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ON THE SOLUTIONS OF *N*-TH ORDER NONLINEAR DIFFERENTIAL EQUATION IN $L^2(0, \infty)$

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In 1950 A. Wintner in his paper [1] stated the conditions that no solution of the differential equation $y''+f(t)y=0$ belongs to $L^2(0, \infty)$.

In 1974 J. Detki in his paper [2] generalized A. Wintner's result for the nonlinear differential equation $y''+f(t)g(y)=0$ and he considered a similar problem interchanging $L^2(0, \infty)$ with $L^p(0, \infty)$, $(p>1)$.

In the present paper J. Detki's result will be generalized for the nonlinear differential equation $y^{(n)}+f(t)g(y)=0$ and a corollary will be deduced for the linear differential equation $y^{(n)}+f(t)y=0$.

We consider the differential equation

$$y^{(n)}+f(t)g(y)=0, \tag{1}$$

where $f(t)\in C[0, \infty)$, $g(u)\in C^1(-\infty, \infty)$, $g(0)=0$ and $|g(y)|\leqq\beta|y|$, for $|y|<c$, c and $\beta>0$ are constants.

Theorem 1. *If*

$$\int_0^\infty t^{2n-1}f^2(t)\,\mathrm{d}t<\infty, \tag{2}$$

then

$$\int_0^\infty g^2(y(t))\,\mathrm{d}t<\infty \tag{3}$$

cannot hold for any solution of the differential equation (1).

Proof. Assume that there exists a solution $y(t)$ of (1) such that $\int_0^\infty g^2(y(t))\,\mathrm{d}t<$ $<\infty$. We shall prove that this assumption leads to a contradiction. Let t_1, $t_2>0$; integrating (1) from t_1 to t_2, we get

$$y^{(n-1)}(t_2)-y^{(n-1)}(t_1)+\int_{t_1}^{t_2}f(t)g(y(t))\,\mathrm{d}t=0.$$

Using the Schwarz Inequality, we obtain

$$|y^{(n-1)}(t_2)-y^{(n-1)}(t_1)|\leqq\int_{t_1}^{t_2}|f(t)|\,|g(y(t))|\,\mathrm{d}t\leqq$$

$$\leqq\left(\int_{t_1}^{t_2}f^2(t)\,\mathrm{d}t\int_{t_1}^{t_2}g^2(y(t))\,\mathrm{d}t\right)^{1/2}.$$

Since according to (2) $f\in L^2(1,\infty)$ and (3) holds, it is clear from this inequality that for every $\varepsilon>0$ there is $T>0$ such that for all $t_1>t_2>T$

$$|y^{(n-1)}(t_2)-y^{(n-1)}(t_1)|<\varepsilon.$$

But this is the Bolzano—Cauchy condition for the existence of the proper limit $\lim\limits_{t\to\infty}y^{(n-1)}(t)$. We shall prove that $\lim\limits_{t\to\infty}y^{(n-1)}(t)=\infty$. We assume that $\lim\limits_{t\to\infty}y^{(n-1)}(t)=$ $=\alpha$, where $\alpha>0$. Then for every $\varepsilon>0$ there is $T>0$ such that every $t\in[T,\infty)$ is $|y^{(n-1)}(t)-\alpha|<\varepsilon$. Let us choose $\varepsilon>0$ such that $\alpha-\varepsilon>0$. Then for all $t\in[T,\infty)$

$$0<\alpha-\varepsilon<y^{(n-1)}(t)<\alpha+\varepsilon.$$

Integrating the last inequality $(n-1)$-times, we shall get the inequality

$$\frac{\alpha-\varepsilon}{(n-1)!}(t-T)^{n-1}\leqq y(t)+\sum_{j=0}^{n-2}\frac{y^{(j)}(T)}{j!}(t-T)^j,$$

i.e.

$$y(t)\geqq\frac{\alpha-\varepsilon}{(n-1)!}(t-T)^{n-2}-\sum_{j=0}^{n-2}\frac{y^{(j)}(T)}{j!}(t-T)^j.$$

Since $\alpha-\varepsilon>0$, it follows that $\lim\limits_{t\to\infty}y(t)=\infty$. If $\alpha<0$, we shall likewise get that $\lim\limits_{t\to\infty}y(t)=\infty$. In both cases from the properties of the function g we obtain $\int_0^\infty g^2(y(t))\,\mathrm{d}t=\infty$, which contradicts (2). Thus $\lim\limits_{t\to\infty}y^{(n-1)}(t)=0$. Therefore,

$$y^{(n-1)}(t)=\int_t^\infty f(s)g(y(s))\,\mathrm{d}s \tag{4}$$

holds.

We shall prove that $\int_0^\infty|y^{(n-1)}(s)|\,\mathrm{d}s<\infty$. Let $A>0$ be an arbitrary number. Then

$$\int_0^A|y^{(n-1)}(s)|\,\mathrm{d}s\leqq\int_0^A\left\{\int_s^\infty|f(u)g(y(u))|\,\mathrm{d}u\right\}\mathrm{d}s\leqq \tag{5}$$

$$\leqq\left[s\int_s^\infty|f(u)g(y(u))|\,\mathrm{d}u\right]_0^A+\int_0^A s|f(s)g(y(s))|\,\mathrm{d}s=$$

$$= A\int_A^\infty |f(u)g(y(u))|\,\mathrm{d}u + \int_0^A s|f(s)g(y(s))|\,\mathrm{d}s.$$

As for every $t_1,\ t_2>0$

$$\int_{t_1}^{t_2} s|f(s)|\,|g(y(s))|\,\mathrm{d}s \leqq \left(\int_{t_1}^{t_2} s^2 f^2(s)\,\mathrm{d}s \int_{t_1}^{t_2} g^2(y(s))\,\mathrm{d}s\right)^{1\ 2}$$

holds and the improper integrals $\int_0^\infty s^2 f^2(s)\,\mathrm{d}s$, $\int_0^\infty g^2(y(s))\,\mathrm{d}s$ exist, we get that

$$\lim_{A\to\infty} \int_0^A s|f(s)|\,|g(y(s))|\,\mathrm{d}s = \int_0^\infty s|f(s)|\,|g(y(s))|\,\mathrm{d}s,$$

exists, too.

For $A \leqq u$ there holds

$$A\int_A^\infty |f(u)|\,|g(y(u))|\,\mathrm{d}u \leqq \int_A^\infty u|f(u)|\,|g(y(u))|\,\mathrm{d}u,$$

where the integral $\int_A^\infty u|f(u)|\,|g(y(u))|\,\mathrm{d}u \to 0$ if $A\to\infty$. Then, from the last inequality it follows that

$$\lim_{A\to\infty} A\int_A^\infty |f(u)|\,|g(y(u))|\,\mathrm{d}u = 0. \tag{6}$$

From inequality (5) it follows that $\int_0^\infty |y^{(n-1)}(s)|\,\mathrm{d}s<\infty$. As for every $t,\ x>0$

$$|y^{(n-2)}(x) - y^{(n-2)}(t)| = \left|\int_t^x y^{(n-1)}(u)\,\mathrm{d}u\right| \leqq \int_t^x |y^{(n-1)}(u)|\,\mathrm{d}u$$

and the improper integral $\int_0^\infty |y^{(n-1)}(u)|\,\mathrm{d}u$ exists, it follows that the proper limit $\lim_{t\to\infty} y^{(n-1)}(t)$ exists. Likewise, as above for $y^{(n-1)}(t)$, it can be proved that $\lim_{t\to\infty} y^{(n-2)}(t)=0$.

Integrating (4) from t_1 to t_2 we get

$$y^{(n-2)}(t_2) - y^{(n-2)}(t_1) = \int_{t_1}^{t_2} \left\{\int_t^\infty f(s)g(y(s))\,\mathrm{d}s\right\}\mathrm{d}t =$$

$$= \left[t\int_t^\infty f(s)g(y(s))\,\mathrm{d}s\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} tf(t)g(y(t))\,\mathrm{d}t =$$

$$= t_2\int_{t_1}^{t_2} f(s)g(y(s))\,\mathrm{d}s - t_1\int_{t_1}^{\infty} f(s)g(y(s))\,\mathrm{d}s + \int_{t_1}^{t_2} tf(t)g(y(t))\,\mathrm{d}t.$$

Using (6) we obtain $\lim_{t_2\to\infty} t_2\int_{t_2}^{\infty} f(s)g(y(s))\,ds=0$. Hence

$$-y^{(n-2)}(t)=\int_t^{\infty}(s-t)f(s)g(y(s))\,ds. \tag{7}$$

Likewise, as before, it can be proved that $\int_0^{\infty}|y^{(n-2)}(s)|\,ds<\infty$. From this it follows that $\lim_{t\to\infty} y^{(n-3)}(t)$ exists and $\lim_{t\to\infty} y^{(n-3)}(t)=0$.

Integrating (7) from t_1 to t_2 and from the foregoing we get

$$y^{(n-3)}(t)=\int_t^{\infty}\frac{(s-t)^2}{2!}f(s)g(y(s))\,ds.$$

Successively it can be proved that

$$(-1)^{n-2}y'(t)=\int_t^{\infty}\frac{(s-t)^{n-2}}{(n-2)!}f(s)g(y(s))\,ds, \tag{8}$$

$$\int_0^{\infty}|y'(s)|\,ds<\infty \quad\text{and}\quad \lim_{t\to\infty} y(t)=0.$$

Integrating (8) from t_1 to t_2 we get

$$(-1)^{n-2}[y(t_2)-y(t_1)]=\int_{t_1}^{t_2}\left(\int_s^{\infty}\frac{(u-s)^{n-2}}{(n-2)!}f(u)g(y(u))\,du\right)ds.$$

Interchanging the order of integration we obtain

$$(-1)^{n-2}y(t_2)-(-1)^{n-2}y(t_1)=\int_{t_2}^{\infty}\left(\int_{t_1}^{t_2}\frac{(u-s)^{n-2}}{(n-2)!}f(u)g(y(u))\,ds\right)du+$$

$$+\int_{t_1}^{t_2}\left(\int_{t_1}^{u}\frac{(u-s)^{n-2}}{(n-2)!}f(u)g(y(u))\,ds\right)du=\frac{1}{(n-1)!}\left\{\int_{t_1}^{\infty}[(u-t_1)^{n-1}-\right.$$

$$\left.-(u-t_2)^{n-1}]f(u)g(y(u))\,du+\int_{t_1}^{t_2}(u-t_1)^{n-1}f(u)g(y(u))\,du\right\}=$$

$$=\frac{1}{(n-1)!}\int_{t_1}^{\infty}(u-t_1)^{n-1}f(u)g(y(u))\,du-\frac{1}{(n-1)!}\int_{t_2}^{\infty}(u-t_2)^{n-1}f(u)g(y(u))\,du.$$

Since for every t_1, $t_2>0$ there holds

$$\int_{t_1}^{t_2}s^{n-1}|f(s)||g(y(s))|ds\leqq\left(\int_{t_1}^{t_2}s^{2_n-2}f^2(s)ds\int_{t_1}^{t_2}g^2(y(s))\,ds\right)^{1/2}$$

and the improper integrals $\int_0^{\infty}s^{2n-2}f^2(s)\,ds$, $\int_0^{\infty}g^2(y(s))\,ds$ exist, we get that

$$\lim_{A\to\infty}\int_0^A s^{n-1}|f(s)|\,|g(y(s))|\,\mathrm{d}s=\int_0^\infty s^{n-1}|f(s)|\,|g(y(s))|\,\mathrm{d}s<\infty$$

exists, too.

Further, for $t_2>0$ we have

$$\left|\int_{t_2}^\infty (s-t_2)^{n-1}f(s)g(y(s))\,\mathrm{d}s\right|\leqq\int_{t_2}^\infty s^{n-1}|f(s)g(y(s))|\,\mathrm{d}s<\infty,$$

therefore $\int_{t_2}^\infty (s-t_2)^{n-1}f(s)g(y(s))\,\mathrm{d}s\to 0$ if $t_2\to\infty$.

From the above and (8) we obtain that

$$(-1)^{n-1}y(t)=\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}f(s)g(y(s))\,\mathrm{d}s. \tag{9}$$

From (9) we get for $t>0$

$$|y(t)|\leqq\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}|f(s)|\,|g(y(s))|\,\mathrm{d}s\leqq$$

$$\leqq\int_t^\infty (s-t)^{n-1}|f(s)|\,|g(y(s))|\,\mathrm{d}s\leqq\int_t^\infty s^{n-1}|f(s)|\,|g(y(s))|\,\mathrm{d}s.$$

From this it follows that

$$\int_t^\infty y^2(s)\,\mathrm{d}s\leqq\int_t^\infty\left[\int_s^\infty u^{n-1}|f(u)|\,|g(y(u))|\,\mathrm{d}u\right]^2\mathrm{d}s\leqq$$

$$\leqq\int_t^\infty\left(\int_s^\infty u^{2n-2}f^2(u)\,\mathrm{d}u\int_s^\infty g^2(y(u))\,\mathrm{d}u\right)\mathrm{d}s.$$

Since for $s>0$

$$s\int_s^\infty u^{2n-2}f^2(u)\,\mathrm{d}u\leqq\int_s^\infty u^{2n-1}f^2(u)\,\mathrm{d}u,$$

from this and from assumption (2) it follows that

$$\lim_{s\to\infty}s\int_s^\infty u^{2n-1}f^2(u)\,\mathrm{d}u=0.$$

Then

$$\int_t^\infty\int_s^\infty u^{2n-2}f^2(u)\,\mathrm{d}u\,\mathrm{d}s=\left[s\int_s^\infty u^{2n-2}f^2(u)\,\mathrm{d}u\right]_t^\infty+$$

$$+\int_t^\infty s^{2n-1}f^2(s)\,\mathrm{d}s=\lim_{s\to\infty}s\int_s^\infty u^{2n-2}f^2(u)\,\mathrm{d}u+$$

$$+\int_t^\infty (s-t)s^{2n-2}f^2(s)\,\mathrm{d}s=\int_t^\infty (s-t)s^{2n-2}f^2(s)\,\mathrm{d}s\leqq\int_t^\infty s^{2n-1}f^2(s)\,\mathrm{d}s.$$

Hence

$$\int_t^\infty y^2(s)\,ds \leqq \int_t^\infty g^2(y(u))\,du \int_t^\infty \int_s^\infty u^{2n-2} f^2(u)\,du \leqq$$

$$\leqq \int_t^\infty g^2(y(u))\,du \int_t^\infty s^{2n-1} f^2(s)\,ds.$$

Since $\lim_{t\to\infty} y(t)=0$, there is a number T such that for all $t \geqq T$ is $|y(t)| \leqq c$, where c is a positive constant. According to the assumption concerning the function g, there holds for all $t \geqq T$

$$|g(y(t))| \leqq \beta |y(t)|.$$

From this and from the above it follows that

$$\int_t^\infty g^2(y(s))\,ds \leqq \beta^2 \int_t^\infty y^2(s)\,ds \leqq \beta^2 \int_t^\infty g^2(y(s))\,ds \int_t^\infty s^{2n-1} f^2(s)\,ds.$$

Whith regard to the conditions for s^{2n-1} and since $y(t) \neq 0$ implies that $\int_t^\infty g^2(y(s))\,ds \neq 0$, we get

$$1 \leqq \beta^2 \int_t^\infty s^{2n-1} f^2(s)\,ds.$$

But this is a contradiction with $\beta^2 \int_t^\infty s^{2n-1} f^2(s)\,ds < 1$ for sufficiently large t. The proof of Theorem 1 is complete.

If we put $n=2$, then equation (1) has the form

$$y^{(2)} + f(t)g(y) = 0. \tag{10}$$

From Theorem 1 we get

Corollary 1. *If* $\int_0^\infty t^3 f^2(t)\,dt < \infty$, *then for any solution of equation (10)* $\int_0^\infty g^2(y(t))\,dt < \infty$ *cannot hold.*

Corollary 1 is Theorem 1 in [2].

If we put $g(u)=u$, Then equation (1) has the form

$$y^{(n)} + f(t)y = 0. \tag{11}$$

From Theorem 1 we get

Corollary 2. *If* $\int_0^\infty t^{2n-1} f^2(t)\,dt < \infty$, *then* $\int_0^\infty y^2(t)\,dt < \infty$ *cannot hold for any solution of equation (11).*

If we put $n=2$ in equation (11), then we get the equation

$$y''+f(t)y=0. \tag{12}$$

From Theorem 1 we get

Corollary 3. *If* $\int_0^\infty t^3 f^2(t)\,\mathrm{d}t<\infty$, *then* $\int_0^\infty y^2(t)\,\mathrm{d}t<\infty$ *cannot hold for any solution of equation (12).*

Remark 1. The notion (L^2)-solution for equation (11) or (12) can be introduced as follows. Let $y(t)$ be the solution of equation (11) or (12). If $0<\int_0^\infty |y(t)|^2\,\mathrm{d}t<\infty$, then $y(t)$ is called the (L^2)-solution of equation (11) or (12).

Then Corollary 2 and Corollary 3 can be expressed as follows

Corollary 2′. *If* $\int_0^\infty t^{2n-1} f^2(t)\,\mathrm{d}t<\infty$, *then equation (11) cannot have the* (L^2)-*solution.*

Corollary 3′. *If* $\int_0^\infty t^3 f^2(t)\,\mathrm{d}t<\infty$, *then equation (12) cannot have the* (L^2)-*solution.*

Corollary 3′ is the result of A. Wintner's paper [1].

Remark 2. The condition in Corollary 2′ is the best in the sense that it cannot be replaced by

$$\int_0^\infty t^{2n-1-\varepsilon}|f(t)|^2\,\mathrm{d}t<\infty \quad \text{and} \quad \int_0^\infty t^{2n-1}|f(t)|^{2+\varepsilon}\,\mathrm{d}t<\infty$$

if $\varepsilon>0$. In fact, both these integrals converge for every $\varepsilon>0$ if

$$f(t)=\frac{b}{t^n}, \quad b \text{ is a constant.}$$

It can be proved that in this case equation (11) has the solution $y(t)=\frac{1}{t^\alpha}$ and that the exponent $\alpha=\alpha(b)$ can be chosen arbitrarily large if b is suitable.

Remark 3. Let p and q be positive numbers such that $\frac{1}{p}+\frac{1}{q}=1$ and let

$$\int_0^\infty t^{2q-1}f^q(t)\,\mathrm{d}t<\infty;$$

then $\int_0^\infty g^p(y(t))\,\mathrm{d}t<\infty$ cannot hold for any solution of equation (1). The proof can be done as before if instead of the Schwarz Inequality we use Hölder's Inequality.

Remark 4. If the function $g(u)$ has the property that $\int_0^\infty y^2(s)\,\mathrm{d}s<\infty$, it implies

$\int_0^\infty g^2(y(s))\, ds < \infty$; then according to Theorem 1 it can be asserted that equation (1) has no solution belonging to $L^2(0, \infty)$ (we say that the solution $y(t)$ of equation (1) belongs to $L^2(0, \infty)$ if $0 < \int_0^\infty (y(s))\, ds < \infty$).

REFERENCES

[1] WINTNER, A : A criterion for the nonexistence of (L^2)-solutions of a nonoscillatory different al equation, J. London Math. Soc. 25 (1950), 347—351.

[2] DETKI, J.: On the solvability of certain nonlinear ordinary second order differential equation in $L^p(0, \infty)$, Math Balkanica, (4) 21 (1974), 115—119.

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О РЕШЕНИЯХ НЕЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ПОРЯДКА n В $L^2(0, \infty)$

Йозеф Элиаш

Резюме

В работе приведено достаточное условие, которое обеспечивает, что ни одно решение дифференциального уравнения $y^{(n)} + f(t)g(y) = 0$ не принадлежит $L^2(0, \infty)$, и приведены следствия, которые обобщают результаты авторов А. Винтнер для линейного дифференциального уравнения $y'' + f(t)y = 0$ и Й. Детки для нелинейного уравнения $y'' + f(t)g(y) - 0$.