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ON A FAMILY OF WEIGHTED SPACES

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ABSTRACT. Let G denote a locally compact Abelian group with character group $\hat{G}$. For $1 \le p \le \infty$, $A_p(G)$ denotes the vector space of all complex-valued functions in $L^1(G)$ whose Fourier transforms $\hat{f}$ belong to $L^p(\hat{G})$. $A_p(G)$ is a Banach convolution algebra with the sum norm $\|f\|^p = \|f\|_1 + \|\hat{f}\|_{p'}$, [14].

Research on the spaces $A_p(G)$ was initiated by Warner [22]. Later, a number of authors such as, e.g., Larsen, Liu and Wang [14], Martin and Yap [17], and Lai [13] worked on these spaces and a generalization to the weighted case was given by Gürkanlı [10], Feichtinger and Gürkanlı [8]. The present paper deals with the more general case in which we also use the generalized Fourier transform of [5]: $A^{p,q}_{w,\omega}(G)$ denotes the (Banach) space of functions in $L^p_w(G)$, a weighted L^p-space, with Fourier transform in $L^q_\omega(\hat{G})$, equipped with the sum norm $\|f\|^{p,q}_{w,\omega} = \|f\|_{p,w} + \|\hat{f}\|_{q,\omega}$.

0. Introduction

Through this paper, G denotes a locally compact (non-compact, non-discrete) abelian group with dual group $\hat{G}$, both written additively. Certain well-known terms such as Banach function space (BF-space), Banach module, Banach ideal, strong translation- and character-invariance, generalized character will be used frequently in the sequel; their definitions may be found, e.g., in [3], [6], [8], [18], [21]. $C_c(G)$ denotes the space of complex-valued, continuous functions with compact support. The main tool in the present paper is the Fourier transform denoted by $\hat{}$ or by F; two kinds of Fourier transforms will be used: The classical Fourier transform, cf. eg., [18], [19], and a generalized Fourier transform (cf. [5]) discussed below. The Fourier algebra $\{\hat{f} \mid f \in L^1(\hat{G})\}$ is denoted by $A(G)$ and is given the norm $\|\hat{f}\|_A = \|f\|_1$; here, $\hat{f}$ is the classical Fourier transform of $f \in L^1(G)$.

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Let now Q be a fixed open and relatively compact subset of G. We define

$$S_0(G) = \{f \mid f = \sum L_{y_n} f_n\,,\ y_n \in G\,,\ f_n \in A_Q\,,\ n \geq 1\,,\ \text{and } \sum \|f_n\|_A < \infty\}\,, \tag{1}$$

where $A_Q = \{h \in A(G) \mid \operatorname{supp}(h) \subset Q\}$, and where L_y denotes the translation operator given by $L_y f(x) = f(x-y)$. Any representation of f in the form (1) is called an *admissible representation.* It is known that $S_0(G)$ is independent of Q, ([4]). Endowed with the norm

$$\|f\|_{S_0} = \inf\{\sum \|f_n\|_A \mid f = \sum L_{y_n} f_n \text{ admissible}\}\,,$$

$S_0(G)$ is the smallest strongly character invariant Segal algebra on G. It is well-known that the Fourier transforms induce an isomorphism between $S_0(G)$ and $S_0(\hat{G})$, hence their transposes induce isomorphisms of their dual spaces (e.g., for the weak topologies). Thus the generalized Fourier transform is defined by

$$\langle \hat{\sigma}, f\rangle = \langle \sigma, \hat{f}\rangle \qquad \text{for} \quad f \in S_0(\hat{G})\,,\ \ \sigma \in S_0'(G)\,.$$

Below we present some further details concerning this known construction.

Through this note, we also will use *Beurling weights*, i.e., real-valued, measurable and locally bounded functions w on a locally compact abelian group G which satisfy

$$1 \leq w(x)\,, \quad w(x+y) \leq w(x)w(y) \qquad \text{for} \quad x, y \in G\,.$$

For $1 \leq p < \infty$, we set

$$L^p_w(G) = \{f \mid fw \in L^p(G)\}\,.$$

Under the norm $\|f\|_{p,w} = \|fw\|_p$, this is a Banach space embedded into $L^p(G)$. For $p = 1$, $L^1_w(G)$ is a Banach algebra under convolution, called *Beurling algebra*, cf. [18]. It is known that the maximal ideal space of $L^1_w(G)$ can be identified with the space of all generalized characters η on G such that $\eta \in L^\infty_w(G)$ and $\eta(x) \leq w(x)$ l.a.e., cf. [21] and [16].

1. The spaces $A^{p,q}_{w,\omega}(G)$

The first lemma is a sharpening of a known result [5] and will be used in a number of arguments in the sequel.

Lemma 1.1. *Let $1 \leq p \leq \infty$ be given. For $f \in L^p(G)$ and $h \in S_0(G)$, the integral*

$$\langle f, h\rangle := \int_G f(x)h(x)\,\mathrm{d}x$$

converges, and $(f,h) \mapsto \langle f,h\rangle$ *is a continuous bilinear form on* $L^p(G) \times S_0$. *Hence* $f \mapsto \langle f,\cdot\,\rangle$ *is a continuous linear embedding* $L^p(G) \to \big(S_0'(G), \|\cdot\|\big)$.

Proof. The space $S_0(G)$ is continuously embedded into any of the $L^q(G)$ spaces with $1 \le q \le \infty$.

The lemma plays a role in the generalization of the classical Fourier transform to spaces such as $L^p(G)$, $p > 1$, in the following manner (see [5] and the remark in the introduction): It is known that the classical Fourier transforms F_G and $F_{\hat G}$ induce isomorphisms $S_0(G) \to S_0(\hat G)$ and $S_0(\hat G) \to S_0(\hat{\hat G})$. Accordingly, their transposes ${}^tF_G$ and ${}^tF_{\hat G}$ yield isomorphisms of the dual spaces $S_0'(\hat G)$ and $S_0'(G)$. In fact, these maps are isometries of the dual spaces. One now defines

$$F\colon L^p(G) \to S_0'(\hat G)$$

as the composition of the map of Lemma 1.1 with

$${}^tF_{\hat G}\colon S_0'(G) \to S_0'(\hat G).$$

This is the precise meaning of the map $f \mapsto \hat f$ of the introduction, i.e., of the identity $\langle \hat f, h\rangle = \langle f, \hat h\rangle$. For $p = 1$ or $p = 2$ the generalized Fourier transform coincides with the classical one. □

COROLLARY 1.2. *For any weight function* w *on* G, *the form* $\langle\cdot\,,\cdot\rangle$ *restricts to a continuous bilinear form on* $L^p_w(G) \times S_0$.

DEFINITION 1.3. Let $1 \le p,q < \infty$ and let w, ω be weight functions on G, $\hat G$, respectively. We set

$$A^{p,q}_{w,\omega}(G) = \{f \in L^p_w(G) \mid \hat f \in L^q_\omega(\hat G)\}$$

and equip this space with the norm

$$\|f\|^{p,q}_{w,\omega} = \|f\|_{p,w} + \|\hat f\|_{q,\omega},$$

where $(\hat{\ })$ is the generalized Fourier transform [5].

We remark here that $A^{p,q}_{w,\omega}(G)$ clearly is continuously embedded into $L^p_w(G)$, hence also into $L^p(G)$. Thus whenever the latter is continuously embedded into the locally convex vector space B, then so is $A^{p,q}_{w,\omega}(G)$. As an example, we mention the following result, needed later:

PROPOSITION 1.4. *Let* $A_c(G) = A(G) \cap C_c(G)$ *with the inductive limit topology induced by* $C_c(G)$. *Then* $A^{p,q}_{w,\omega}(G)$ *is continuously embedded into* A_c' *with its* w^**-topology.*

Indeed, it is known that $A_c' = Q(G)$, the space of quasimeasures on G, and that $L^p(G)$ is continuously embedded into $Q(G)$ (with its weak topology as the dual of $D(G)$) ([15]).

The first structure result is the following:

THEOREM 1.5. $\left(A^{p,q}_{w,\omega}(G), \|\cdot\|^{p,q}_{w,\omega}\right)$ *is a Banach space.*

Proof. Let (f_n) be a Cauchy sequence. Clearly, (f_n) is a Cauchy sequence in $L^p_w(G)$, and $(\hat{f}_n)$ is Cauchy in $L^q_\omega(\hat{G})$. Therefore, (f_n) converges to some $f \in L^p_w(G)$ while $(\hat{f}_n)$ converges to $g \in L^q_\omega(\hat{G})$. To prove the theorem, we need to show that $\hat{f} = g$.

But this is obvious since $f \to \hat{f}$ is continuous from $L^p_w(G)$ into $\left(S'_0(\hat{G}), \|\cdot\|\right)$.

□

LEMMA 1.6. *For every $f \in L^p_w(G)$, $x \mapsto L_x f$ is continuous $G \to L^p_w(G)$, and if $\|L_x\|_{p,w}$ denotes the operator norm, then $\|L_x\|_{p,w} \le w(x)$, for $1 \le p < \infty$.*

Proof. Since $\|L_x f\|^p_{p,w} \le w^p(x)\|f\|^p_{p,w}$, this proves second claim. Recall that $C_c(G)$ is dense in $L^p_w(G)$, and this lemma is true for all $g \in C_c(G)$. Hence a simple calculation shows that $x \mapsto L_x f$ is continuous for all $f \in L^p_w(G)$. □

PROPOSITION 1.7. *For every $0 \ne f \in L^p_w(G)$, there exists $c(f) > 0$ such that $c(f)w(x) \le \|L_x f\|_{p,w} \le w(x)\|f\|_{p,w}$. Moreover, there is $c > 0$ such that $cw(x) \le \|L_x\|_{p,w} \le w(x)$.*

Proof. The upper estimate already is established in Lemma 1.6. For the lower one, recall from [8; Lemma 2.2] that for each $f \ne 0$, there is $c_1 > 0$ such that $c_1 w(x) \le \|L_x f\|_{pw}$, i.e., such that $c_1 \le \|L_x f\|_{pw}/w(x)$ (for every $x \in G$). Accordingly, the set of such positive constants c_1 is bounded, and we let $c(f)$ be its supremum. Clearly, then, $c(f)w(x) \le \|L_x f\|_{p,w}$ for $x \in G$, proving the first part of the proposition.

The inequality $c(f) \le \|L_x f\|_{p,w}/w(x)$ leads to:

$$c(f) \le \|L_x\|_{p,w}/w(x)$$

for every $f \ne 0$ such that $\|f\|_{p,w} \le 1$. Here, the bound on the right is independent of f, and thus, the proof is complete if we set

$$c = \sup\{c(f) \mid \|f\|_{p,w} \le 1\}.$$

□

Before returning to the spaces $A^{p,q}_{w,\omega}(G)$, we establish the following result concerning the multiplication operators M_t, $t \in \hat{G}$, on $L^p_w(G)$:

LEMMA 1.8. *$L^p_w(G)$ is strongly character invariant, and $t \mapsto M_t f$ is a continuous map $\hat{G} \to L^p_w(G)$ for every $f \in L^p_w(G)$ for $1 \le p < \infty$.*

Proof. Since this lemma is true for all $g \in C_c(G)$, and $C_c(G)$ is dense in $L^p_w(G)$, the proof of this lemma is easy. □

THEOREM 1.9. *$A^{p,q}_{w,\omega}(G)$ is translation and character invariant. Furthermore, for each $f \neq 0$, there exists constant $c(f)$, $c(\hat{f}) > 0$ such that*

$$c(f)w(x) \le \|L_x f\|^{p,q}_{w,\omega} \le w(x)\|f\|^{p,q}_{w,\omega}\,, \qquad c(\hat{f})\omega(t) \le \|M_t f\|^{p,q}_{w,\omega} \le \omega(t)\|f\|^{p,q}_{w,\omega}\,.$$

Lastly, there are $C, D > 0$ such that

$$Cw(x) \le \|L_x\|^{p,q}_{w,\omega} \le w(x)\,,$$
$$D\omega(t) \le \|M_t\|^{p,q}_{w,\omega} \le \omega(t)\,.$$

Proof. Since $\|L_x f\|_{q,\omega} = \|\hat{f}\|_{q,\omega}$ and $\|M_t f\|_{p,w} = \|f\|_{p,w}$, $w \ge 1$ and $\omega \ge 1$ yield the upper estimates immediately in view of Lemma 1.6.

For the lower estimates, note that

$$c(f)w(x) \le c(\hat{f})w(x) + \|f\|_{q,\omega} \le \|L_x f\|_{p,w} + \|\hat{f}\|_{q,\omega}\,,$$

and similarly for M_t, where $c(f)$, $c(\hat{f})$ are the constants from Proposition 1.7. Set now

$$C = \sup\{c(f) \mid \|f\|^{p,q}_{w,\omega} \le 1\}\,,$$
$$D = \sup\{c(\hat{f}) \mid \|f\|^{p,q}_{w,\omega} \le 1\}\,,$$

and the lower estimates follow immediately. □

It is easy to prove the following proposition by Lemma 1.6. and Lemma 1.8.

PROPOSITION 1.10. *For every $f \in A^{p,q}_{w,\omega}(G)$, the maps $x \mapsto L_x f$, and $t \mapsto M_t f$, are continuous from G respectively $\hat{G}$ into $A^{p,q}_{w,\omega}(G)$.*

PROPOSITION 1.11.

a) *For $1 \le p < \infty$ and any weight function w on G, $L^p_w(G)$ is a Banach convolution module* ([8]) *over $L^1_w(G)$ under $(f,g) \mapsto f{*}g$.*

b) *For $1 \le q < \infty$ and any weight function ω on $\hat{G}$, $A^{p,q}_{w,\omega}(G)$ is a Banach convolution module over $L^1_w(G)$.*

Proof. It is easy to see that $L^p_w(G)$ and $A^{p,q}_{w,\omega}(G)$ are BF-spaces (i.e., $L^p_w(G)$ and $A^{p,q}_{w,\omega}(G)$ are continuously embedded into $L^1_{\mathrm{loc}}(G)$). Hence, the proof is completed by Lemma 1.6, Proposition 1.10 in this paper, and Lemma 1.5 in [6].

Recall that $A^{1,q}_{w,\omega}(G)$ is a Banach convolution algebra [8]. By Proposition 1.6. (b), $A^{1,q}_{w,\omega}(G)$ is a Banach ideal within $L^1_w(G)$, (i.e., $A^{1,q}_{w,\omega}(G)$ is continuously embedded into $L^1_w(G)$). A trivial consequence of this: $A^{p,q}_{w,\omega}(G)$ is a Banach convolution module over $A^{1,q}_{w,\omega}(G)$. Set

$$\Lambda^K_W = \{f \in L^1_w \mid \operatorname{supp}(\hat{f}) \text{ is compact}\}\,.$$

Then it is clear that $\Lambda^K_W \subset A^{p,q}_{w,\omega}(G)$. □

LEMMA 1.12. *Suppose that w satisfies the Beurling-Domar condition* (BD). *Then a bounded approximate identity* (*shortly* BAI, [8]) $(f_a) \subset \Lambda_W^K \subset L_w^1(G)$ *also yields an approximate identity* (*shortly* AI) *for* $L_w^p(G)$.

Proof. Since the vector space $C_c(G)$ of continuous functions on G with compact support is everywhere dense in $L_w^p(G)$, then, given $g \in L_w^p(G)$ and $\varepsilon > 0$, there exists a function $f \in C_c(G)$ such that $\|f - g\| < \frac{\varepsilon}{3C}$, where $C = \sup_\alpha \|e_\alpha\|_{1,w}$. Using the above result we have

$$\|e_a^* f - e_* g\|_{p,w} \le \|e_\alpha\|_{1,w} \|f - g\|_{p,w} < C \frac{\varepsilon}{3C} = \frac{\varepsilon}{3}\,. \tag{1}$$

It is known that $L_w^1(G)$ has another approximate identity $(u_\beta)_{\beta \in J}$ with compact support, and $(u_\beta)_{\beta\in J}$ is also an approximate identity in $L_w^p(G)$ ([11]). Now, using the inequality

$$\begin{aligned} &\|e_a^* f - f\|_{p,w} \\ &\le \|e_a^* f - (e_a^* f)^* u_\beta\|_{p,w} + \|e_a^*(f^* u_\beta) - f^* u_\beta\| + \|f^* u_\beta - f\|_{p,w} \\ &\le \|e_a^*\|_{1,w} \|f - f^* u_\beta\|_{p,w} + \|e_a^* f - f\|_{1,w} \|u_\beta\|_{p,w} + \|f^* u_\beta - f\|_{p,w} \end{aligned} \tag{2}$$

and choosing a fixed $\beta_0 \in J$ such that

$$\|e_a\|_{1,w} \|f - f^* u_{\beta_0}\|_{p,w} < \frac{\varepsilon}{3}\,, \tag{3}$$

and

$$\|f^* u_{\beta_0} - f\|_{p,w} < \frac{\varepsilon}{3}\,, \tag{4}$$

one can find $\alpha_0 \in I$ such that $\|e_a^* f - f\| < \varepsilon$ for all $\alpha \ge \alpha_0$. Consequently, if one combines (1) and (4), one obtains

$$\|e_a^* g - g\|_{p,w} \le \|e_a^* g - e_a^* f\|_{p,w} + \|e_a^* f - f\|_{p,w} + \|g - f\|_{p,w} < \varepsilon\,. \tag{5}$$

□

PROPOSITION 1.13. *Suppose that w satisfies* (BD). *Then*

a) $A_{w,\omega}^{p,q}(G)$ *admits an approximate identity, bounded in $L_w^1(G)$ and with compactly supported Fourier transforms. Furthermore $A_{w,\omega}^{p,q}(G)$ is an essential Banach module* ([8]) *over $L_w^1(G)$.*

b) $A_{w,\omega}^{p,q}(G)$ *does not admit a* BAI *in its own norm for $1 \le p < \infty$.*

Proof.

a) Let $f \in A_{w,\omega}^{p,q}(G) \subset L_w^p(G)$. By Lemma 1.12, the BAI$(f_n) \subset \Lambda_W^K(G)$ is an approximate identity for $L_w^p(G)$, and so, given $\varepsilon > 0$, $\|f_n^* f - f\|_{p,w} \le \varepsilon$ for sufficiently large n.

Since (f_n) is bounded in $L^1_w(G)$, the techniques used in the proof of [8; Theorem 4.2] apply and show that also $\|\widehat{f_n^* f} - \hat{f}\|_{q,\omega} = \|\hat{f}_n^* \hat{f}\|_{q,\omega} < \varepsilon$ for sufficiently large n. Evidently, this proves the first claim.

One now concludes from [9; Corollary 2.3] that $A^{p,q}_{w,\omega}(G)$ is an essential Banach module over $L^1_w(G)$.

b) By assumption, G is non-discrete, i.e., $\hat{G}$ is non-compact. Therefore, $\operatorname{vol}(K)$ for $K \subset \hat{G}$ compact becomes arbitrarily large. On the other hand, the proof of Theorem 4.2 of [8] shows that $\hat{f}_n \to 1$ uniformly on compact subsets of $\hat{G}$. Thus, if K is compact, $\hat{f}_n \chi_K \to \chi_K$ uniformly, and hence, in $L^q_\omega(G)$; then $\|\hat{f}_n \chi_K\|^q_{q,\omega} \to \|\chi_K\|^q_{q,\omega} = \int\limits_K \omega \geq \operatorname{vol}(K)$. Lastly,

$$\|\hat{f}_n \chi_K\|^q_{q,\omega} \leq \int\limits_{\hat{G}} |\hat{f}_n(t)|^q \omega^q(t)\, dt = \|\hat{f}_n\|^q_{q,\omega} .$$

If these last norms were bounded, the set $\{\operatorname{vol}(K) \mid K \subset \hat{G}\}$ would be bounded, which is a contradiction. Therefore, (f_n) cannot be $\|\cdot\|^{p,q}_{w,\omega}$-bounded. $\square$

We now collect a few results concerning Banach module structures on $A^{p,q}_{w,\omega}(G)$ with respect to pointwise multiplication by elements of suitable function algebras on G. As a beginning, we list some of these algebras:

Recall that $A(G)$ is the Fourier algebra $F(L^1(\hat{G}))$ of G. We also set $A_c = A(G) \cap C_c(G)$, equipped with the inductive limit topology of its subspaces $A_K = A(G) \cap C_K(G)$ topologized by $\|\cdot\|_A$. This makes $A_c(G)$ into a T_2 locally convex vector space whose dual we simply denote by A'_c; unless otherwise specified, A'_c is given its weak topology $\sigma(A'_c, A_c)$, a T_2 locally convex topology. It may be worthwhile, at this point, to recall that $\{g \in L^1(\hat{G}) \mid g \in A_c(G)\}$ is norm-dense in $L^1(G)$, cf. [16; F7.c]. Consequently, $A_c(G)$ is dense in $(A(G), \|\cdot\|_A)$.

Generalizing these standard constructions, we next introduce the Banach spaces $A^\omega(G) = F(L^1_\omega(\hat{G}))$, ω an arbitrary weight function on $\hat{G}$, with the norm $\|\hat{g}\|_\omega = \|g\|_{1,\omega}$. With this, $A^\omega(G)$ is a Banach algebra under pointwise multiplication. By analogy with the earlier definition, we set $A^\omega_c(G) = A^\omega(G) \cap C_c(G)$, again equipped with the inductive limit topology τ_ω of the subspaces

$$A^\omega_K(G) = A^\omega(G) \cap C_K(G)$$

equipped with their norms. $A^\beta_c(G)$ is non-trivial if ω satisfies (BD). We let $A^\omega_c(G)'$ be the dual with the weak topology.

We now define the vector space $A^{q,p}_{\omega,w}(\hat{G})$ and its norm $\|\cdot\|^{q,p}_{\omega,w}$ as in Definition 1.3 and establish the following important results concerning the Banach modules $A^{p,q}_{w,\omega}(G)$.

THEOREM 1.14. *If w is symmetric (i.e., $w(-x) = w(x)$), then the Fourier transform is an isometry between the spaces $A^{p,q}_{w,\omega}(G)$ and $A^{q,p}_{\omega,w}(\hat{G})$.*

Proof. It is known that Fourier transform is linear and one to one. Now, take any function $g \in A^{q,p}_{\omega,w}(\hat{G})$. Since $\hat{g} \in L^p_w(G)$ and w is symmetric, then $\tilde{\hat{g}} \in L^p_w(G)$, where $\tilde{\hat{g}} = \hat{g}(-x)$. Hence we have $\hat{\tilde{\hat{g}}} = g$. That means the Fourier transform is onto. One can also write $\|f\|^{p,q}_{w,\omega} = \|\hat{f}\|^{q,p}_{\omega,w}$ for all $f \in A^{p,q}_{w,\omega}(G)$. This proves the theorem. □

THEOREM 1.15.

a) *If ω is symmetric, then $A^{p,q}_{w,\omega}(G)$ is a Banach module over $A^{\omega}(G)$.*
b) *If, in addition, ω satisfies (BD), then the $A^{\omega}(G)$-module $A^{p,q}_{w,\omega}(G)$ admits an approximate identity and is an essential Banach module.*

Proof. Since $A^{q,p}_{\omega,w}(G)$ is a Banach convolution module over $L^1_{\omega}(G)$ and the generalized Fourier transform turns convolution into pointwise multiplication, then (a) is easily shown by Theorem 1.14; (b) is a consequence of Proposition 1.13. □

THEOREM 1.16. *If w satisfies (BD), then under a natural map, the maximal ideal space of $L^1_w(G)$ is homeomorphic to the one of $A^{1,q}_{w,\omega}(G)$.*

Proof. As usual, (regular) maximal ideals are identified with multiplicative linear forms on the Banach algebras under consideration, and hence, the map of the theorem becomes the restriction map.

First of all, since w satisfies (BD), then $A^{1,q}_{w,\omega}(G)$ is dense in $L^1_w(G)$, and Theorem 24.B of [16] applies and shows that under the restriction map, the space of multiplicative functionals on $L^1_w(G)$ is homeomorphic to a closed subspace of the maximal ideal space of $A^{1,q}_{w,\omega}(G)$. Therefore, it suffices to show that restriction is surjective:

In turn, this will follow if we can show that a multiplicative linear form F on $A^{1,q}_{w,\omega}(G)$ remains continuous in the $L^1_w(G)$-topology. Since $|F(f)| \le \left(\|f\|^{1,q}_{w,\omega}\right)_{\mathrm{sp}}$, this amounts to estimating the spectral norm in terms of the L^1_w-norm.

Let f^{*n} denote the nth "convolution power" of f. Then

$$\begin{aligned}\|f^{*n}\|^{1,q}_{w,\omega} &\le \|f^{*(n-1)}\|_{1,w}\|f\|_{1,w} + \|\hat{f}^{*(n-1)}\hat{f}\|_{q,\omega}\\ &= \|f^{*(n-1)}\|_{1,w}\|f\|_{1,w} + \|\hat{f}^{(n-1)}\hat{f}\|_{q,\omega},\end{aligned}$$

and we wish to obtain a suitable bound on the second summand: Since

$$\|\hat{f}\|_{\infty} \le \|f\|_1 < \infty,$$

$$\begin{aligned}\|\hat{f}^{(n-1)}\hat{f}\|_{q,\omega} &\le \|\hat{f}^{(n-1)}\|_{\infty}\|f\|_{q,\omega} \le \|f^{*(n-1)}\|_1\|\hat{f}\|_{q,\omega} \le \|f\|_1^{n-1}\|\hat{f}\|_{q,\omega}\\ &\le \|f\|_{1,w}^{n-1}\|\hat{f}\|_{q,\omega},\end{aligned}$$

and so we see that

$$\left(\|f\|_{w,\omega}^{1,q}\right)_{\mathrm{sp}} \le \|f\|_{1,w} ,$$

indeed, $\|f^{*n}\|_{w,\omega}^{1,q} \le \|f\|_{1,w}^{n-1}\|f\|_{w,\omega}^{1,q}$ yields the last inequality after taking nth roots and the limit as $n \to \infty$.

Returning to F as above, the inequality now implies that $|F(f)| \le \|f\|_{1,w}$, and we are done. □

Remark 1.17. Because of Theorems 1.16 and 5.6 in [21] the maximal ideal space of $A_{w,\omega}^{1,q}(G)$ may be identified with the space of "generalized characters" (with respect to w).

The last part of Section 1 is concerned with inclusion relations between spaces of the form $A_{w,\omega}^{p,q}(G)$, and the first result is the following one:

LEMMA 1.18. *If $A_{w_1,\omega_1}^{p_1,q_1}(G) \subset A_{w_2,\omega_2}^{p_2,q_2}(G)$, then the inclusion is continuous: There exists a constant $c > 0$ such that*

$$\|f\|_{w_2,\omega_2}^{p_2,q_2} \le c\|f\|_{w_1,\omega_1}^{p_1,q_1}$$

for every $f \in A_{w_1,\omega_1}^{p_1,q_1}(G)$.

P r o o f. Put $\|f\| = \|f\|_{w_1,\omega_1}^{p_1,q_1} + \|f\|_{w_2,\omega_2}^{p_2,q_2}$. Then a sequence (f_n) is $\|\cdot\|$ Cauchy if and only if it is Cauchy for both original norms. Since $A_{w_1,\omega_1}^{p_1,q_1}(G)$ and $A_{w_2,\omega_2}^{p_2,q_2}(G)$ are continuously embedded into $L^{p_1}(G)$ and $L^{p_2}(G)$ respectively, it follows that $\left(A_{w_1,\omega_1}^{p_1,q_1}(G), \|\cdot\|\right)$ is complete.

It is clear that the $\|\cdot\|$-topology is finer than the original one, i.e., the two Banach topologies are comparable, hence coincide by the Closed Graph Theorem. This shows that the two norms are equivalent, and, in particular, that there is $c > 0$ such that $\|\cdot\| \le c\|\cdot\|_{w_1,\omega_1}^{p_1,q_1}$, and this implies the required inequality. □

THEOREM 1.19. *Let p and q be fixed numbers satisfying $1 \le p, q < \infty$. Then $A_{w_1,\omega_1}^{p,q}(G) \subset A_{w_2,\omega_2}^{p,q}(G)$ if and only if $w_2 < w_1$ and $\omega_2 < \omega_1$.*

P r o o f. The sufficiency of the condition is obvious. The necessity follows using the preceding lemma: There now is $c > 0$ such that $\|f\|_{w_2,\omega_2}^{p,q} \le c\|f\|_{w_1,\omega_1}^{p,q}$ for $f \in A_{w_1,\omega_1}^{p,q}(G)$.

Using the result of Theorem 1.9 one sees that there are constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 w_1(x) \le \|L_x f\|_{w_1,\omega_1}^{p,q} \le w_1(x)\|f\|_{w_1,\omega_1}^{p,q}$$

as well as

$$c_2 w_2(x) \le \|L_x f\|_{w_2,\omega_2}^{p,q} \le w_2(x)\|f\|_{w_2,\omega_2}^{p,q}$$

for $x \in G$. Therefore,

$$c_2 w_2(x) \le c\|L_x f\|_{w_1,\omega_1}^{p,q} \le c w_1(x)\|f\|_{w_1,\omega_1}^{p,q} .$$

Taking the supremum over $\|f\|^{p,q}_{w_1,\omega_1} \le 1$ yields $w_2(x) \le \frac{c}{c_2} w_1(x)$ for $x \in G$. The relation $w_2 < w_1$ is obtained in the same manner. □

COROLLARY 1.20. *Let p and q be fixed numbers satisfying $1 \le p,q < \infty$. Then $A^{p,q}_{w_1,\omega_1}(G) = A^{p,q}_{w_2,\omega_2}(G)$ if and only $w_1 \approx w_2$ and $\omega_1 \approx \omega_2$ (i.e., there exist $M_1, M_2, K_1, K_2 > 0$ such that $w_1(x) \le M_1 w_2(x)$, $w_2(x) \le M_2 w_1(x)$ and $\omega_1(x) \le K_1 \omega_2(x)$, $\omega_2(x) \le K_2 \omega_1(x)$ for all $x \in G$).*

Remark 1.21. Lemma 1.18 and the proof of Proposition 1.19 also show that, if $A^{p,q}_{w_1,\omega_1}(G) \subset A^{p,q}_{w_2,\omega_2}(G)$, then already $w_2 < w_1$ and $\omega_2 < \omega_1$ for $1 \le p_1, q_1, p_2, q_2 < \infty$.

The next proposition involves spaces $A^{p,q}_{w,\omega}(G)$ with fixed weights w, ω rather than fixed p, q:

PROPOSITION 1.22. *Assume that w satisfies (BD) and that $\omega(t) \to \infty$ as $t \to \infty$ in $\hat{G}$. If*

$$A^{p_1,q_1}_{w,\omega}(G) \subset A^{p_2,q_2}_{w,\omega}(G),$$

then $q_2 \le q_1$.

P r o o f. The proof is a copy of the proof of Theorem 3.6 in [8] and is omitted. □

THEOREM 1.23. *Assume that $1 \le p_1, p_2, q_1, q_2 < \infty$, w_1, w_2 satisfy (BD) and symmetric and $w_i \to \infty$, $\omega_i \to \infty$ at infinity for $i = 1,2$. Then $A^{p_1,q_1}_{w_1,\omega_1}(G) = A^{p_2,q_2}_{w_2,\omega_2}(G)$ if and only if $w_1 \approx w_2$, $\omega_1 \approx \omega_2$ and $p_1 = p_2$, $q_1 = q_2$.*

P r o o f. If the corresponding weights are equivalent, and the exponents are equal, then equality of these spaces is evident. For the converse, if we use Remark 1.21, we obtain that the corresponding weights are equivalent. Then, by Proposition 1.21, we have $q_1 = q_2$ since w_1 and w_2 are symmetric, and the spaces $A^{p_1,q_1}_{w_1,\omega_1}(G)$, $A^{q_1,p_1}_{\omega_1,w_1}(\hat{G})$ and $A^{p_2,q_2}_{w_2,\omega_2}(G)$, $A^{q_2,p_2}_{\omega_2,w_2}(\hat{G})$ are isometric. Hence, $A^{p_1,q_1}_{w_1,\omega_1}(G) = A^{p_2,q_2}_{w_2,\omega_2}(G)$ if and only if $A^{q_1,p_1}_{\omega_1,w_1}(\hat{G}) = A^{q_2,p_2}_{\omega_2,w_2}(\hat{G})$. Once again using Proposition 1.22 we obtain that $p_1 = p_2$. This completes the proof. □

EXAMPLES. There are several interesting special cases of the space $A^{p,q}_{w,\omega}(G)$. The most important is the space

$$A^{2,2}_{w,\omega}(\mathbb{R}^m) = L^2_w(\mathbb{R}^m) \cap L^2_s(\mathbb{R}^m),$$

where $L^2_s(\mathbb{R}^m)$ denotes the Bessel potential spaces of order s (see [20]) with

$$\omega(y) = \left(1 + |y|^2\right)^{\frac{s}{2}}.$$

For more examples, we refer the reader to [8] and [10].

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