Bohdan Zelinka

Disconnected neighbourhood graphs

Mathematica Slovaca, Vol. 36 (1986), No. 2, 109--110

Persistent URL: http://dml.cz/dmlcz/128747

Math. Slovaca 36, 1986, No. 2, 109—110

DISCONNECTED NEIGHBOURHOOD GRAPHS

BOHDAN ZELINKA

Let G be an undirected graph without loops and multiple edges. Let v be a vertex of G. The subgraph of G induced by the set of all vertices adjacent to v in G is called the neighbourhood graph of v in G and denoted by $N_G(v)$.

At the Symposium on Graph Theory in Smolenice [1] in 1963 B. A. Trahtenbrot and A. A. Zykov suggested the problem: Which graphs H have the property that there exists a graph G such that $N_G(v) \cong H$ for each vertex v of G.

The class of the above mentioned graphs H will be denoted by $\mathcal{N}$. We shall study the case when a graph from $\mathcal{N}$ is disconnected.

The direct product $G_1 \times G_2$ of two graphs G_1, G_2 with vertex sets $V(G_1)$, $V(G_2)$ is defined in the usual way; its vertex set is the Cartesian product $V(G_1) \times V(G_2)$ and two vertices (u_1, u_2), (v_1, v_2) are adjacent in it if and only if either $u_1 = v_1$ and u_2, v_2 are adjacent in G_2, or $u_2 = v_2$ and u_1, u_1 are adjacent in G_1. This definition can be easily extended for an arbitrary finite number of factors.

Theorem. *If H is a disconnected graph with the connected components $H_1, \ldots, H_k$, where k is an arbitrary integer greater than one, and if $H_u \in \mathcal{N}$ for $i = 1, \ldots, k$, then $H \in \mathcal{N}$, but not vice versa.*

Proof. The proof of the implication is easy. For $i = 1, \ldots, k$ let G_i be a graph with the property that $N_{G_i}(v) \cong H_i$ for each vertex v of G_i. Let $G \cong G_1 \times \ldots \times G_k$. Then it is easy to verify that $N_G(v) \cong H$ for an arbitrary vertex v of G.

Now we shall show an example of a disconnected graph H which belongs to $\mathcal{N}$, while none of its connected components does. Let m, n, p be three positive integers such that $m < n < p$. The graph H has three connected components which are the complete bipartite graphs $K_{m,n}$, $K_{m,p}$, $K_{n,p}$. Now we shall describe a graph G. The vertex set V of the graph G is the set of all ordered sextuples $(a_1, a_2, a_3, b_1, b_2, b_3)$, where $a_1 \in \{1, 2, 3\}$, $a_2 \in \{1, 2, 3\}$, $a_3 \in \{1, 2, 3\}$, $b_1 \in \{1, \ldots, m\}$, $b_2 \in \{1, \ldots, n\}$, $b_3 \in \{1, \ldots, p\}$. If $a_1 + a_2 + a_3 \equiv 0 \pmod 3$, then each vertex $(a_1, a_2, a_3, b_1, b_2, b_3)$ is adjacent to all vertices $(a_1+1, a_2, a_3, b_1, x_1, b_3)$, $(a_1+2, a_2, a_3, b_1, b_2, y_1)$, $(a_1, a_2+1, a_3, x_2, b_2, b_3)$, $(a_1, a_2+2, a_3, b_1, b_2, y_2)$, $(a_1, a_2, a_3+1, x_3, b_2, b_3)$, $(a_1, a_2, a_3+2, b_1, y_3, b_3)$, where $x_1 \in \{1, \ldots, n\}$, $y_1 \in \{1, \ldots, p\}$, $x_2 \in \{1, \ldots, m\}$, $y_2 \in \{1, \ldots, p\}$, $x_3 \in \{1, \ldots, m\}$, $y_3 \in \{1, \ldots, n\}$ and the sums are taken modulo 3. If $a_1 + a_2 + a_3 \equiv 1 \pmod 3$, then each vertex $(a_1, a_2, a_3, b_1, b_2, b_3)$ is adjacent to all vertices $(a_1+1, a_2, a_3, x_1, b_2, b_3)$, $(a_1+2, a_2, a_3, b_1, b_2, y_1)$, $(a_1, a_2+1, a_3, x_2, b_2, b_3)$, $(a_1, a_2+2, a_3, b_1, y_2, b_3)$, $(a_1, a_2, a_3+1, b_1, x_3, b_3)$, $(a_1, a_2, a_3+2, b_1, b_2, y_3)$, where

$x_1 \in \{1, \dots, m\}$, $y_1 \in \{1, \dots, p\}$, $x_2 \in \{1, \dots, m\}$, $y_2 \in \{1, \dots, n\}$, $x_3 \in \{1, \dots, n\}$, $y_3 \in \{1, \dots, p\}$ and the sums are again taken modulo 3. If $a_1 + a_2 + a_3 \equiv 2 \pmod 3$, then each vertex $(a_1, a_2, a_3, b_1, b_2, b_3)$ is adjacent to all vertices $(a_1 + 1, a_2, a_3, x_1, b_2, b_3)$, $(a_1 + 2, a_2, a_3, b_1, y_1, b_2)$, $(a_1, a_2 + 1, a_3, b_1, x_2, b_3)$, $(a_1, a_2 + 2, a_3, b_1, b_2, y_2)$, $(a_1, a_2, a_3 + 1, x_3, b_2, b_3)$, $(a_1, a_2, a_3 + 2, b_1, b_2, y_3)$, where $x_1 \in \{1, \dots, m\}$, $y_1 \in \{1, \dots, n\}$, $x_2 \in \{1, \dots, n\}$, $y_2 \in \{1, \dots, p\}$, $x_3 \in \{1, \dots, m\}$, $y_3 \in \{, \dots, p)$ and the sums are again taken modulo 3. For each vertex v of G we have $N_G(v) \cong H$ and hence $H \in \mathcal{N}$.

Now suppose that $K_{m,n} \in \mathcal{N}$, i.e. that there exists a graph G_0 such that $N_{G_0}(v) \cong K_{m,n}$ for each vertex v of G_0. Let u_1 be a vertex of G_0, let $\{v_1, \dots, v_m\}$, $\{w_1, \dots, w_n\}$ be the bipartition classes of $N_{G_0}(u_1)$. Now consider $N_{G_0}(v_1)$. This is also a graph isomorphic to $K_{m,n}$ and contains an independent set $\{w_1, \dots, w_n\}$. In $K_{m,n}$ with $m < n$ there is exactly one independent set with n vertices and this is a bipartition class of $K_{m,n}$. Hence $\{w_1, \dots, w_n\}$ is the bipartition class of $N_{G_0}(v_1)$ with n vertices. The other bipartition class of $N_{G_0}(v_1)$ contains u_1 and no vertex from $\{v_1, \dots, v_m\}$. Therefore there exist vertices $u_2, \dots, u_m$ such that $\{u_1, u_2, \dots, u_m\}$ is a bipartition class of $N_{G_0}(v_1)$. Now consider $N_{G_0}(w_1)$. This graph contains a subgraph isomorphic to $K_{m,n}$ with the bipartition classes $\{u_1, \dots, u_m\}$, $\{v_1, \dots, v_m\}$. As $N_{G_0}(w_1)$ is isomorphic to $K_{m,n}$, one of these sets, say $\{u_1, \dots, u_m\}$, is one of its bipartition classes and there exist vertices $v_{m+1}, \dots, v_n$ such that $\{v_1, \dots, v_n\}$ is the other bipartition class of $N_{G_0}(w_1)$. Each of the vertices $v_{m+1}, \dots, v_n$ is adjacent to all vertices $u_1, \dots, u_m$ and is different from the vertices $v_1, \dots, v_m$, $w_1, \dots, w_n$. But then these vertices belong to $N_{G_0}(u_1)$ and this is a contradiction with the assumption that the vertex set of $N_{G_0}(u_1)$ is $\{v_1, \dots, v_m, w_1, \dots, w_n\}$. Hence G_0 does not exist and $K_{m,n} \notin \mathcal{N}$. Analogously $K_{m,p} \notin \mathcal{N}$ and $K_{n,p} \notin \mathcal{N}$.

REFERENCE

[1] Theory of Graphs and Its Applications. Proc. Symp. Smolenice 1963 (ed. by M. Fiedler), Academia, Prague 1964.

Received February 1, 1984

Katedra tváření a plastů
Vysoké školy strojní a textilní
Studentská 1292
460 01 Liberec 1

НЕСВЯЗНЫЕ ГРАФЫ ОКРЕСТНОСТЕЙ

Bohdan Zelinka

Резюме

Статья занимается классом $\mathcal{N}$ графов H, обладающих тем свойством, что существует граф G, в котором окрестность каждой вершины порождает граф, изоморфный графу H. Доказано, что если все компоненты графа H принадлежат классу $\mathcal{N}$, то граф H принадлежит классу $\mathcal{N}$, но не обратно.