}(x) - f(x)|, \dots, |f_{m+p}(x) - f(x)|\} < \varepsilon$$

and the sequence $(f_n)$ quasiuniformly converges to $f$.

Problem. In [4] it is shown that every cliquish function $f\colon \mathbb{R} \to \mathbb{R}$ is a pointwise limit of a sequence of Darboux quasicontinuous functions. Is it true also for quasiuniform convergence?

## REFERENCES


[1] DOBOŠ, J.: *Some generalizations of the notion of continuity and quasi-uniform convergence*, Časopis Pěst. Mat. **106** (1981), 431-434.

[2] DOBOŠ, J.—ŠALÁT, T.: *Cliquish functions, Riemann integrable functions and quasi-uniform convergence*, Acta Math. Univ. Comenian. **40–41** (1982), 219-223.

[3] GRANDE, Z.: *Sur la quasi-continuité et la quasi-continuité approximative*, Fund. Math. **129** (1988), 167-172.

[4] GRANDE, Z.—SOLTYSZIK, L.: *On sequences of real functions with the Darboux property*, Math. Slovaca **40** (1990), 261-265.

[5] MARCUS, S.: *Sur les fonctions quasicontinuous au sens de S. Kempisty*, Colloq Math. **8** (1961), 47-53.

[6] SIKORSKI, R.: *Real Functions I.* (Polish), PWN, Warszawa, 1958.


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