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AN EXTENSION OF KOTZIG'S THEOREM ON THE MINIMUM WEIGHT OF EDGES IN 3–POLYTOPES

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ABSTRACT. It is proved that if no triangular faces in a 3-polytope touch each other, then there exist two adjacent vertices with the degree sum at most 8. On the other hand, an example of a 3-polytope is constructed in which every two triangles have at most one point in common, but the degree sum of every two adjacent vertices is greater than 8.

The weight of an edge in a 3-polytope is defined to be the degree sum of its end vertices. Kotzig proved [1] that in each 3-polytope there exists an edge of the weight at most 13, the bound being sharp. For such bipartite 3-polytopes, where each face is a quadrangle, he strengthened this bound to 8 [2], that is also the best possible as shown by the dual of the $(3,5,3,5)$-Archimedian solid.

In the present note, it is proved that the bound 8 for the minimum weight of edges is valid also in a broader class of 3-polytopes in which no triangular faces touch each other, but becomes invalid as soon as every two triangles are allowed to touch each other in at most one point. The second part of this statement follows from the figure below, while the first one is represented by

THEOREM. *If P is a 3-polytope in which no triangular faces touch each other, then there exists an edge in P of the weight at most 8.*

Proof. Let a 3-polytope P contradict to our Theorem. The Euler formula

$$|V| - |E| + |F| = 2$$

for P may be rewritten [3] as

$$\sum_{v\in V}\big(s(v)-6\big) + \sum_{i\geq 3}(2i-6)|F_i| = -12\,, \tag{1}$$

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where $s(v)$ denotes the degree of a vertex v and F_i is the set of i-faces. We define a function $g\colon V \cup F \to R$ as follows:

$$g(v) = s(v) - 6 \quad \text{for each} \quad v \in V\,;$$
$$g(f) = 2i - 6 \quad \text{for each} \quad f \in F_i\,.$$

Now (1) may be rewritten as

$$\sum_{x \in V \cup F} g(x) = -12\,. \tag{2}$$

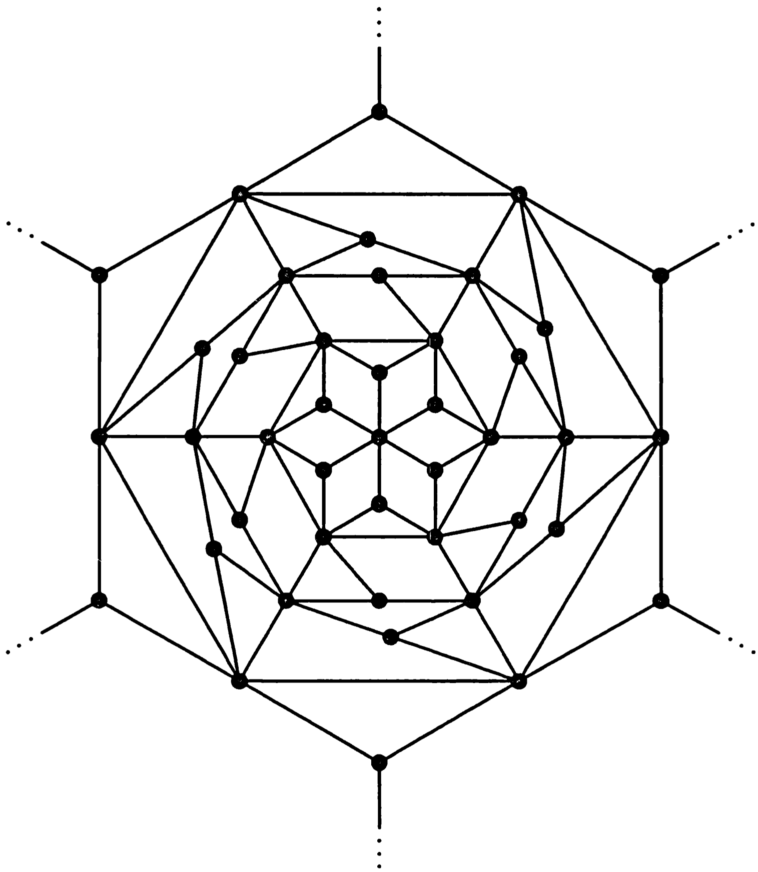

Figure 1.

We want to construct a function $g^*\colon V \cup F \to R$ with the following mutually excluding properties:

$$\sum_{x\in V\cup F} g^*(x) = \sum_{x\in V\cup F} g(x) = -12\,; \tag{3}$$

$$g^*(x) \geq 0 \qquad \text{for each} \quad x \in V \cup F\,; \tag{4}$$

this contradiction will complete the proof of our Theorem.

First a few definitions. For a vertex v, we denote by $v_1, v_2, \ldots, v_{s(v)}$ the vertices adjacent to v in a cyclic order. A 6-vertex v is called ***particular*** if the face $[\ldots v_1 v v_2 \ldots]$ is a triangle, $s(v_2) = \cdots = s(v_r) = 3$, where $r \geq 2$, and $s(v_i) \geq 6$ for $i > r$. We remind that according to the properties of our P, no vertex is incident with more than one 3-face and no 3-face is incident with more than one 3-vertex. The initial *charge* of each vertex or face, x, of P, is defined to be $g(x)$.

At the first stage of constructing g^*, every nontriangular face, f, transfers parts of its charge to incident vertices.

Namely, if $f = [u_1 u_2 u_3 u_4]$, then u_1 receives from f:

0 if $s(u_1) \geq 6$, $s(u_2) \geq 6$, and $s(u_4) \geq 6$;
3/4 if $s(u_1) \geq 6$, $s(u_2) \geq 6$, and $s(u_4) = 3$;
1/2 in all other cases;

if the size of f is greater than four, then f transfers 3/4 to each incident vertex.

Now for every face f the value of $g^*(f)$ is completely defined and equal to the resulting charge on f. To construct g^* for the vertices, another distribution of charges is required: Each vertex, w, of degree greater than four transfers to each adjacent vertex, v, the following charge:

1/4 if $s(v) = 4$ and the edge wv is incident with a 3-face;
1/2 if $s(v) = 3$ and wv is not incident with a 3-face;
3/4 if $s(v) = 3$ and wv is incident with a 3-face.

After the second distribution of charges, the function $g^*\colon V \cup F \to R$ is completely defined. By construction, (3) is satisfied. It remains to verify (4):

First, let $f = [u_1 u_2 u_3 u_4]$ be a 4-face. If f does not give more than 1/2 to any incident vertices, then

$$g^*(f) \geq 2 - 4 \times 1/2 = 0\,.$$

Otherwise, suppose u_1 receives 3/4 from f. Then it may be assumed in addition that $s(u_4) = 3$ and $s(u_2) \geq 6$. Furthermore, $s(u_3) \geq 6$ due to the property of P, therefore f gives nought to u_2 and

$$g^*(f) \geq 2 - 2 \times 3/4 - 1/2 = 0\,.$$

If f is an i-face where $i > 4$, then

$$g^*(f) = g(f) - i \times 3/4 = 2i - 6 - 3i/4 = (i - 24/5) \times 5/4 > 0.$$

Now consider a vertex $v \in V$. Assume first that $s(v) = 3$. If v is incident with a 3-face, then it receives at least $1/2$ from each of the two nontriangular faces, $3/4$ from adjacent vertices along each of the two edges incident with 3-faces, and also $1/2$ along the edge not incident with a 3-face. This implies

$$g^*(v) \geq s(v) - 6 + 2 \times 1/2 + 2 \times 3/4 + 1/2 = 0.$$

If v is not incident with a 3-face, then there holds

$$g^*(v) \geq -3 + 3 \times 1/2 + 3 \times 1/2 = 0.$$

Let $s(v) = 4$; if v is incident with a 3-face, then

$$g^*(v) \geq -2 + 3 \times 1/2 + 2 \times 1/4 = 0,$$

otherwise

$$g^*(v) \geq -2 + 4 \times 1/2 = 0.$$

Consider the case $s(v) = 5$. If v is incident with a 3-face, then

$$g^*(v) \geq -1 + 4 \times 1/2 - 1/4 > 0.$$

(We make use of the fact that v is not adjacent to 3-vertices here.) If v is not incident with a 3-face, then

$$g^*(v) \geq -1 + 5 \times 1/2 > 0.$$

At last, assume $s(v) \geq 6$. Let also v be incident with t faces of the size three (of course, $0 \leq t \leq 1$) and p such nontriangular faces that give nought to v, i.e., are of the type $[vu_1u_2u_3]$ where $s(u_1) \geq 6$ and $s(u_3) \geq 6$. Then v receives from the faces $s(v) - t - p$ times $1/2$ or $3/4$. Our next purpose is to estimate how much should v transfer totally to adjacent vertices of degree 3 and 4.

For $t = 0$ we clearly have

$$g^*(v) \geq s(v) - 6 + \big(s(v) - p\big) \times 1/2 - \big(s(v) - p\big) \times 1/2 \geq 0.$$

Assume from now on that $t = 1$, i.e. v is incident with a 3-face, say $[vv_1v_2]$. Observe that at most one of the vertices v_1, v_2 may be of degree not greater than four, hence v_1 and v_2 receive from v at most $3/4$ totally. Evidently, at most $s(v)-2-p$ of the vertices $v_3, v_4, \dots, v_s(v)$ may have the degree 3 or 4. It follows that

$$g^*(v) \geq s(v)-6+\big(s(v)-1-p\big)\times 1/2-\big(s(v)-2-p\big)\times 1/2-3/4 = s(v)-6-1/4\,.$$

So, the target inequality $g^*(v) \geq 0$ remains still unproved only under the following assumptions: $s(v) = 6$; all the faces incident with v except $[vv_1v_2]$ are 4-faces; precisely $4-p$ vertices among $v_3, \dots, v_6$ receive $1/2$ from v each, i.e. are 3-vertices; finally, one of v_1, v_2, say v_2, receives $3/4$, i.e. is a 3-vertex. But it follows that among $v_3, \dots, v_6$ precisely p vertices have degree at least 6, hence $s(v_i) \geq 6$ for $7-p \leq i \leq 6$. Besides, $s(v_i) = 3$ for $2 \leq i \leq 6-p$. In other words, if v is not particular, then $g^*(v) \geq 0$ is already proved. However, if v is particular, then it receives $3/4$ from certain incident face; in our case from $[v_1vv_6z]$ if $p=0$ and from $[v_{7-p}vv_{6-p}z]$ if $p>0$: $s(z) \geq 6$ since z is adjacent to the 3-vertex v_{6-p}. Therefore

$$g^*(v) \geq (4-p)\times 1/2+3/4-(4-p)\times 1/2-3/4 = 0\,.$$

So, (4) is proved. Now from (3) and (4) we get a contradiction

$$0 \leq \sum_{x \in V \cup F} g^*(x) = \sum_{x \in V \cup F} g(x) = -12,$$

which completes the proof of our Theorem.

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