Piotr Fijałkowski

On equation $P(\mathrm{D})u = f(u^{(m)}) + g\big(t, (u^{(j)})\big)$ on the line

Mathematica Slovaca, Vol. 55 (2005), No. 3, 283--294

Persistent URL: `http://dml.cz/dmlcz/128697`

Math. Slovaca, **55** (2005), No. 3, 283–294

ON EQUATION $P(\mathrm{D})u = f(u^{(m)}) + g\big(t, (u^{(j)})\big)$ ON THE LINE

Piotr Fijałkowski

(*Communicated by Michal Fečkan*)

ABSTRACT. This paper deals with the existence of a real solution for the ordinary differential equation

$$P(\mathrm{D})u = f(u^{(m)}) + g\big(t, (u^{(j)})\big)$$

in the Sobolev space $H_n(\mathbb{R})$ where n is the degree of the linear differential operator $P(\mathrm{D})$.

1. Introduction

We shall consider an ordinary differential equation of the following form:

$$P(\mathrm{D})u = f(u^{(m)}) + g\big(t, (u^{(j)})_{j=j_1,\dots,j_l}\big) . \tag{1}$$

Above, $P(\mathrm{D})$ is a linear differential operator in $\mathbb{R}$ with a polynomial P of one variable and, as in [8],

$$\mathrm{D} = -\,\mathrm{id} = -\,\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} ,$$

for which the polynomial $P(-\,\mathrm{id})$ of the variable d has real coefficients. We shall consider two cases of m: $m = 2k-1$, $m = 2k$. Other assumptions on P and the values of j will be precised in the theorems corresponding to these cases.

Let us assume that the function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and that there are positive constants ε_0 and K, such that

$$|f(x)| \le K|x| \qquad \text{for} \quad |x| \le \varepsilon_0 . \tag{2}$$

Let us suppose that the function $g\colon \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ satisfies Carathéodory condition in the following form: $g(t,\cdot)$ is continuous a.e. with respect to t and

2000 Mathematics Subject Classification: Primary 34B40, 34C11.
Keywords: ordinary differential equation, fixed point, topological degree.

$g(\cdot, y_1, \ldots, y_l)$ is measurable for all $y_1, \ldots, y_l$. (For $l = 0$, we assume simply that $g = g(t)$ is measurable.)

Let us assume also that there is a function $h \in L_2(\mathbb{R})$ such that

$$|g(t, y_1, \ldots, y_l)| \le h(t) \tag{3}$$

for $t, y_1, \ldots, y_l \in \mathbb{R}$. (Note that we do not assume any growth condition for the function f.)

We shall look for real solutions of equation (1) in the Sobolev space $H_n(\mathbb{R})$ where n denotes the degree of P. Thus the problem can be treated as a kind of an infinite interval boundary value one. Such an approach can be found in [1], [4] and [5].

We define the Sobolev space $H_s(\mathbb{R})$ for non-negative s as the space of tempered distributions v on $\mathbb{R}$ for which

$$\|v\|_s^2 := (2\pi)^{-1} \int_{-\infty}^{+\infty} |(\mathcal{F}v)(\xi)|^2 (1 + |\xi|^2)^s \, \mathrm{d}\xi < +\infty \tag{4}$$

where $\mathcal{F}$ denotes the Fourier Transformation. Note that $H_0(\mathbb{R}) = L_2(\mathbb{R})$. Consequently, we shall denote the norm of $L_2(\mathbb{R})$ as $\|\cdot\|_0$.

Let us note the following important lemma (see [8; Corollary 7.9.4]):

Lemma 1. *Let s be a real number and j an integer for which $0 \le j < s - 1/2$.*

Then any $v^{(j)}$ is (i.e. may be represented as) a continuous bounded function if $v \in H_s(\mathbb{R})$, and there exists a constant C such that

$$\sup_{t \in \mathbb{R}} |v^{(j)}(t)| \le C \|v\|_s .$$

In particular, we have:

Lemma 2. *Every function $v \in H_1(\mathbb{R})$ is continuous, vanishing at $-\infty$, $+\infty$, and*

$$\sup_{t \in \mathbb{R}} |v(t)| \le \|v\|_1 .$$

P r o o f. The lemma is obvious by the identity

$$v^2(t) = \int_{-\infty}^{t} v(s) v'(s) \, \mathrm{d}s$$

and the Schwarz inequality. □

Note that, under our assumptions on the function f, the following lemma is valid:

Lemma 3. *The mapping $v \mapsto f \circ v$ maps continuously $H_1(\mathbb{R})$ into $L_2(\mathbb{R})$.*

Proof. By Lemma 2, any function $v \in H_1(\mathbb{R})$ is bounded, vanishes at infinity and clearly, $v \in L_2(\mathbb{R})$. Hence, by (2), $f \circ v \in L_2(\mathbb{R})$.

Let $v_j \to v_0$, as $j \to \infty$, in $H_1(\mathbb{R})$. Let $0 < \varepsilon \leq \varepsilon_0/2$. We have, for a certain j_1,

$$\int_{-\infty}^{+\infty} |v_j(t) - v_0(t)|^2 \, \mathrm{d}t \leq \varepsilon \tag{5}$$

and

$$|v_j(t) - v_0(t)| \leq \varepsilon, \qquad t \in \mathbb{R}, \tag{6}$$

if $j \geq j_1$.

Lemma 2 implies the existence of a constant α such that

$$|v_0(t)| \leq \varepsilon \qquad \text{for} \quad |t| \geq \alpha \tag{7}$$

and

$$\int_{-\infty}^{-\alpha} |v_0(t)|^2 \, \mathrm{d}t \,, \int_{\alpha}^{+\infty} |v_0(t)|^2 \, \mathrm{d}t \leq \varepsilon \,. \tag{8}$$

By Lemma 2, $v_j \to v_0$ uniformly, which implies the uniform convergency $f\big(v_j(t)\big) \to f\big(v_0(t)\big)$ for $t \in [-\alpha, \alpha]$. Thus, for a certain j_2,

$$\int_{-\alpha}^{\alpha} \big|f\big(v_j(t)\big) - f\big(v_0(t)\big)\big|^2 \, \mathrm{d}t \leq \varepsilon \tag{9}$$

if $j \geq j_2$.

Suppose $j \geq \max\{j_1, j_2\}$. From (6) and (7),

$$|v_0(t)|, |v_j(t)| \leq \varepsilon_0 \qquad \text{for} \quad |t| \geq \alpha \,.$$

From (5) and (8),

$$\int_{\infty}^{\alpha} |v_j(t)|^2 \, \mathrm{d}t \leq 2 \int_{-\infty}^{-\alpha} |v_j(t) - v_0(t)|^2 \, \mathrm{d}t + 2 \int_{-\infty}^{-\alpha} |v_0(t)|^2 \, \mathrm{d}t \leq 4\varepsilon \,. \tag{10}$$

Thus, by (2), (8), and (10),

$$\begin{aligned} \int_{\infty}^{\alpha} \big|f\big(v_j(t)\big) - f\big(v_0(t)\big)\big|^2 \, \mathrm{d}t &\leq 2 \int_{-\infty}^{-\alpha} \big|f\big(v_j(t)\big)\big|^2 \, \mathrm{d}t + 2 \int_{-\infty}^{-\alpha} \big|f\big(v_0(t)\big)\big|^2 \, \mathrm{d}t \\ &\leq 2K^2 \int_{-\infty}^{-\alpha} |v_j(t)|^2 \, \mathrm{d}t + 2K^2 \int_{-\infty}^{-\alpha} |v_0(t)|^2 \, \mathrm{d}t \\ &\leq 10K^2\varepsilon \,. \end{aligned}$$

Estimating the integral

$$\int_{\alpha}^{+\infty} \left|f\big(v_j(t)\big) - f\big(v_0(t)\big)\right|^2 \, dt$$

in the similar way and making use of (9), we obtain

$$\int_{-\infty}^{+\infty} \left|f\big(v_j(t)\big) - f\big(v_0(t)\big)\right|^2 \, dt \le (20K^2 + 1)\varepsilon ,$$

which ends the proof. □

By $H_s^{\mathrm{loc}}(\mathbb{R})$, we denote a local space corresponding to the space $H_s(\mathbb{R})$. this means the space of all distributions $v \in D'(\mathbb{R})$ for which $\phi v \in H_s(\mathbb{R})$ if $\phi \in C_0^\infty(\mathbb{R})$ where $C_0^\infty(\mathbb{R})$ is the space of smooth functions with compact supports in $\mathbb{R}$. The space $H_s^{\mathrm{loc}}(\mathbb{R})$ is a Frechét space with the topology defined by the system of the seminorms $\|\phi v\|_s$, $\phi \in C_0^\infty(\mathbb{R})$.

We shall use the following theorem (see, for example, [8; Theorem 10.1.27]):

Theorem 1. *For $0 \le s_1 < s_2$, the embedding $H_{s_2}(\mathbb{R}) \to H_{s_1}(\mathbb{R})$ is continuous and the embedding $H_{s_2}^{\mathrm{loc}}(\mathbb{R}) \to H_{s_1}^{\mathrm{loc}}(\mathbb{R})$ is compact, this means it is continuous and maps bounded sets onto precompact ones.*

1. Main results

We shall prove, under some additional assumptions, the existence of a solution of equation (1) for $m = 2k - 1$.

Theorem 2. *Assume that all assumptions from Introduction are valid and the degree of the polynomial $\operatorname{Re} P$ is equal to $2n_1$ with*

$$n_1 \ge 1 .$$

Suppose that $\operatorname{Re} P$ has no real roots, hence there exists a positive constant C_1 for which

$$(1 + \xi^2)^{n_1} \le C_1 |\operatorname{Re} P(\xi)| . \tag{11}$$

Then the equation

$$P(\mathrm{D})u = f\big(u^{(2k-1)}\big) + g\big(t, u^{(k)}, \dots, u^{(k+n_1-1)}\big) \tag{12}$$

with

$$1 \le k \le n_1$$

has a solution $u \in H_n(\mathbb{R})$ *for which*

$$\|u^{(k)}\|_{n_1} \le C_1 \|h\|_0 . \tag{13}$$

Proof. We shall show that if equation (12) has a real solution $u \in H_n(\mathbb{R})$, then estimation (13) holds. Indeed, we have

$$\begin{aligned}
\int_{-\infty}^{+\infty} P(\mathrm{D})u(t)u^{(2k)}(t)\,\mathrm{d}t &= \int_{-\infty}^{+\infty} \overline{P(\mathrm{D})u(t)}u^{(2k)}(t)\,\mathrm{d}t \\
&= (2\pi)^{-1} \int_{-\infty}^{+\infty} \overline{\mathcal{F}(P(\mathrm{D})u)(\xi)}\mathcal{F}(u^{(2k)})(\xi)\,\mathrm{d}\xi \\
&= (2\pi)^{-1} \int_{-\infty}^{+\infty} (\mathrm{i}\xi)^k \overline{\mathcal{F}(P(\mathrm{D})u)(\xi)}\mathcal{F}(u^{(k)})(\xi)\,\mathrm{d}\xi \\
&= (2\pi)^{-1}(-1)^k \int_{-\infty}^{+\infty} \overline{P(\xi)}\,\big|\mathcal{F}(u^{(k)})(\xi)\big|^2\,\mathrm{d}\xi \\
&= (2\pi)^{-1}(-1)^k \int_{-\infty}^{+\infty} \operatorname{Re} P(\xi)\big|\mathcal{F}(u^{(k)})(\xi)\big|^2\,\mathrm{d}\xi .
\end{aligned}$$

From the above equality, estimation (11), and definition (4), we have

$$C_1^{-1}\|u^{(k)}\|_{n_1}^2 \le \left| \int_{-\infty}^{+\infty} P(\mathrm{D})u(t)u^{(2k)}(t)\,\mathrm{d}t \right| . \tag{14}$$

On the other hand, by condition (3) and the Schwarz inequality, we have

$$\begin{aligned}
&\left| \int_{-\infty}^{+\infty} P(\mathrm{D})u(t)u^{(2k)}(t)\,\mathrm{d}t \right| \\
&= \left| \int_{-\infty}^{+\infty} f\big(u^{(2k-1)}(t)\big)u^{(2k)}(t)\,\mathrm{d}t + \int_{-\infty}^{+\infty} g\big(t, u^{(k)}(t), \dots, u^{(k+n_1-1)}(t)\big)u^{(2k)}(t)\,\mathrm{d}t \right|
\end{aligned}$$

$$= \left| \int_0^0 f(x)\,\mathrm{d}x + \int_{-\infty}^{+\infty} g\big(t, u^{(k)}(t), \dots, u^{(k+n_1-1)}(t)\big)\, u^{(2k)}(t)\,\mathrm{d}t \right|$$

$$= \left| \int_{\infty}^{+\infty} g\big(t, u^{(k)}(t), \dots, u^{(k+n_1-1)}(t)\big) u^{(2k)}(t)\,\mathrm{d}t \right|$$

$$\le \int_{\infty}^{+\infty} |h(t) u^{(2k)}(t)|\,\mathrm{d}t$$

$$\le \|h\|_0 \|u^{(2k)}\|_0 \le \|h\|_0 \|u^{(k)}\|_k \le \|h\|_0 \|u^{(k)}\|_{n_1}\,. \tag{15}$$

Observe that, from Lemma 3, $f \circ u^{(2k-1)} \in L_2(\mathbb{R})$, which warrants the above calculation.

From (14) and (15), we obtain estimation (13) for solutions $u \in H_l(\mathbb{R})$ of equation (1).

Let us define the function f_1 in the following way:

$$f_1(x) = \begin{cases} f\big(-C_1\|h\|_0\big) & \text{for } x < -C_1\|h\|_0\,, \\ f(x) & \text{for } -C_1\|h\|_0 \le x \le C_1\|h\|_0\,, \\ f\big(C_1\|h\|_0\big) & \text{for } x > C_1\|h\|_0\,. \end{cases} \tag{16}$$

and consider the equation with a positive integer j and $\lambda \in [0,1]$:

$$P(\mathrm{D})v = \lambda\Big(f_1\big(v^{(2k-1)}\big) + g\big(t, v^{(k)}, \dots, v^{(k+n_1-1)}\big)\Big)\chi_{[-j,j]} \tag{17}$$

where $\chi_{[-j,j]}$ is the characteristic function of the interval $[-j,j]$.

We shall compute an a priori bound for real solutions $v \in H_n(\mathbb{R})$ of equation (17). In the same way as above, we obtain

$$C_1^{-1}\|v^{(k)}\|_{n_1}^2 \le \left| \int_{-\infty}^{+\infty} P(\mathrm{D})v(t) v^{(2k)}(t)\,\mathrm{d}t \right|. \tag{18}$$

Now, we shall estimate the right hand side of equation (17):

$$
\begin{aligned}
&\Big|\int_{-\infty}^{+\infty} P(\mathrm{D})v(t)v^{(2k)}(t)\ \mathrm{d}t\Big| \\
&= \lambda\Big|\int_{-j}^{j} f_1\big(v^{(2k-1)}(x)v^{(2k)}(t)\big)v^{(2k)}(t)\ \mathrm{d}t \\
&\qquad\qquad + \int_{-j}^{j} g\big(t, v^{(k)}(t),\dots,v^{(k+n_1-1)}(t)\big)v^{(2k)}(t)\ \mathrm{d}t\Big| \\
&= \lambda\Big|\int_{v^{(2k-1)}(-j)}^{v^{(2k-1)}(j)} f_1(x)\ \mathrm{d}x + \int_{-j}^{j} g\big(t, v^{(k)}(t),\dots,v^{(k+n_1-1)}(t)\big)v^{(2k)}(t)\ \mathrm{d}t\Big| \\
&\le \big|v^{(2k-1)}(j) - v^{(2k-1)}(-j)\big| \sup_{y\in\mathbb{R}} |f_1(y)| + \|h\|_0 \|v^{(2k)}\|_0 \\
&\le 2\|v^{(2k-1)}\|_1 \sup_{y\in\mathbb{R}} |f_1(y)| + \|h\|_0 \|v^{(k)}\|_{n_1} \\
&< \Big(2\sup_{y\in\mathbb{R}} |f_1(y)| + \|h\|_0\Big)\|v^{(k)}\|_{n_1}\,.
\end{aligned}
$$

Thus we obtain the a priori bound for real solutions $v \in H_n(\mathbb{R})$ of equations (17):

$$
\|v^{(k)}\|_{n_1} \le C_1\Big(2\sup_{y\in\mathbb{R}} |f_1(y)| + \|h\|_0\Big)\,. \tag{19}
$$

Now, we observe that equation (17) in the space $H_n(\mathbb{R})$ is equivalent to

$$
P\mathcal{F}v = \lambda\mathcal{F}\Big(\big(f_1\big(v^{(2k-1)}(\cdot)\big) + g\big(\cdot, v^{(k)}(\cdot),\dots,v^{(k+n_1)}(\cdot)\big)\big)\chi_{[-j,j]}\Big). \tag{20}
$$

Since the polynomial $\operatorname{Re} P$ has no real roots, hence the same is for the polynomial P. Thus, from (20), we have

$$
v^{(k)} = \lambda\mathcal{F}^{-1}\Big(\frac{(\mathrm{i}\,\cdot)^k}{P}\mathcal{F}\Big(\big(f_1\big(v^{(2k-1)}(\cdot)\big) + g\big(\cdot, v^{(k)}(\cdot),\dots,v^{(k+n_1-1)}(\cdot)\big)\big)\chi_{[-j,j]}\Big)\Big).
$$

Setting $w := v^{(k)}$, we have:

$$
w = \lambda\mathcal{F}^{-1}\Big(\frac{(\mathrm{i}\,\cdot)^k}{P}\mathcal{F}\Big(\big(f_1\big(w^{(k-1)}(\cdot)\big) + g\big(\cdot, w(\cdot),\dots,w^{(n_1-1)}(\cdot)\big)\big)\chi_{[-j,j]}\Big)\Big). \tag{21}
$$

Let

$$
T_j(w) = \mathcal{F}^{-1}\Big(\frac{(\mathrm{i}\,\cdot)^k}{P}\mathcal{F}\Big(\big(f_1\big(w^{(k-1)}(\cdot)\big) + g\big(\cdot, w(\cdot),\dots,w^{(n_1-1)}(\cdot)\big)\big)\chi_{[-j,j]}\Big)\Big).
$$

Thus we may rewrite equation (21) as

$$(I - \lambda T_j)(w) = 0\,, \tag{22}$$

where I stands for the identity mapping.

We shall prove that T_j is a compact mapping from $H_{n_1}(\mathbb{R})$ into itself. By Lemma 1, the mapping $v \mapsto (f \circ v)\chi_{[-j,j]}$ maps continuously $H_{n_1-1/4}(\mathbb{R})$ into $L_2(\mathbb{R})$. We can prove the continuity of the Nemytzkii operator

$$H_{n_1-1}(\mathbb{R}) \ni w \mapsto g\big(\cdot, w(\cdot), \dots, w^{(n_1-1)}(\cdot)\big) \in L_2(\mathbb{R}) \tag{23}$$

in the standard way, using (3) (see for example [3; Appendix], where the case of $n_1 - 1 = 0$ is considered). By Theorem 1, the above operator is also continuous as a mapping from $H_{n_1-1/4}(\mathbb{R})$ into $L_2(\mathbb{R})$.

The operator

$$v \mapsto \mathcal{F}^{-1}\left(\frac{(\mathrm{i}\,\cdot)^k}{P}\mathcal{F}v\right)$$

maps continuously $L_2(\mathbb{R})$ into $H_{n_1}(\mathbb{R})$ (even into $H_{n-k}(\mathbb{R})$). From above and continuity of the embedding $H_{n_1}(\mathbb{R}) \to H_{n_1-1/4}(\mathbb{R})$, $T_j\colon H_{n_1}(\mathbb{R}) \to H_{n_1}(\mathbb{R})$ is continuous. If B is a bounded set in $H_{n_1}(\mathbb{R})$, then B is bounded in $H^{\mathrm{loc}}_{n_1}(\mathbb{R})$, hence, by Theorem 1, it is precompact in $H^{\mathrm{loc}}_{n_1-1/4}(\mathbb{R})$. By the factor $\chi_{[-j,j]}$, T continuously maps $H^{\mathrm{loc}}_{n_1-1}(\mathbb{R})$ into $H_{n_1}(\mathbb{R})$, hence the set $T(B)$ is precompact in $H_{n_1}(\mathbb{R})$.

Now, we treat $I - \lambda T_j$ as a mapping from the ball of the center at zero and the radius

$$C_1\Big(2\sup_{y\in\mathbb{R}}|f_1(y)| + \|h\|_0\Big) + \varepsilon$$

in the space H_{n_1} into H_{n_1}.

From the a priori bound (19), we know that

$$(I - \lambda T_j)(w) \neq 0$$

for

$$\|w\|_{n_1} = C_1\Big(2\sup_{y\in\mathbb{R}}|f_1(y)| + \|h\|_0\Big) + \varepsilon\,,$$

hence the Leray-Schauder degree of the mapping $I - \lambda T_j$ with respect to zero is equal to 1 — the degree of I. From the Leray-Schauder degree theory (see for example [10]), equation (22) has a solution $w_j \in H_{n_1}$.

By (19) and Theorem 1, the sequence $\{w_j\}_{j=1}^{\infty}$ is precompact in $H^{\mathrm{loc}}_{n_1-1/4}(\mathbb{R})$. Take a subsequence $\{w_{j_m}\}_{m=1}^{\infty}$ convergent to a certain w in the topology of $H^{\mathrm{loc}}_{n_1-1/4}(\mathbb{R})$. Since $\{w_{j_m}\}_{m=1}^{\infty}$ is bounded in $H_{n_1}(\mathbb{R})$, hence it is also bounded in $H_{n_1-1/4}(\mathbb{R})$. Thus we have $w \in H_{n_1-1/4}(\mathbb{R})$.

We shall demonstrate that w is a solution of the equation

$$w = \mathcal{F}^{-1}\Big(\frac{(\mathrm{i}\,\cdot)^k}{P}\mathcal{F}\big(f_1(w^{(k-1)}(\cdot)) + g(\cdot, w(\cdot), \dots, w^{(n_1-1)}(\cdot))\big)\Big). \tag{24}$$

Observe that

$$\big(f_1(w_{j_m}^{(k-1)}(\cdot)) + g(\cdot, w_{j_m}(\cdot), \dots, w_{j_m}^{(n_1-1)}(\cdot))\big)\chi_{[-j_m, j_m]} \longrightarrow f_1(w^{(k-1)}(\cdot)) + g(\cdot, w(\cdot), \dots, w^{(n_1-1)}(\cdot))$$

in the topology of the space $S'(\mathbb{R})$ of tempered distributions on $\mathbb{R}$.

In fact, for any $\phi \in S(\mathbb{R})$, from the boundness of f_1, (3), and the Lebesgue Theorem, we have

$$\int_{-\infty}^{+\infty} \phi(t)\big(f_1(w_{j_m}^{(k-1)}(t)) + g(t, w_{j_m}(t), \dots, w_{j_m}^{(n_1-1)}(t))\big)\chi_{[-j_m, j_m]}(t)\,\mathrm{d}t$$
$$\longrightarrow \int_{-\infty}^{+\infty} \phi(t)\big(f_1(w^{(k-1)}(t)) + g(\cdot, w(t), \dots, w^{(n_1-1)}(t))\big)\,\mathrm{d}t\,.$$

The convergence in $H^{\mathrm{loc}}_{n_1-1}(\mathbb{R})$ implies the convergence in $S'(\mathbb{R})$. Since $\mathcal{F}$ is an homeomorphism of $S'(\mathbb{R})$, (24) may be obtained from (21) if $m \to +\infty$.

Let

$$u := \mathcal{F}^{-1}\Big(\frac{1}{P}\mathcal{F}\big(f_1(w^{(k-1)}(\cdot)) + g(\cdot, w(\cdot), \dots, w^{(n_1-1)}(\cdot))\big)\Big)\,.$$

It is easy to see that $u \in H_n(\mathbb{R})$, $u^{(k)} = w$, hence u is a solution of equation

$$P(\mathrm{D})u = f_1(u^{(2k-1)}) + g(t, u^{(k)}, \dots, u^{(k+n_1-1)})\,,$$

which have the same a priori bound for solution (13) as equation (12). From definition (16) of function f_1, we conclude that u is a solution of equation (12), which ends the proof. □

Now, we shall formulate a theorem for the case of $m = 2k$:

THEOREM 3. *Suppose that all assumptions from Introduction are satisfied. Let*

$$P(\xi) = \operatorname{Re} P(\xi) + \mathrm{i}\xi^{2n_3+1}Q(\xi)\,,$$

and suppose that the degree of Q is equal to $2n_2$ with

$$n_2 \ge 1\,.$$

Suppose $\operatorname{Re} P(0) \ne 0$ and Q has no real roots, hence there exists a positive constant C_2 for which

$$(1+\xi^2)^{n_2} \le C_2|Q(\xi)|\,. \tag{25}$$

Then the equation

$$P(\mathrm{D})u = f\big(u^{(2k)}\big) + g\big(t, u^{(k+n_3+1)}, \dots, u^{(k+n_2+n_3)}\big) \tag{26}$$

with

$$n_3 + 1 \le k \le n_3 + n_2$$

has a solution $u \in H_n(\mathbb{R})$ *for which*

$$\|u^{(k+n_3+1)}\|_{n_2} \le C_2 \|h\|_0 \,. \tag{27}$$

Proof. The proof of Theorem 3 is similar to the proof of Theorem 2. A priori bound (27) instead of (13) for real solutions $u \in H_n(\mathbb{R})$ of equation (26) is the unique essential difference between them. We shall demonstrate that if equation (26) has a real solution $u \in H_n(\mathbb{R})$, then estimation (27) holds. In fact, we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} P(\mathrm{D})u(t)u^{(2k+1)}(t)\,\mathrm{d}t \\
&\quad = \int_{-\infty}^{+\infty} \overline{P(\mathrm{D})u(t)}u^{(2k+1)}(t)\,\mathrm{d}t \\
&\quad = (2\pi)^{-1} \int_{-\infty}^{+\infty} \overline{\mathcal{F}\big(P(\mathrm{D})u\big)(\xi)}\mathcal{F}\big(u^{(2k+1)}\big)(\xi)\,\mathrm{d}\xi \\
&\quad = (2\pi)^{-1} \int_{-\infty}^{+\infty} (\mathrm{i}\xi)^k \overline{\mathcal{F}\big(P(\mathrm{D})u\big)(\xi)}\mathcal{F}\big(u^{(k)}\big)(\xi)\,\mathrm{d}\xi \\
&\quad = (2\pi)^{-1} \int_{-\infty}^{+\infty} \overline{\big(\operatorname{Re} P(\xi) + \mathrm{i}\xi^{2n_3+1}Q(\xi)\big)}(\mathrm{i}\xi)^{2k+1}|\mathcal{F}(u)(\xi)|^2\,\mathrm{d}\xi \\
&\quad = (-1)^k(2\pi)^{-1} \int_{-\infty}^{+\infty} \big((\mathrm{i}\xi)^{2k+1}\operatorname{Re} P(\xi) - \mathrm{i}^{2k+2}\,\xi^{2k+2n_3+2}Q(\xi)\big)|\mathcal{F}(u)(\xi)|^2\,\mathrm{d}\xi \\
&\quad = (-1)^k(2\pi)^{-1} \int_{-\infty}^{+\infty} \xi^{2k+2n_3+2}Q(\xi)|\mathcal{F}(u)(\xi)|^2\,\mathrm{d}\xi \\
&\quad = (-1)^k(2\pi)^{-1} \int_{-\infty}^{+\infty} Q(\xi)\big|\mathcal{F}\big(u^{(k+n_3+1)}\big)(\xi)\big|^2\,\mathrm{d}\xi \,.
\end{aligned}$$

From the above equality, estimation (25), and definition (4), we have

$$C_1^{-1}\|u^{(k+n_3+1)}\|_{n_2}^2 \le \left|\int_{-\infty}^{+\infty} P(\mathrm{D})u(t)u^{(2k+1)}(t)\,\mathrm{d}t\right|,$$

and (27) may be obtained as in the proof of Theorem 2.

Observe that, under our assumptions,

$$P(\xi) \neq 0$$

for $\xi \in \mathbb{R}$.

Setting $w := v^{(k+n_3+1)}$, we obtain a continuous Nemytzkii operator

$$H_{n_2-1}(\mathbb{R}) \ni w \mapsto g\big(\cdot, w(\cdot), \dots, w^{(n_2-1)}(\cdot)\big) \in L_2(\mathbb{R})$$

instead of (23).

Thus it is easy to see that Theorem 3 may be proved as Theorem 2. □

3. Applications

We shall give two simple examples of applications of Theorems 2 and 3.

EXAMPLE 1. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and differentiable at zero, and $h \in L_2(\mathbb{R})$. Let us consider the equation

$$u^{(4)} + u^{(3)} + u = f(u') + g(t, u', u'') \tag{28}$$

with the function g satisfying the assumptions from Introduction.

We have

$$f(x) = f(0) + f_1(x)$$

and, from differentiability of f at zero, f_1 satisfies condition (2). Setting $v := u - f(0)$, we obtain the equation

$$v^{(4)} + v^{(3)} + v = f(v') + h(t)\,. \tag{29}$$

We shall apply Theorem 2. We have

$$P(\xi) = \operatorname{Re} P(\xi) = \xi^4 + 1 \ge (\xi^2 + 1)^2)/2\,,$$

hence $n_1 = 2$ and $C_1 = 2$. From Theorem 2, equation (29) has a solution $v \in H_4(\mathbb{R})$ for which

$$\|v'\|_1 \le 2\|h\|_0\,.$$

Thus equation (28) has a solution u for which $u - f(0) \in H_4(\mathbb{R})$ and

$$\|u'\|_1 \le 2\|h\|_0\,.$$

EXAMPLE 2. It is easy to see that the equation

$$u^{(5)} - u^{(3)} - u'' + u = f\big(u^{(4)}\big) + g\big(t, u^{(4)}\big)\,, \tag{30}$$

with functions f and g satisfying assumptions from Introduction, satisfies the assumptions of Theorem 3. Indeed, we have

$$P(\xi) = 1 + \mathrm{i}\xi^3(\xi^2 + 1)\,,$$

and $n = 5$, $n_2 = 1$, $n_3 = 1$, $C_2 = 1$, $k = 2$.

From Theorem 3, equation (30) has a solution $u \in H_5(\mathbb{R})$ for which

$$\|u^{(4)}\|_1 \le \|h\|_0\,.$$

REFERENCES

[1] ANDRES, J.—GABOR, G.—GÓRNIEWICZ, L.: *Boundary value problems on infinit intervals*, Trans. Amer. Math. Soc. **351** (1999), 4861–4903.

[2] BERNSTEIN, S. N.: *Sur les équations du calcul des variations*, Ann. Sci. École Norm. Sup. **29** (1912), 431–485.

[3] BROWDER, F. E.: *Nonlinear functional analysis and nonlinear integral equations of Hammerstein and Urysohn type.* In: Contributions to Nonlinear Functional Analysis (E. H. Zarantonello, ed.), Academic Press, New York, 1971, pp. 425 500.

[4] CONSTANTIN, A.: *On an infinite interval boundary value problem*, Ann. Mat. Pura Appl. (4) **176** (1999), 379–396.

[5] FIJAŁKOWSKI, P.: *On the solvability of nonlinear elliptic equations in Sobolev spaces*, Ann. Polon. Math. **56** (1992), 149–156.

[6] FIJAŁKOWSKI, P.—PRZERADZKI, B.: *On a boundary value problem for a nonlocal elliptic equation*, J. Appl. Anal. (To appear).

[7] GRANAS, A.—GUENTHER, R.—LEE, J.: *Nonlinear boundary value problems for ordinary differential equations*, Dissertationes Math. (Rozprawy Mat.) **244** (1985).

[8] HÖRMANDER, L.: *The Analysis of Linear Partial Differential Operators II: Differential Operators with Constant Coefficients.* Grundlehren Math. Wiss. 257, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.

[9] KRASNOSEL'SKII, M. A.—PUSTYL'NIK, E. I. SOBOLEVSKII, P. E. ZABREIKO, P. P.: *Integral Operators of Summable Functions*, Nauka, Moscow, 1966. (Russian)

[10] LLOYD, N. G.: *Degree Theory.* Cambridge Tracts in Math. 73, Cambridge Univ. Press, Cambridge, 1978.

[11] YOSIDA, K.: *Functional Analysis*, Springer, Berlin, 1980.

Received December 18, 2003
Revised February 23, 2004

Faculty of Mathematics
University of Łódź
ul. S. Banacha 22
PL 90-238 Łódź
POLAND
E-mail: fijalkowski@math.uni.lodz.pl