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COMPLEMENTARILY DOMATIC NUMBER OF A GRAPH

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The domatic number of a graph was introduced by E. J. Cockayne and S. T. Hedetniemi [1]. Further a lot of variants of this concept were introduced and studied, e. g. the total domatic number [2], the connected domatic number [3] etc. Here we shall introduce a new variant of the domatic number, namely the complementarily domatic number. First we recall the definition of the domatic number.

Let G be an undirected graph, let D be a subset of the vertex set $V(G)$ of G. The set D is called dominating in G if to each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ adjacent to x. A domatic partition of G is a partition of $V(G)$, all of whose classes are dominating sets in G. The maximum number of classes of a domatic partition of G is called the domatic number of G and denoted by $d(G)$.

Now we turn to the definition of the complementarily domatic number. Let again D be a subset of $V(G)$. The set D is called complementarily dominating if to each vertex $x \in V(G) - D$ there exist vertices $y \in D$, $z \in D$ such that y is adjacent and z is non-adjacent to x in G. A complementarily domatic partition of G is a partition of $V(G)$, all of whose classes are complementarily dominating sets in G. The maximum number of classes of a complementarily domatic partition of G is called a complementarily domatic number of G and denoted by $d_{cp}(G)$.

Note that a complementarily dominating set of G is simultaneously a dominating set of G and a dominating set of its complement $\bar{G}$.

The following propositions are clear from the definition.

Proposition 1. *Let G be a graph, let $\bar{G}$ be its complement. Then* $d_{cp}(G) = d_{cp}(\bar{G})$.

Proposition 2. *Let G be a graph with n vertices. Then* $d_{cp}(G) \leqq n/2$.

Proposition 3. *Let G be a graph, let $\bar{G}$ be its complement. Then* $d_{cp}(G) \leqq \min(d(G), d(\bar{G}))$.

In [1] it was proved that $d(G) \leqq \delta(G) + 1$ for any graph G, where $\delta(G)$ is the minimum degree of a vertex in G. The following proposition follows directly from this fact.

Proposition 4. *Let G be a graph with n vertices, let $\delta(G)$ be the minimum degree* *of a vertex in G, let $\Delta(G)$ be the maximum degree of a vertex in G. Then* $d_{cp}(G) \leqq \min(\delta(G) + 1, n - \Delta(G))$.

Now we shall proved some theorems.

Theorem 1. *Let G be a disconnected graph. Then $d_{cp}(G) = d(G)$.*

Proof. Let $\mathscr{D}$ be a domatic partition of G with $d(G)$ classes. Each class of $\mathscr{D}$ has non-empty intersections with vertex sets of all connected components of G. Let $D \in \mathscr{D}$, $x \in V(G) - D$. Let C be the connected component of G which contains x, let C' be another connected component of G. As D is a dominating set of G, there exists $y \in D$ adjacent to x; evidently y is in C. Now in C' there exists a vertex $z \in D$; it is evidently non-adjacent to x. Hence $\mathscr{D}$ is a complementarily domatic partition of G with $d(G)$ vertices. As $d_{cp}(G) \leqq d(G)$, we have $d_{cp}(G) = d(G)$.

Corollary 1. *Let G be a graph whose complement $\bar{G}$ is disconnected. Then $d_{cp}(G) = d(\bar{G})$.*

According to [1], every graph has the domatic number at least 1; if it has no isolated vertex, it has the domatic number at least 2. Obviously also every graph has the complementarily domatic number at least 1. But the assertion about the complementarily domatic number at least 2 is not so simple.

A vertex of G is called saturated if it is adjacent to all other vertices of G.

Theorem 2. *Let G be a graph containing either an isolated vertex, or a saturated one. Then $d_{cp}(G) = 1$.*

Proof. If G contains an isolated vertex, then $d(G) = 1$ and, according to Proposition 3, also $d_{cp}(G) = 1$. If G contains a saturated vertex, then its complement $\bar{G}$ contains an isolated vertex and the assertion follows from Proposition 1.

Theorem 3. *Let G be a disconnected graph without isolated vertices. Then $d_{cp}(G) \geqq 2$.*

Proof. The assertion follows from the quoted assertion from [1] and from Theorem 1.

Corollary 2. *Let G be a graph without saturated vertices whose complement is disconnected. Then $d_{cp}(G) \geqq 1$.*

Theorem 4. *Let G be a connected graph of the diameter at least 4. Then $d_{cp}(G) \geqq 2$.*

Proof. Let u be a vertex of G with the property that there exists at least one vertex having the distance from u equal to the diameter a of G. For $i = 0, 1, \ldots, a$ let M_i be the set of all vertices of G having the distance i from u. Let M_2' be the subset of M_2 consisting of vertices adjacent to at least one vertex of M_3, let $M_2'' = M_2 - M_2'$. Now let D_1 be the union of M_0, M_2'' and all M_i for odd $i \geqq 3$, let D_2 be the union of M_1, M_2' and all M_i for even $i \geqq 4$. Let $x \in V(G) - D_1 = D_2$. If $x \in M_1$, then it is adjacent to $u \in D_1$ and non-adjacent to any vertex of $M_3 \subseteq D_1$. If $x \in M_2'$, then it is adjacent to a vertex of $M_3 \subseteq D_1$ and non-adjacent to $u \in D_1$. If $x \in M_i$ for even $i \geqq 4$, then it is adjacent to a vertex of $M_{i-1} \subseteq D_1$ and non-adjacent to $u \in D_1$. Hence D_1 is a complementarily dominating set in G. Now let $x \in V(G) - D_2 = D_1$. If $x \in M_0$, then $x = u$ and is adjacent to any vertex of $M_1 \subseteq D_2$ and non-adjacent to any vertex of $M_2' \subseteq D_2$. If $x \in M_2''$ then x is adjacent to a vertex of $M_1 \subseteq D_2$ and non-adjacent to any vertex of $M_4 \subseteq D_2$. If $x \in M_3$, then x is adjacent to a vertex of $M_2' \subseteq D_2$ and non-adjacent to any vertex of $M_1 \subseteq D_2$. If $x \in M_i$ for odd $i \geqq 5$, then x is adjacent to a vertex of $M_{i-1} \subseteq D_2$ and non-adjacent to any vertex of $M_1 \subseteq D_2$.

Corollary 3. *Let G be a graph whose complement is connected and has the diameter at least 4. Then $d_{cp}(G) \geqq 2$.*

Theorem 5. *There exist connected graphs G_1, G_2 of the diameter 2 such that their complements G_1, G_2 have also the diameter 2 and $d_{cp}(G_1) = 1$, $d_{cp}(G_2) \geqq 2$.*

Proof. The graph G_1 is the circuit of the length 5. Let its vertices be u_1, u_2, u_3, u_4, u_5, let its edges be u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_1. Suppose that $d_{cp}(G_1) \geqq 2$. As the union of complementarily dominating sets is again a complementarily dominating set, we may suppose that there exists a complementarily domatic partition $\{D_1, D_2\}$ of G_1. Without loss of generality we may suppose $u_1 \in D_1$. The vertices adjacent to u_1 are u_2 and u_5; therefore at least one of these vertices must be in D_2. Without loss of generality we may suppose that $u_2 \in D_2$. The vertices non-adjacent to u_1 are u_3 and u_5; therefore at least one of them must be in D_2. First suppose $u_3 \in D_2$. The vertices adjacent to u_3 are u_2 and u_4 and $u_2 \in D_2$, therefore $u_4 \in D_1$. The vertex u_5 cannot be in D_1, because it is adjacent only to vertices of D_1. It cannot be in D_2, because there is no vertex of D_1 non-adjacent to it. We have a contradiction. Now suppose $u_4 \in D_2$. The vertices adjacent to u_3 are u_2 and u_4 and they are both in D_2; therefore $u_3 \in D_1$. The vertex u_5 cannot be in D_1, because then there would be no vertex of D_2 non-adjacent to u_3. It cannot be in D_2, because then there would be no vertex of D_1 non-adjacent to u_2. We have again a contradiction and thus $d_{cp}(G_1) = 1$.

The graph G_2 is the Petersen graph in Fig. 1. The vertices of D_1 are denoted by 1, the vertices of D_2 by 2. The partition $\{D_1, D_2\}$ is a complementarily domatic partition of G_2.

Theorem 6. *There exist connected graphs G_1, G_2 of the diameter 3 such that their complements $\bar{G}_1$, $\bar{G}_2$ have also the diameter 3 and $d_{cp}(G_1) = 1$, $d_{cp}(G_2) \geqq 2$.*

Proof. The graph G_1 is in Fig. 2. Suppose that there exists a complementarily domatic partition $\{D_1, D_2\}$ of G_1. Without loss of generality we may suppose that $u_1 \in D_1$. The unique vertex adjacent to u_1 is u_2, therefore $u_2 \in D_2$. The unique vertex non-adjacent to u_2 is u_4, therefore $u_4 \in D_1$. The unique vertex adjacent to u_4 is u_3, therefore $u_3 \in D_2$. The vertex u_5 cannot be in D_1, because there is no vertex of D_2 non-adjacent to it. It cannot be in D_2, because there is no vertex of D_1 adjacent to it. We have a contradiction and thus $d_{cp}(G_1) = 1$.

The graph G_2 is the path of the length 3. It is in Fig. 3; again the vertices of

D_1 are denoted by 1, the vertices of D_2 by 2 and $\{D_1, D_2\}$ is a complementarily domatic partition of G_2.

Theorem 7. *Let P_n be the path of the length n. Then $d_{cp}(P_1) = d_{cp}(P_2) = 1$, $d_{cp}(P_n) = 2$ for $n \geqq 3$.*

Proof. The paths P_1 and P_2 contain saturated vertices, thus the assertions for them follow from Theorem 2. The assertion for P_3 was proved in the proof of Theorem 6. For P_n with $n \geqq 4$ it follows from Theorem 4 and Proposition 3.

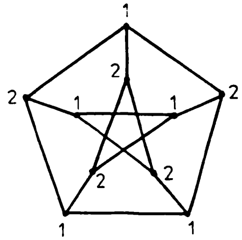

Fig. 1

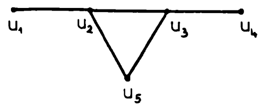

Fig. 2

Fig. 3

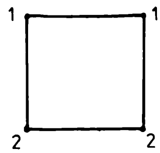

Fig. 4

Theorem 8. *Let C_n be the circuit of the length n. Then $d_{cp}(C_3) = 1$, $d_{cp}(C_4) = 2$, $d_{cp}(C_5) = 1$, $d_{cp}(C_n)$ for $n \geqq 6$ divisible by 3, $d_{cp}(C_n) = 2$ for $n \geqq 7$ non-divisible by 3.*

Proof. The assertion for C_3 follows from Theorem 2. For C_5 it was proved in the proof of Theorem 5. As it was proved by E. J. Cockayne and S. T. Hedetniemi, $d(C_n) = 2$ for n non-divisible by 3 and $d(C_n) = 3$ for n divisible by 3. Therefore the complementarily domatic number of these circuits cannot be greater. For C_4 a complementarily domatic partition is shown in Fig. 4. Let C_n for $n \geqq 6$ have the vertices $u_1, \ldots, u_n$ and edges $u_i u_{i+1}$ for $i = 1, \ldots, n-1$ and $u_n u_1$. If n is divisible by 3, then we may put $D_i = \{u_j | j \equiv i \pmod 3\}$ for $i = 1, 2, 3$ and $\{D_1, D_2, D_3\}$ is a complementarily domatic partition of C_n. If n is not divisible by 3, then we may put $D_i = \{u_j | j \equiv i \pmod 2\}$ for $i = 1, 2$ and $\{D_1, D_2\}$ is a complementarily domatic partition of C_n.

We have met a graph G such that $d_{cp}(G) < \min(d(G), d(\bar{G}))$. We shall prove an existence theorem.

Theorem 9. *For any integer $k \geqq 5$ there exists a graph G with $4k$ vertices such that $d(G) = d(\bar{G}) = k + 1$, $d_{cp}(G) = k$.*

Proof. Let V_1, V_2, V_3, V_4 be pairwise disjoint sets, $|V_1| = |V_2| = |V_3| = |V_4| = k$. We construct a graph G with the vertex set $V = V_1 \cup V_2 \cup V_3 \cup V_4$. Two vertices of G are adjacent if and only if either they both belong to $V_1 \cup V_2$, or one belongs to V_1 and the other to V_3, or one belongs to V_2 and the other to V_4. There exists a domatic partition $\mathscr{D}$ of G with $k + 1$ classes such that one class is $V_3 \cup V_4$ and all the other classes of $\mathscr{D}$ are two-element sets consisting of one vertex of V_1 and one vertex of V_2. As $\delta(G) = k$, we have $d(G) = k + 1$. The complement $\bar{G}$ of G is evidently isomorphic to G, hence also $d(\bar{G}) = k + 1$. Now let D be a complementarily dominating set in G. If $D \cap V_1 = \emptyset$, then $V_3 \subseteq D$, because the vertices of V_3 are adjacent only to vertices of V_1. Analogously if $D \cap V_2 = \emptyset$, then $V_4 \subseteq D$. If $D \cap V_3 = \emptyset$, then $V_2 \subseteq D$, because the vertices of V_2 are non-adjacent only to vertices of V_3. Analogously if $D \cap V_4 = \emptyset$, then $V_1 \subseteq D$. Hence if D is disjoint with one of the sets V_1, V_2, V_3, V_4, then $|D| \geqq k \geqq 5$. If D has non-empty intersections with all of them, then $|D| \geqq 4$. Hence there exists no complementarily domatic partition of G with more than k classes. There exists a complementarily domatic partition of G with exactly k classes; it is an arbitrary partition of G such that each of its classes contains exactly one vertex from each of the sets V_1, V_2, V_3, V_4. Therefore $d_{cp}(G) = k$.

Finally we shall describe a construction of graphs G with the maximal possible complementarily domatic number, i.e. with $d_{cp}(G) = \lfloor n/2 \rfloor$.

Construction C. Let $n = 2k$, where k is a positive integer. Let $V = \{u_1, \ldots, u_k, v_1, \ldots, v_k\}$. If i, j are two numbers of the set $\{1, \ldots, n\}$ and $i \neq j$, we join either u_i with u_j and v_i with v_j, or u_i with v_j and v_i with u_j by an edge. Further we may add edges $u_i v_i$ for arbitrary numbers i.

Let $n = 2k + 1$, where k is a positive integer. We perform the construction for $n = 2k$. Then we add a vertex w and for any $i \geqq 2$ we join it with exactly one of the vertices u_i, v_i. Further we may join w with u_1 or v_1 or with both of them.

Proposition 5. *Construction* **C** *yields exactly all graphs G with $d_{cp}(G) = \lfloor n/2 \rfloor$, where n is the number of vertices of G.*

Proof is left to the reader.

Now we present two problems.

Problem 1. *Consider the class of graphs G with the property that the diameters of both G and $\bar{G}$ are* 2 *or* 3. *Characterize graphs from this class whose complementarily domatic number is* 1.

Problem 2. *How large can the difference* $\min(d(G), d(\bar{G})) - d_{cp}(G)$ *be?*

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ДОПОЛНИТЕЛЬНО ДОМАТИЧЕСКОЕ ЧИСЛО ГРАФА

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Резюме

Дополнительно доминирующим множеством в графе G называется подмножество D множества $V(G)$ вершин графа G, обладающее тем свойством, что для каждой вершины $x \in V(G) - D$ существует вершина $y \in D$, смежная с x, и вершина $z \in D$, несмежная с x. Максимальное число классов разбития множества $V(G)$, все классы которого являются дополнительно доминирующими множествами, называется дополнительно доматическим числом графа G. В работе описаны его основные свойства.