Mathematica Slovaca

Milan Paštéka

Some properties of Buck's measure density

Mathematica Slovaca, Vol. 42 (1992), No. 1, 15--32

Persistent URL: http://dml.cz/dmlcz/128687

Math. Slovaca, 42 (1992), No. 1, 15–32

Mathematica Slovaca

SOME PROPERTIES OF BUCK'S MEASURE DENSITY

MILAN PAŠTÉKA

ABSTRACT. The purpose of this paper is to describe some properties of Buck's measure density.

1. Introduction

Denote by $\mathbb{N}$ the set of all positive integers and by $P(\mathbb{N})$ the system of all subsets of $\mathbb{N}$. Let the symbol $a + \langle d \rangle$ for a nonnegative and $d \in \mathbb{N}$ denote the arithmetic sequence $\{a + dn, \quad n = 0, 1, 2 \dots\}$. We shall write $\langle d \rangle$ instead of $0 + \langle d \rangle$. The symbol $a + \langle d \rangle$ will be also used to denotes the set of elements of this sequence.

For two sets B_1, B_2 let the symbol $B_1 \dot{\subset} B_2$ denote that the set $B_1 \setminus B_2$ is finite. Instead of the facts $B_1 \dot{\subset} B_2$ and $B_2 \dot{\subset} B_1$ we shall write $B_1 \doteq B_2$.

In the paper [1], the measure density of a set $A \in P(\mathbb{N})$ has been introduced in the following way: Let D_0 be the system of all subsets $S \in \mathbb{N}$ such that there exists a finite number of arithmetic sequences $a_1 + \langle d_1 \rangle, \dots, a_k + \langle d_k \rangle$ such that

$$S \doteq a_1 + \langle d_1 \rangle \cup \dots \cup a_k + \langle d_k \rangle.$$

Now we introduce on D_0 a real function Δ as follows: For every disjoint union of arithmetic sequences

$$S = a_1 + \langle d_1 \rangle \dot{\cup} \dots \cup a_k + \langle d_k \rangle$$

we put

$$\Delta(S) = \frac{1}{d_1} + \dots + \frac{1}{d_k}.$$

AMS Subject Classification (1991): Primary 11B05. Secondary 28E99.

Key words: Measure density, Arithmetic progression, Uniform distribution, Darboux property.

And for each $S' \doteq S$ we put $\Delta(S') = \Delta(S)$. It can be easily seen that $\Delta(S)$ does not depend on the representation of S as union of disjoint arithmetic sequences.

If $A \in P(\mathbb{N})$, then the value

$$\mu^*(A) = \inf\{\Delta(S);\ A \dot{\subset} S \text{ and } S \in D_0\}$$

will be called the measure density of the set A.

The purpose of this paper is to describe some properties of the function μ^*.

In the next part we shall prove a formula for the evaluation of measure density and some corollaries will be deduced from it. The algebra of measurable sets will be the object of investigation in the third section. In particular, we show there that the measure density has the Darboux property on the algebra of measurable sets. The last part is devoted to the relationship between the measure density and uniform distribution. We also give a characterization of the algebra of measurable sets based on the notion of the uniform distribution in $\mathbb{Z}$.

In what follows we will employ the following notation

$$D(A) = \limsup_{n\to\infty} \frac{1}{n} \sum_{\substack{k\le n\\ k\in A}} 1$$

and

$$d(A) = \liminf_{n\to\infty} \frac{1}{n} \sum_{\substack{k\le n\\ k\in A}} 1$$

for $A \in P(\mathbb{N})$.

Obviously

$$d(A) \le D(A) \le \mu^*(A) \tag{1}$$

for every $A \in P(\mathbb{N})$. $D(A)$ will be called the upper asymptotic density and $d(A)$ the lower asymptotic density of the set A.

2. Limit formula

In this section we shall prove one formula for evaluation of the measure density. Using this formula we established some properties of μ^*. For $a, b \in \mathbb{N}$, denote by $a \bmod b$ the least nonnegative remainder of a after division by b. For the set $S \in P(\mathbb{N})$ and $b \in \mathbb{N}$ we put

$$S \bmod b = \{s \bmod b;\ s \in S\}.$$

The set $S \bmod b$ will be called the system of representatives of the set S modulo b. If $R(S,b)$ denotes the number of elements of the set $S \bmod b$, then the measure density can be evaluated according to the following theorem:

THEOREM 1. *Let $\{B_n\}$ be a sequence of positive integers for which the following condition is satisfied :*

(i) *For every $d \in \mathbb{N}$ there exists n_0 such that for $n > n_0$ we have $d \mid B_n$. Then*

$$\mu^*(S) = \lim_{n\to\infty} \frac{R(S, B_n)}{B_n}$$

for every $S \in P(\mathbb{N})$.

P r o o f. Suppose that $\{a_1^{(n)}, \dots, a_{k_n}^{(n)}\}$ is the system of representatives of the set S modulo $B_n (n = 1, 2, \dots)$. Then

$$S \subset \bigcup_{i=1}^{k_n} a_i + \langle B_n \rangle$$

and

$$k_n = R(S, B_n).$$

Since the arithmetic progressions on the right-hand side are disjoint, the definition of μ^*, gives that

$$\mu^*(S) \leq \frac{R(S, B_n)}{B_n} \tag{2}$$

for every $n = 1, 2, \dots$.

Lower bound. Let $\varepsilon > 0$. Then, according to the definition of $\mu(S)$, there exists a disjoint system of arithmetic sequences $a_1 + \langle d_1 \rangle, \dots, a_k + \langle d_k \rangle$ such that

$$S \mathrel{\dot{\subset}} a_1 + \langle d_1 \rangle \cup \dots \cup a_k + \langle d_k \rangle \tag{3}$$

and

$$\frac{1}{d_1} + \dots + \frac{1}{d_k} - \varepsilon \leq \mu^*(S). \tag{4}$$

Condition (i) implies that there exists n_0 such that for $n \geq n_0$ we have $d_i \mid B_n$, $i = 1, 2, \dots, k$. This divisibility relation implies that the arithmetic progression $a_i + \langle d_i \rangle$, $i = 1, 2, \dots, k$, can be represented as a disjoint union of the arithmetic progressions of the form

$$a_i + \langle d_i \rangle = \bigcup_{r=0}^{k_i^{(n)}} a_i + r d_i + \langle B_n \rangle,$$

where $k_i^{(n)} = \frac{B_n}{d_i} - 1$, $i = 1, 2, \ldots, k$, $n \geq n_0$. Consequently

$$a_1 + \langle d_1 \rangle \cup \cdots \cup a_k + \langle d_k \rangle = \bigcup_{j=0}^{R_n} b_j^{(n)} + \langle B_n \rangle, \tag{5}$$

where $b_j^{(n)} \in \mathbb{N}$, $j = 1, 2, \ldots, R_n$, $n > n_0$ and

$$\frac{1}{d_1} + \cdots + \frac{1}{d_k} = \frac{R_n}{B_n}. \tag{6}$$

From (3) there follows that the set

$$H = S \setminus \bigcup_{i=1}^{k} a_i + \langle d_i \rangle$$

is finite. Denote the number of its elements by h.

From (3) and (5) we have

$$S \setminus H \subset \bigcup_{j\ 1}^{R_n} b_j + \langle B_n \rangle, \qquad n \geq n_0. \tag{7}$$

The system of repre entatives of the set $S \setminus H$ modulo B_n has at least $R(S, B_n) - h$ elements. Two integers contained in the same arithmetic sequence $b + \langle B_n \rangle$ are congruent modulo B_n, (7) implies that

$$R_n \geq R(S, B_n) - h, \qquad n \geq n_0.$$

From the last inequality and from (2), (4) and (6) we have for $n \geq n_0$

$$\frac{R(S, B_n) - h}{B_n} - \varepsilon \leq \mu^*(S) \leq \frac{R(S, B_n)}{B_n}.$$

From this

$$0 \leq \frac{R(S, B_n)}{B_n} - \mu^*(S) \leq \frac{h}{B_n} + \varepsilon, \qquad n > n_0.$$

On the other hand $\lim_{n \to \infty} \frac{h}{B_n} = 0$ and the proof of Theorem is complete.

Note that the system of sequences satisfying the condition (i) is non-empty: One of such sequence is B_n, with $B_n = n!$ $(n - 1, 2, \ldots)$.

The second one can be constructed in the following manner : Let $2 = p_1 < p_2 < \dots$ be the increasing sequence of all primes. Let

$$p(n) = p_1 p_2 \dots p_n \tag{8}$$

for $n = 1, 2, \dots$. Then the sequence p(n) satisfies the condition (i) too.

There follows from Theorem 1 that μ^* has the properties of the so called strong submeasure, i.e.:

$$A \subset B \implies \mu^*(A) \leq \mu^*(B) \tag{ii}$$

$$\mu^*(A \cup B) + \mu^*(A \cap B) \leq \mu^*(A) + \mu^*(B) \tag{iii}$$

for every $A, B \in P(\mathbb{N})$.

The following four corollaries are immediate consequences of Theorem 1.

COROLLARY 1. *Let $A \in P(\mathbb{N})$. Then $\mu^*(A) = 1$ if and only if for every couple a, d, ($a \geq 0$, $d \in \mathbb{N}$) the set $a + \langle d \rangle \cap A$ is non-empty.*

If for $S \in P(\mathbb{N})$ and $a \in \mathbb{N}$, denote

$$a + S = \{a + s;\ s \in S\}$$

$$aS = \{as;\ s \in S\}.$$

Then Theorem 1 in turn implies :

COROLLARY 2. *If $S \in P(\mathbb{N})$ and $a \in \mathbb{N}$, then*

$$\mu^*(a + S) = \mu^*(S).$$

COROLLARY 3. *If $S \in P(\mathbb{N})$ and $a \in \mathbb{N}$, then*

$$\mu^*(aS) = \frac{\mu^*(S)}{a}.$$

Proof of Corollary 3. With any sequence $\{B_n\}$ of positive integers also the sequence $\{aB_n\}$ satisfies condition (i). Since

$$s_1 \equiv s_2 \pmod{B_n} \iff as_1 \equiv as_2 \pmod{aB_n}$$

for every $s_1, s_2 \in \mathbb{N}$, then

$$R(aS, aB_n) = R(S, B_n)$$

and thus

$$\frac{R(aS, aB_n)}{aB_n} = \frac{1}{a} \frac{R(S, B_n)}{B_n}.$$

Theorem 1 finishes the proof.

COROLLARY 4. *If for* $A_1, A_2 \in P(\mathbb{N})$ *there exists an arithmetic sequence* $a + \langle d \rangle$ *with* $A_1 \subset a + \langle d \rangle$ *and* $A_2 \cap a + \langle d \rangle = 0$, *then*

$$\mu^*(A_1 \cup A_2) = \mu^*(A_1) + \mu^*(A_2).$$

Proof. Let the sequence $\{B_n\}$ satisfy the condition (i). Then there exists n_0 such that for $n \geq n_0$ we have $d \mid B_n$. Therefore for $n \geq n_0$

$$R(A_1 \cup A_2, B_n) = R(A_1, B_n) + R(A_2, B_n)$$

and the assertion follows.

3. Measurable sets

In [1] the following concept of a measurable set $A \in P(\mathbb{N})$ has been intro duced: A set $A \in P(\mathbb{N})$ is called measurable if

$$\mu^*(A) + \mu^*(\mathbb{N} \setminus A) = 1. \tag{9}$$

The system of all measurable sets in $P(\mathbb{N})$ will be denoted D_μ.

Let D be the class of all the $A \in P(\mathbb{N})$ possessing the asymptotic density, i.e. for which $d(A) = D(A)$ holds. In [1] it is proved that $D_\mu \subset D$. This result can be concerning in the following form:

THEOREM 2. *If* A *is an arbitrary set from* D_μ, *then*

$$d(A) = \mu^*(A) = D(A).$$

Proof. If

$$A(n) = \sum_{\substack{k \leq n \\ k \in A}} 1 \qquad (n = 1, 2, \dots)$$

for $A \in P(\mathbb{N})$, then trivially $A(n) \leq R(A, n)$, for $n = 1, 2, \dots$. On the other hand $R(\mathbb{N} \setminus A, n)$ denotes the number of the remainder classes modulo n, having a non-empty intersection with the set $\mathbb{N} \setminus A$. Therefore the value $n - R(\mathbb{N} \setminus A, n)$ is the number of the remainder classes modulo n which are disjoint with $\mathbb{N} \setminus A$, i.e. which are contained in set A. From this we have for $n - 1, ?, \dots$

$$n - R(\mathbb{N} \setminus A, n) \leq A(n) \leq R(A, n).$$

Now let $\{k_n\}$ be an arbitrary sequence of positive integers, and $p(n)$ defined by (8). Then Theorem 1 yields for the sequence $\{k_n p(n)\}$ that

$$1-\mu^*(\mathbb{N}\setminus A) \le \liminf_{n\to\infty} \frac{A(k_n p(n))}{k_n p(n)} \le \limsup_{n\to\infty} \frac{A(k_n p(n))}{k_n p(n)} \le \mu^*(A).$$

Therefore, if $A \in D_\mu$, then (9) implies

$$\lim_{n\to\infty} \frac{A(k_n p(n))}{k_n p(n)} = \mu^*(A). \tag{10}$$

Put

$$K_n = \max\{k;\ p(k) \le n\}.$$

It can be seen that

$$\lim_{n\to\infty} K_n = \infty. \tag{11}$$

Every $n \in \mathbb{N}$ can be represented in the form

$$\begin{aligned} n &= k_n p(K_n) + r'_n, && 0 \le r'_n < p(K_n) \\ n &= k_n p(K_n-1) + r_n, && 0 \le r_n < p(K_n-1). \end{aligned}$$

Since $p(K_n-1) \mid p(K_n)$, we have

$$r_n \equiv r'_n (\operatorname{mod} p(K_n-1)).$$

The last congruence implies

$$r_n \le r'_n, \qquad n = 1,2,\dots\ . \tag{12}$$

Put

$$B_n = k'_n p(K_n-1). \tag{13}$$

From this it follows that $n = B_n + r_n$ and

$$\frac{A(n)}{n} = \frac{\dfrac{A(B_n)}{B_n} + O\left(\dfrac{r_n}{B_n}\right)}{1+\dfrac{r_n}{B_n}}. \tag{14}$$

According to (12) it holds that $B_n \ge k_n p(K_n) \ge p(K_n)$. Now we have

$$0 \le \frac{r_n}{B_n} \le \frac{p(K_n-1)}{p(K_n)} \le p_{K_n}^{-K_n} \to 0$$

for $n \to \infty$. From (11), (10) and (14) it follows

$$\lim_{n\to\infty} \frac{A(n)}{n} = \mu^*(A)$$

and the proof is finished.

Note that the inclusion $D \subset D_\mu$ does not hold. To see this consider the set

$$A = \{n + n!;\ n = 1, 2, \dots\}.$$

According to Theorem 1 we have $\mu^*(A) = 1$. But it is easily seen that $d(A) = D(A) = 0$. Therefore $A \in D$ and $A \notin D_\mu$.

From the properties (ii) and (iii) we can deduce that D_μ is an algebra of sets and that the function

$$\mu = \mu^* \Big|_{D_\mu}$$

is a finitely-additive probability measure on D_μ. In [1] it is proved that

$$\{\mu(S);\ S \in D_\mu\} = [0, 1]. \tag{15}$$

Seeing ideas of the proof of (15) a more precise result can be established. Before stating it, we reproduce here for the convenience of the reader the following result [1], p. 562 relation (ii)]:

LEMMA. *$H \in D_\mu$ if and only if for every $\varepsilon > 0$ there exist the sets $S_1, S_2 \in D_\mu$ such that*

$$S_1 \dot\subset H \dot\subset S_2 \quad \text{and} \quad \Delta(S_2) - \Delta(S_1) < \varepsilon.$$

THEOREM 3. *Let $\{H_n\}$, $n = 1, 2 \dots$ be a system of disjoint measurable sets. Let*

$$\lim_{n\to\infty} \mu^*\left(\bigcup_{k=n}^{\infty} H_k\right) = 0. \tag{16}$$

Then the set $H = \bigcup\limits_{k=1}^{\infty} H_k$ belongs to D_μ and

$$\mu(H) = \sum_{k=1}^{\infty} \mu(H_k). \tag{17}$$

Proof. Let $\varepsilon > 0$. Then according to (16) there exists n_0 such that for $n > n_0$ we have

$$\mu^*\left(\bigcup_{k=n}^{\infty} H_k\right) \le \frac{\varepsilon}{2}.$$

Therefore by the definition of μ^* there exists a set $G \in D_0$ such that for $n \geq n_0$ we have

$$\bigcup_{k=n}^{\infty} H_k \subset G$$

and

$$\Delta(G) < \varepsilon.$$

For $n \geq n_0$ we have evidently

$$H_1 \cup \cdots \cup H_n \subset H \subset H_1 \cup \cdots \cup H_n \cup G.$$

Lemma implies that $H \in D_\mu$. But

$$\sum_{j=1}^{n} \mu(H_j) \leq \mu(H) \leq \sum_{j=1}^{n} \mu(H_j) + \varepsilon$$

for $n > n_0$. Letting $n \to \infty$ we obtain

$$\sum_{j=1}^{\infty} \mu(H_j) \leq \mu(H) \leq \sum_{j=1}^{\infty} \mu(H_j) + \varepsilon.$$

This being true for every $\varepsilon > 0$ implies (17) and the proof is complete.

It can be easily seen that the condition (16) cannot be omitted. To see this take again the set

$$A = \{n + n!,\ n = 1, 2, \dots\} \tag{18}$$

and the disjoint system of the sets $H_n = n + n!$, $n = 1, 2, \dots$. Then $H_n \in D_\mu$, for every $n = 1, 2, \dots$ but

$$\mu^*\left(\bigcup_{k=n}^{\infty} H_k\right) = 1.$$

However, the set $A = \bigcup_{k=1}^{\infty} H_k$ does not belong to D_μ, as we see above.

Corollary. *Let $H_i \in D_\mu$, $i = 1, 2, \dots$ be a system of disjoint sets. Let $B \in D_\mu$ be a set such that $H_i \subset B$ for $i = 1, 2, \dots$ and*

$$\sum_{i=1}^{\infty} \mu(H_i) = \mu(B). \tag{19}$$

Then the set $H = \bigcup_{i=1}^{\infty} H_i$ *belongs to* D_μ *and*

$$\mu(H) = \mu(B).$$

P r o o f . It is obvious that

$$\bigcup_{i=n}^{\infty} H_i \subset B \setminus (H_1 \cup \cdots \cup H_{n-1})$$

for $i = 1, 2, \ldots$. Then (19) implies

$$\lim_{n\to\infty} \mu^*\left(\bigcup_{k=n}^{\infty} H_k\right) = 0.$$

Then according to Theorem 3 we have $H \in D_\mu$ and

$$\mu(H) = \sum_{i=1}^{\infty} \mu(H_i) = \mu(B).$$

The proof is complete.

Using Theorem 3 we can establish a stronger result, than that of (15), namely that the measure m has the Darboux property on the algebra D_μ . A different proof of this result can be found in [7].

THEOREM 4. *Let* $A \in D_\mu$. *Then for every* $\alpha \in [0, \mu(A)]$ *there exists a set* $B \in D_\mu$ *such that* $B \subset A$ *and* $\mu(B) = \alpha$.

P r o o f . If $\alpha = \mu(A)$, then the assertion is trivial. Let $\alpha < \mu(A)$. Then there exists an $\varepsilon > 0$ such that

$$\alpha < \mu(A) - \varepsilon.$$

It follows from lemma that there exists the sets $H_1, H_2 \in D_\mu$ such that

$$H_1 \dot{\subset} A \dot{\subset} H_2$$

and

$$\Delta(H_2) - \Delta(H_1) < \frac{\varepsilon}{2} . \tag{20}$$

Then there exists a finite set S with

$$H \setminus S \subset A \tag{21}$$

whereas the set $H = H_1 \setminus S$ is a union of a finite number of arithmetic sequences. Let d be the least common multiple of the moduli of these arithmetic sequences. Then H can be represented as a disjoint union

$$H = a_1 + \langle d \rangle \cup \cdots \cup a_k + \langle d \rangle,$$

where

$$\frac{k}{d} = \Delta(H_1) = \Delta(H).$$

But (20) implies $\Delta(H) = \Delta(H_1) > \Delta(H_2) - \frac{\varepsilon}{2} > \mu(A) - \varepsilon$. Therefore

$$\alpha < \frac{k}{d}. \tag{22}$$

Let

$$\alpha = \sum_{j=1}^{\infty} \frac{c_j}{d^j}, \qquad 0 \leq c_j < d, \quad j = 1, 2, \ldots$$

be the d-adic expansion of the number α. Then (22) implies that $c_1 < k$. If

$$S_1 = \bigcup_{i=1}^{c_1} a_i + \langle d \rangle,$$

then $S_1 \subset H$, $S_1 \cap a + \langle d \rangle = 0$ and $\mu(S_1) = \frac{c_1}{d}$. Now denote

$$S_n = \bigcup_{j=1}^{c_n} a_k + j d^{n-1} + \langle d^n \rangle$$

for $n = 2, 3, \ldots$. (If $c_n = 0$, then $S_n = 0$). The union on the right-hand side is disjoint and therefore

$$\mu(S_n) = \frac{c_n}{d^n}, \qquad n = 2, 3, \ldots. \tag{23}$$

It is obvious that for $n \geq 2$ we have

$$S_n \subset a_k + \langle d^{n-1} \rangle. \tag{24}$$

We claim that the sets S_n are disjoint. Suppose on the contrary that the intersection $S_m \cap S_n$ is non-empty for some $1 \le m < n$. Then there exist numbers $j, j_1, h, h_1 \in \mathbb{N}$ such that

$$1 \le j, \quad j_1 < d$$

and

$$jd^{m-1} + hd^m = j_1 d^{n-1} + h_1 d^n.$$

However, this yields $d \mid jd^{m-1}$, which is impossible. Therefore the sets S_n, $n = 1, 2, \dots$ are disjoint and moreover

$$S_n \subset H \subset A, n = 1, 2, \dots . \tag{25}$$

Denote

$$B = \bigcup_{n=1}^{\infty} S_n.$$

Then (24) implies that

$$S_n \subset a_k + \langle d^m \rangle$$

for $n > m$. Thus

$$\lim_{m \to \infty} \mu\left(\bigcup_{n=m}^{\infty} S_n \right) = 0.$$

Theorem 3 and (23) implies $B \in D_\mu$ and $\mu(B) = \alpha$. Moreover (25) implies that $B \subset A$ and the proof is complete.

Consider the set $A = \{n + n!;\ n = 1, 2, \dots\}$. Using (15) and Theorem 1 we prove the following result:

THEOREM 5. $\{\mu^*(S);\ S \subset A\} - [0, 1]$.

Proof. Let $\alpha \in [0, 1]$. From (15) we can deduce that there exists a set $B \in D_\mu$ such that

$$\mu(B) = \alpha. \tag{26}$$

Let $B = \{a_1 < a_2 < \dots\}$ and

$$S = \{a_k + (a_k)!;\ k = 1, 2, \dots\}.$$

Then $S \subset A$. If S is finite, then so is B is finite and consequently $\mu(B) = \mu^*(S) = 0$. Therefore suppose that S is an infinite set. Then

$$\lim_{n \to \infty} a_n - \infty$$

and the sequence $\{(a_n)!\}$ satisfies the condition (i). For $n \geq k$ we have

$$a_n \equiv a_n + (a_n)! \pmod{a_k!}.$$

Therefore

$$R(S, a_k!) = R(B, a_k!) + O(k)$$

for every $k \in \mathbb{N}$. The relation (26) and Theorem 1 yield $\mu^*(S) = \alpha$ and the proof is complete.

Note that the system $\{S;\ S \subset A\}$ is small as can be seen from the well-known characterization based on the dyadic mapping defined as follows: For $S \in P(\mathbb{N})$ we put

$$\Gamma(S) = \sum_{k \in S} 2^{-k}.$$

Then the system is measured using the Hausdorf dimension of the image [5, p. 19]. Thus if $\dim C$ denotes the Hausdorf dimension of the subset $C \subset \langle 0, 1 \rangle$, then Theorem 1 of [6, p. 20] immediately implies that $\dim \{\Gamma(S);\ S \subset A\} = 0$.

Denote by S^0 the system of all the sets $S \in P(\mathbb{N})$ with $D(S) = 0$. Then Theorem 5 implies in turn the next result:

COROLLARY. $\{\mu(S);\ S \in S^0\} = [0, 1]$.

4. Uniform distribution

In this part we will use the concept of uniform distribution in $\mathbb{Z}$. The reader is referred to [4, p. 335]. For more details it is proved in [2] that if the sequence $A = \{a_1, a_2, \dots\}$ is uniformly distributed in $\mathbb{Z}$, then $\mu^*(A) = 1$. This fact follows also from our corollary 1 of Theorem 1. [2] contains more precise results, e.g.

1. If $A \in D_\mu$, $\mu(A) = 1$ and $A = \{a_1 < a_2 < \dots\}$, then A is uniformly distributed in $\mathbb{Z}$.
2. There exists a sequence having the measure density 1, but which is not uniformly distributed in $\mathbb{Z}$.

The following theorem is closely related to these results:

THEOREM 6. *Let $S \in P(\mathbb{N})$. Then $\mu^*(S) = 1$ if and only if S can be rearranged into a sequence which is uniformly distributed in $\mathbb{Z}$.*

For the proof we shall need the following lemma :

LEMMA. *Let $\{x_n\}$ be a sequence of positive integers, such that $x_n \equiv n \pmod{n!}$, for $n = 1, 2, \dots$. Then $\{x_n\}$ is uniformly distributed in $\mathbb{Z}$.*

Proof. Let $m \in \mathbb{N}$. Then there exists n_0 such that $m \mid n!$ for $n > n_0$. Thus for $n > n_0$

$$x_n \equiv n \pmod{m}.$$

Then for $N \geq n_0$ and $j \in \mathbb{N}$, $0 \leq j < m$ we have

$$\frac{1}{N} \sum_{\substack{n \leq N \\ x_n \equiv j \pmod{m}}} 1 = \frac{1}{N} \sum_{\substack{n \leq n_0 \\ x_n \equiv j \pmod{m}}} 1 + \frac{1}{N} \sum_{\substack{n_0 < n \leq N \\ x_n \equiv j \pmod{m}}} 1 = \frac{1}{N} \sum_{\substack{n \leq N \\ n \equiv j \pmod{m}}} 1 + O\left(\frac{1}{N}\right) \longrightarrow \frac{1}{m}$$

as $N \to \infty$ and lemma follows.

Proof of Theorem 6. If $S = \{x_1, x_2, \dots\}$ is a uniformly distributed sequence in $\mathbb{Z}$, then by virtue of Corollary 1 of Theorem 1 we have $\mu^*(S) = 1$.

If $\mu^*(S) = 1$, then S has a non-empty intersection with every arithmetic sequence. Therefore for every $n \in \mathbb{N}$ there exists $y_n \in S$ such that

$$y_n \equiv n \pmod{n!}.$$

Then lemma implies that $\{y_n\}$ is uniformly distributed in $\mathbb{Z}$. We can assume that the sequence $\{y_n\}$ is increasing. If the set $S \setminus \{y_n;\ n = 1, 2, \dots\}$ is finite, then the proof is complete.

Suppose therefore that the set

$$S \setminus \{y_n;\ n = 1, 2, \dots\} = \{y'_n;\ n = 1, 2, \dots\}$$

is infinite. Define

$$x_n = \begin{cases} y_n, & \text{for } n \neq k^2, \\ y_{k^2} & \text{for } n = (2k)^2, \\ y'_k & \text{for } n = (2k+1)^2, \end{cases}$$

for $n = 1, 2, \dots$. Clearly $\{x_n;\ n = 1, 2, \dots\} = S$. Let $j, m \in \mathbb{N}$. Then for $N \to \infty$

$$\frac{1}{N} \sum_{\substack{n \leq N \\ x_n \equiv j \pmod{m}}} 1 = \frac{1}{N} \sum_{\substack{n \leq N \\ y_n \equiv j \pmod{m}}} 1 + O(N^{-\frac{1}{2}}) \longrightarrow \frac{1}{m}.$$

Thus the sequence $\{x_n\}$ is uniformly distributed in $\mathbb{Z}$. The proof of Theorem is complete.

We shall finish this paper pointing out one more analogy between the uniform distribution in $\mathbb{Z}$ and uniform distribution mod 1.

Let $\{x_n\}$ be a sequence of positive integers. Given $A \in P(\mathbb{N})$ and $k \in \mathbb{N}$ let

$$Q(A, \{x_n\}, k) = \sum_{\substack{n \le k \\ x_n \in A}} 1.$$

Then we have immediately : $\{x_n\}$ is uniformly distributed in $\mathbb{Z}$ if and only if for every $H \in D_0$ there holds

$$\lim_{k \to \infty} \frac{Q(H, \{x_n\}, k)}{k} = \Delta(H). \tag{27}$$

This result can be extended over the whole algebra D_μ than the next result says:

THEOREM 7. *The sequence $\{x_n\}$ of positive integers is uniformly distributed in $\mathbb{Z}$ if and only if for every set $A \in D_\mu$*

$$\lim_{k \to \infty} \frac{Q(A, \{x_n\}, k)}{k} = \mu(A). \tag{28}$$

Proof. The sufficiency of the condition is obvious.

To the opposite direction take $A \in D_\mu$. Let $\varepsilon > 0$. Then by lemma there exist sets $H_1, H_2 \in D_0$ such that

$$H_1 \subset A \subset H_2$$

and

$$\Delta(H_2) - \Delta(H_1) < \varepsilon.$$

Consequently

$$\frac{Q(H_1, \{x_n\}, k)}{k} \le \frac{Q(A, \{x_n\}, k)}{k} \le \frac{Q(H_2, \{x_n\}, k)}{k}$$

for $k = 1, 2, \ldots$. If $\{x_n\}$ is uniformly distributed in $\mathbb{Z}$, then the last inequalities and (27) imply

$$\left| \limsup_{k \to \infty} \frac{Q(A, \{x_n\}, k)}{k} - \mu(A) \right| \le \varepsilon$$

and

$$\left| \liminf_{k \to \infty} \frac{Q(A, \{x_n\}, k)}{k} - \mu(A) \right| \le \varepsilon.$$

Thus for $\varepsilon \to 0^+$ we obtain the required relation (28). The proof is finished.

The condition $A \in D_\mu$ cannot be omitted, which can again be seen using $A = \{n+n!;\ n = 1,2,\dots\}$. The sequence $\{x_n\}$, where $x_n = n+n!$, $n = 1,2,\dots$ is uniformly distributed in $\mathbb{Z}$ and

$$\lim_{k\to\infty} \frac{Q(\mathbb{N} \setminus A, \{x_n\}, k)}{k} = 0.$$

However, (1) implies $\mu^*(\mathbb{N} \setminus A) = 1$.

The concept of the uniform distribution in $\mathbb{Z}$ gives us a further possibility to characterize the algebra D_μ.

Let $A \in P(\mathbb{N})$ and $n \in \mathbb{N}$. A set $A' \subset A$ will be called a remainder system of the set A modulo n if

(vi) For every $a \in A$ there exists an $a' \in A'$ such that $a = a' \pmod n$

(v) For every $a', a'' \in A'$ $a' \equiv a'' \pmod n \implies a' = a''$.

It is obvious that two remainder systems of the set A modulo n have the same number of elements and that this number is equal to the number of elements of the system of representatives of the set A modulo n.

THEOREM 8. *Let $A \in P(\mathbb{N})$. If for every uniformly distributed in $\mathbb{Z}$ sequence $\{x_n\}$ we have*

$$\lim_{k\to\infty} \frac{Q(A, \{x_n\}, k)}{k} = \mu^*(A),$$

then $A \in D_\mu$.

Proof. Let $A \notin D_\mu$. Then

$$1 - \mu^*(\mathbb{N} \setminus A) < \mu^*(A). \tag{29}$$

Suppose that the sequence $\{B_n\}$ satisfied the condition (i). Suppose that this sequence also satisfies the condition

$$B_n \mid B_{n+1}, \qquad n = 1,2,\dots.$$

Let A'_n be a remainder system of the set A modulo B_n, for $n = 1,2,\dots$. Put $A_1 = A'_1$ and

$$A_n = A'_{n-1} \cup \{y \in A'_n;\ \forall x \in A_{n-1},\ x \not\equiv y \pmod{B_n}\}$$

for $n = 2,3,\dots$. In this way an increasing sequence of sets A_n

$$A_1 \subset A_2 \subset \dots \subset A_n \subset \dots,$$

remainder systems of the set A modulo B_n can be constructed.

Similarly, there exists a sequence

$$\overline{A}_1 \subset \overline{A}_2 \subset \cdots \subset \overline{A}_n \ldots$$

such that $\overline{A}_n$ is a remainder system of the set $\mathbb{N}\backslash A$ modulo B_n for $n = 1, 2, \ldots$.

Construct the sequence $\{C(B_n)\}$ of sets, as follows: Set $C(B_n)$ is the complete remainder system modulo B_n $(n = 1, 2, \ldots)$ which consists of the elements of $\overline{A}_n$ and $B_n - R(\mathbb{N} \setminus A, B_n)$ elements of A_n.

Clearly

$$C(B_1) \subset C(B_2) \subset \cdots \subset C(B_n) \subset \ldots .$$

Put $D_1 = B_1$. Let us rearrange the set $C(D_1)$ into a (finite) sequence

$$C'(D_1) = \{x_0, \ldots, x_{D_1-1}\}$$

in such a way that $x_j \equiv j \pmod{D_1}$, for $j = 0, \ldots, D_1 - 1$. Let

$$D_2 = \min\{B_n;\ x_1 < B_n, \ldots, x_{D_1-1} < B_n\}.$$

Rearrange the set $C(D_2)$ into the (finite) sequence

$$C'(D_2) = \{x_0, \ldots, x_{D_1-1}, x_{D_1}, \ldots, x_{D_2-1}\},$$

where $x \equiv j \pmod{D_2}$, $D_1 \leq j < D_2$. In this way we can construct a sequence $\{D_n\}$, for which the condition (i) is satisfied, and the system of finite sequences

$$C'(D_n) = \{x_0, \ldots, x_{D_{n-1}-1}, x_{D_{n-1}}, \ldots, x_{D_n-1}\}$$

in which $x_j \equiv j \pmod{D_n}$, $D_{n-1} \leq j < D_n$.

Consider the sequence

$$\{x_n\} = \bigcup_{n=1}^{\infty} C'(D_n)$$

in which its elements are written in such a way that we begin with elements of the sequence $C'(D_1)$, then follow the remaining elements of the sequence $C'(D_2)$ etc. For $d \in \mathbb{N}$ there exists n_0 such that $d \mid D_{n_0}$. Therefore for $j > D_{n_0}$ we have

$$x_j \equiv j \pmod{d}.$$

This implies that the sequence $\{x_n\}$ is uniformly distributed in $\mathbb{Z}$.

If $n = 1, 2, \ldots$, then

$$Q(A, \{x_j\}, D_n) = D_n - R(\mathbb{N} \setminus A, D_n).$$

Owing to (29) and Theorem 1 we have

$$\lim_{n\to\infty} \frac{Q(A, \{x_j\}, D_n)}{D_n} < \mu^*(A).$$

The proof is complete.

REFERENCES

[1] BUCK, R. C.: *The measure theoretic approach to the density*, Amer J. Math. **LXVIII** (1946), 560 580.

[2] DIJKSMA, A MEIJER, H. G.. *Note on uniformly distributed sequences*, Nieuw Arch. Wisk.(4) **XVII** (1969), 210 213.

[3] HLAWKA, E.: *Theorie der Gleichverteilung*, Bibliographishes Institut, Manheim, Wien, Ztrich, 1979.

[4] KUIPERS, Z.—NIEDERREITER, H.: *Uniform distribution of sequences*, John Wiley et Sons, New York-London Sydney Toronto, 1974.

[5] OXTOBY, J. C.: *Measure and Category*, (Russian translation Moskva, Mir 1971), Springer, Berlin, 1971.

[6] ŠALÁT, T.: *Cantorsche Entwicklungen der reallen Zahlen und das Hausdorffsche Mass*, Publ. of the Math. Inst. of the Hung. Acad. of Sci **VI** (1961), 15–41.

[7] PAŠTÉKA, M.—ŠALÁT, T.: *Buck's measure density and sets of positive integers containing arithmetic progressions*, Math.Slovaca **41** (1991), 283 293.

Received October 11, 1989

Matematický ústav SAV
Štefánikova 49
814 73 Bratislava