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TWO-VALUED STATES ON A CONCRETE LOGIC AND THE ADDITIVITY PROBLEM

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Abstract

We investigate the question of additivity of the integral on σ-classes, as posed by S. Gudder in [2] and analyzed in [3], [4], [8], [1] and [6]. We clarify the question for two-valued measures.

Introduction
Motivation of the problem. Basic notions for the mathematical setup

By a logic of a quantum mechanical system we usually understand an orthomodular partially ordered set (see [7], [4], etc.). A "concrete" logic is a logic representable by a collection of subsets of a set. It was introduced in the realm of quantum mechanical investigations by S. Gudder ([2], [3], [4]) under the name of a σ-class. This notion is thus a generalization of the notion of σ-algebra but certain important "quantum" phenomena not accesible in σ-algebras are still present. For instance, the concept of compatibility or noncompatibility is well modelled because a σ-class need not be closed under the formation of intersections (see [2], [4] for other considerations). If the logic is supposed concrete, the observables become measurable functions and states become the probability measures. A natural question appears: Given a state, is the expectation of the sum of two observables equal to the sum of the respective expectations? The answer to this question is undoubtedly important and there have been several attempts to clarify it (see [2], [3], [4], [5], [6], [1], [8]). It may happen that a sum of two observables is no longer an observable, but even if we assume the observables "summable" (but generally noncompatible), the question still remains intersting from the mathematical point of view. In this paper we discuss in detail the case when the state is two-valued. The two-valued states seem to be of particular importance (due to the hidden variables conjecture, etc.) and therefore it is desirable to know to which extent of generality the additivity of the expectations is valid. We show that the additivity may fail only for a fairly special pair of observables (see the theorem on the page 5 and the comments at the end of the paper). The fact is that additivity does fail sometimes (even for bounded observables — see [6]).

This paper extends and complements the paper [6]. A very special case of our result (both observables finitely valued) can be also derived from the results of [8]. Our method differs completely from that of [8]. In fact, one has to use a new method to achieve the present results as we indicate by an example.

Let us recall basic definitions. Suppose that X is a non-empty set. A *σ-class* C (*on* X) is a collection of subsets of X subject to the following requirements:

(i) $\emptyset \in C$,

(ii) $A \in C \Rightarrow A' \in C$ (A' denotes the complement of A in X),

(iii) if $A_i \in C$, $i \in N$ then $\bigcup_{i \in N} A_i \in C$ whenever A_i, $i \in N$, are mutually disjoint.

Observe that a σ-class is closed under the formation of unions of increasing sequences of sets and under the formation of relative complements (see [4]).

A *probability measure* on a σ-class C is a mapping m: $C \to \langle 0, 1\rangle$ such that

(i) $m(X) = 1$,

(ii) $m\left(\bigcup_{i \in N} A_i\right) = \sum_{i \in N} m(A_i)$, whenever A_i, $i \in N$, are mutually disjoint.

A triple (X, C, m), where C is a σ-class of subsets of X and m is a probability measure on C, is called a *generalized probability space* (abbr. g.p.s.).

A probability measure is called *two-valued* if its only values are 0 and 1. In this article (with a single exception of the comment at the very end of the paper) we have restricted our considerations to two-valued probability measures.

Let $\mathscr{B}(R)$ denote the σ-algebra of Borel sets on the real line R and let (X, C) be a σ-class. A function f: $X \to R$ is called *measurable* if $f^{-1}(A) \in C$ for every set $A \in \mathscr{B}(R)$. One can easily see that $A_f = f^{-1}(\mathscr{B}(R))$ is a sub- σ-algebra of C (see [5]). Therefore the restriction of a given measure m: $C \to \langle 0, 1\rangle$ to A_f is an ordinary probability measure on A_f. Let us denote the restriction by $m|A_f$. We may now define the integral $\int f\, \mathrm{d}m$ of f with respect to the measure m: $C \to \langle 0, 1\rangle$ by putting

$$\int f\, \mathrm{d}m = \int f\, \mathrm{d}(m|A_f).$$

The right-hand side means the Lebesgue integral.

Assume that we are given two measurable functions f, g: $X \to R$. If $f + g$ is again measurable, we call the functions f, g *summable*. The question then arises: Assuming f, g are summable, does the equality

$$\int f\, \mathrm{d}m + \int g\, \mathrm{d}m = \int (f + g)\, \mathrm{d}m$$

always hold? The latter question was raised in [2] and investigated in [3], [4] and [8]. The proof of the additivity for two simple summable functions (i.e. those summable functions attaining only finite number of values) is presented in [8]. Some further investigations are carried on in [1] and [6]. If the measure is supposed to be two-valued we are able to give the proof of the additivity for fairly general case. The assumptions are roughly such that at least one of the functions has a nowhere dense range and at least one of the function is bounded. The latter result approaches the full generality for the additivity to hold as we establish at the end of the paper.

Results

Suppose that f, g are functions on X. We shall use the notation $C_{f,g}$ for the least σ-class on X containing the set $A = A_f \cup A_g \cup A_{f+g}$. Evidently, if f, g are measurable and summable on a g.p.s. (X, C, m) then $C_{f,g} \subset C$.

Proposition. *Suppose that (X, C, m) is a g.p.s. and f, g: $X \to R$ are measurable summable functions. Then $\int f\,\mathrm{d}m + \int g\,\mathrm{d}m = \int (f+g)\,\mathrm{d}m$ if and only if $m|C_{f,g}$ is concentrated in a point.*

Proof. We show the necessity, the sufficiency is trivial. Obviously, there exist r, $s \in R$ such that $m[f^{-1}(\{r\})] = m[g^{-1}(\{s\})] = 1$. Since $f^{-1}(\{r\})$, $g^{-1}(\{s\})$ cannot be disjoint, we may choose a point $y \in f^{-1}(\{r\}) \cap g^{-1}(\{s\})$. Then

$$\int f\,\mathrm{d}m + \int g\,\mathrm{d}m = f(y) + g(y) = r + s$$

and the additivity holds only if $m[(f+g)^{-1}(\{r+s\})] = 1$. As $y \in (f+g)^{-1}(\{r+s\})$, m is concentrated in y if we consider only the generators of $C_{f,g}$, so it must be such on the entire $C_{f,g}$ (the measures on σ-classes extend uniquely from the generators!).

Let us denote by Rf the range of a function f, that is, $Rf = \{f(x)\colon x \in X\}$. If $M \subset R$ then $\bar{M}$ will denote the closure of M in R. The set M is called nowhere dense if $\overline{R - \bar{M}} = R$.

Theorem. *Let f, g be measurable summable functions on a g.p.s. (X, C, m). Let m be a two-valued measure on a σ-class (X, C). Then the equality*

$$\int f\,\mathrm{d}m + \int g\,\mathrm{d}m = \int (f+g)\,\mathrm{d}m$$

holds whenever both sides are defined and the following conditions are fulfilled:

(i) *Rf is nowhere dense in R,*

(ii) $\overline{Rg} \neq R$,

(iii) *at least one of the functions f, g is bounded from above or from below.*

Proof. We shall prove that the assumptions of the latter theorem guarantee all two-valued measures on $C_{f,g}$ concentrated. We shall make use of the sets

$$K(a, b)=[f^{-1}((-\infty, a))\cap g^{-1}((-\infty, b))]\cup \\ \cup[f^{-1}((-\infty, a))\cap(f+g)^{-1}(\langle a+b, +\infty))]\cup \\ \cup[g^{-1}((-\infty, b))\cap(f+g)^{-1}(\langle a+b, +\infty))].$$

It will turn out that if $m|C_{f,g}$ were not concentrated then, for some $a, b\in R$, the value of m on $K(a, b)\in C$ would violate the additivity of the measure m.

The proof will require a few lemmas. Of course, we may assume without any loss of generality that $C=C_{f,g}$.

Lemma 1. *Suppose that*

$$f^{-1}((-\infty, a))\cap g^{-1}((-\infty, b))\cap(f+g)^{-1}(\langle c, +\infty))=\emptyset$$

for some a, b, $c\in R$, $c\leqq a+b$. Then either all the sets $K(a, b)$, $K(c-b, b)$, $K(a, c-a)$ belong to C or none of them. In the former case the following equalities hold:

$$m[K(c-b, b)]+m[(f+g)^{-1}((-\infty, c))]-m[f^{-1}((-\infty, c-b))]= \\ =m[K(a, b)]+m[(f+g)^{-1}((-\infty, a+b))]-m[f^{-1}((-\infty, a))],$$

$$m[K(a, c-a)]+m[(f+g)^{-1}((-\infty, c))]-m[g^{-1}((-\infty, c-a))]= \\ =m[K(a, b)]+m[(f+g)^{-1}((-\infty, a+b))]-m[g^{-1}((-\infty, b))].$$

Proof of Lemma 1. A simple computation yields

$$K(a, b)\cup(f+g)^{-1}(\langle c, a+b))=[f^{-1}((-\infty, a))\cap(f+g)^{-1}(\langle c, +\infty))]\cup \\ \cup[f^{-1}((-\infty, a))\cap g^{-1}((-\infty, b))]\cup[g^{-1}((-\infty, b))\cap(f+g)^{-1}(\langle c, +\infty))]= \\ =K(c-b, b)\cup f^{-1}(\langle c-b, a)).$$

On the opposite sides of the last equality we have unions of two disjoint sets. Therefore we obtain that $K(c-b, b)\in C$ if and only if $K(a, b)\in C$. Moreover,

$$m[K(a, b)]+m[(f+g)^{-1}(\langle c, a+b))]= \\ =m[K(c-b, b)]+m[f^{-1}(\langle c-b, a))].$$

Let us add to both sides of the last equality the factor

$$m[(f+g)^{-1}((-\infty, c))]+m[f^{-1}((-\infty, c-b))].$$

We obtain

$$m[K(a, b)]+m[(f+g)^{-1}((-\infty, a+b))]+m[f^{-1}((-\infty, c-b))]= \\ =m[K(c-b, b)]+m[f^{-1}((-\infty, a))]+m[(f+g)^{-1}((-\infty, c))]$$

and therefore

$$m[K(c-b, b)] + m[(f+g)^{-1}((-\infty, c))] - m[f^{-1}((-\infty, c-b))] =$$
$$= m[K(a, b)] + m[(f+g)^{-1}((-\infty, a+b))] - m[f^{-1}((-\infty, a))].$$

The case of $K(a, c-a)$ argues similarly.

Let us consider the relation $S \subset R^2 \times R^2$ determined by the following requirement:

$[[a, b], [p, q]] \in S$ if there exists a real $c \leqq a+b$ such that

(i) $p = c-b$, $q = b$ or $p = a$, $q = c-a$,

(ii) $f^{-1}((-\infty, a)) \cap g^{-1}((-\infty, b)) \cap (f+g)^{-1}(\langle c, +\infty)) = \emptyset$.

Let $E \subset R^2 \times R^2$ be the least equivalence relation containing S, that is, $[[a, b], [p, q]] \in E$ if and only if there exists a finite chain $[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]$ such that $[a_1, b_1] = [a, b]$, $[a_n, b_n] = [p, q]$ and for all $i = 1, \ldots, n-1$ we have

$$[[a_i, b_i], [a_{i+1}, b_{i+1}]] \in S \quad \text{or} \quad [[a_{i+1}, b_{i+1}], [a_i, b_i]] \in S.$$

Particularly, all points of the set $\bar{J} \times R$, where J is an open interval satisfying $f^{-1}(J) = \emptyset$, are equivalent within the equivalence E. The same observation can be made for the sets of the form $R \times \bar{J}$, where $g^{-1}(J) = \emptyset$, J an open interval.

Lemma 2. *If $K(a, b) \in C$ and $[[a, b], [p, q]] \in E$ then $K(p, q) \in C$.*

Proof of Lemma 2. We use Lemma 1 (possibly repeatedly).

Lemma 3. *Suppose that $K(a, b) \in C$. Then there exists a constant $k \in R$ such that if $[[a, b], [p, q]] \in E$ then*

$$m[K(p,q)] = m[f^{-1}((-\infty, p))] + m[g^{-1}((-\infty, q))] - m[(f+g)^{-1}((-\infty, p+q))] + k.$$

Proof of Lemma 3. We shall prove that

$$k = m[K(a, b)] + m[(f+g)^{-1}((-\infty, a+b))] - m[f^{-1}((-\infty, a))] - m[g^{-1}((-\infty, b))].$$

Suppose that $[[a, b], [p, q]] \in E$. Then there is a chain $[a_0, b_0], [a_1, b_1], \ldots, [a_n, b_n]$ such that $[a_0, b_0] = [a, b]$, $[a_n, b_n] = [p, q]$ and $[[a_{i-1}, b_{i-1}], [a_i, b_i]] \in S$ or $[[a_i, b_i], [a_{i-1}, b_{i-1}]] \in S$ for any $i \leqq n$. We shall proceed by induction over the length of the chain. Suppose first that $n = 1$. Then $[[a, b], [p, q]] \in S$ or $[[p, q], [a, b]] \in S$. Let us discuss the former case. We may suppose that $p = c-b$, $q = b$ or $p = a$, $q = c-a$. Consider the case $p = c-b$, $q = b$, the second possibility discusses analogically. Then by Lemma 1

$$m[K(a, b)] + m[(f+g)^{-1}((-\infty, a+b))] - m[f^{-1}((-\infty, a))] =$$
$$m[K(c-b, b)] + m[(f+g)^{-1}((-\infty, c))] - m[f^{-1}((-\infty, c-b))] =$$
$$= m[K(p, q)] + m[(f+g)^{-1}((-\infty, p+q))] - m[f^{-1}((-\infty, p))].$$

By subtracting the factor $m[g^{-1}((-\infty, q))]$ from both sides of the latter equality we obtain that

$$\begin{aligned} m[K(p, q])&=m[f^{-1}((-\infty, p))]+m[g^{-1}((-\infty, q))]- \\ &-m[(f+g)^{-1}((-\infty, p+q))]+m[K(a, b)]+ \\ &+m[(f+g)^{-1}((-\infty, a+b))]-m[f^{-1}((-\infty, a))]-m[g^{-1}((-\infty, b))]= \\ &=m[f^{-1}((-\infty, p))]+m[g^{-1}((-\infty, q))]-m[(f+g)^{-1}((-\infty, p+q))]+k. \end{aligned}$$

The case $[[p, q], [a, b]]\in S$ leads analogically to the required equation.

Let us now suppose that Lemma 3 holds as soon as $[[a, b], [p, q]]\in E$ can be "connected" by a chain of shorter length than n. Then

$$\begin{aligned} m[K(a_{n-1}, b_{n-1})]&=m[f^{-1}((-\infty, a_{n-1}))]+m[g^{-1}((-\infty, b_{n-1}))]- \\ &-m[(f+g)^{-1}((-\infty, a_{n-1}+b_{n-1}))]+k. \end{aligned}$$

Since $[[a_{n-1}, b_{n-1}], [p, q]]\in S$ or $[[p, q], [a_{n-1}, b_{n-1}]]\in S$ (Lemma 1) then

$$\begin{aligned} m[K(p, q)]&=m[f^{-1}((-\infty, p))]+m[g^{-1}((-\infty, q))]- \\ &-m[(f+g)^{-1}((-\infty, p+q))]+m[K(a_{n-1}, b_{n-1})]+ \\ &+m[(f+g)^{-1}((-\infty, a_{n-1}+b_{n-1}))]-m[f^{-1}((-\infty, a_{n-1}))]-m[g^{-1}((-\infty, b_{n-1}))]= \\ &=m[f^{-1}((-\infty, p))]+m[g^{-1}((-\infty, q))]-m[(f+g)^{-1}((-\infty, p+q))]+k. \end{aligned}$$

The proof of Lemma 3 is finished.

Suppose now that $f(x)>M$ for all $x\in X$ (the other cases would be discussed similarly). Take a pair $[a, b]\in R^2$, $a\leqq M$. Then

$$\begin{aligned} K(a, b)&=g^{-1}((-\infty, b))\cap(f+g)^{-1}(\langle a+b, +\infty))= \\ &=[g^{-1}(\langle b, +\infty))\cup(f+g)^{-1}((-\infty, a+b))]' \end{aligned}$$

belongs to C because the sets in brackets are disjoint. Let us fix the point $[a, b]$ for the rest of the proof.

Let K be the set of cluster points of Rf. Since $\overline{Rg}\neq R$, there exists a point $p\in R$ such that p, together with its open neighbourhood $\mathcal{O}(p)$, belongs to $R-\overline{Rg}$. Consider the set $U=R^2-K\times(R-\mathcal{O}(p))$. Obviously, U is a dense subset of R^2. We claim now that all the points of U are equivalent within the equivalence E. To see that, take a point $[c, d]\in U$. We shall show that $[[c, d], [a, b]]\in E$.

For $c\in K$ we have $d\in\mathcal{O}(p)$ and since $g^{-1}(\mathcal{O}(p))=\emptyset$, we obtain that $[[c, d], [c, p]]\in E$. For $c\in R-K$ there exists an interval $J=(c, c_1)$ such that $f^{-1}(J)=\emptyset$, so $[[c, d], [c, p]]\in E$. As $g^{-1}(\mathcal{O}(p))=\emptyset$, $[[c, p], [a, p]]\in E$. One can easily see that $[[a, p], [a, b]]\in E$ and we have thus obtained that all the elements of U are E-equivalent. Moreover, for all $[p, q]\in U$, we have $K(p, q)\in C$.

Now we shall suppose that m is a two-valued measure on $C_{f,g}$ which is not concentrated. Since any two-valued measure on $\mathcal{B}(R)$ must be concentrated in a point, we see that there exist such r, s, t that

$$m[f^{-1}(\{r\})]=m[g^{-1}(\{s\})]=m[(f+g)^{-1}(\{t\})]=1.$$

If we had $t=r+s$ then m would have been concentrated, so $t\neq r+s$. Since U is dense in R^2, there exist $a, b, c, d\in R$ such that

$$a>r, \quad b<s,$$
$$c>r, \quad d>s,$$
$$t\notin \langle a+b, c+d\rangle,$$

and

$$[[a, b], [c, d]]\in U.$$

Using the equation in Lemma 3, we obtain finally that

$$m[K(c, d)]=2+k-m[(f+g)^{-1}((-\infty, c+d))],$$
$$m[K(a, b)]=k-m[(f+g)^{-1}((-\infty, a+b))].$$

As

$$t\notin \langle a+b, c+d\rangle,$$
$$m[(f+g)^{-1}((-\infty, a+b))]=m[(f+g)^{-1}((-\infty, c+d))]$$

and

$$m[K(c, d)]=m[K(a, b)]+2.$$

We have reached a contradiction for m has been supposed to be a probability measure.

Let us make a few comments on the theorem. The conditions in the theorem are not necessary, but there are examples (see [6]) showing that none of them can be omitted. We may also remark that the additivity does not hold for more than two functions even if they are finitely valued (see [6]). It is a natural question whether the additivity does hold under the assumptions of the theorem if m is not supposed to be two-valued. This would extend the result of [8]. We answer this question in the negative by the following counterexample:

Example. Let $X=Z\times N_0$, where Z is the set of all integers and N_0 is the set of all non-negative integers. For every $[a, b]\in X$ we define

$$f([a, b])=a,$$
$$g([a, b])=b.$$

Then $C_{f,g}=A_f\cup A_g\cup A_{f+g}\cup K\cup K^*\cup L\cup L^*$, where

$$K=\{K(a, b): a, b\in Z\},$$
$$K^*=\{[K(a, b)]': a, b\in Z\},$$
$$L=\{K(a, b)\cup f^{-1}(\{a\}): a, b\in Z\},$$
$$L^*=\{[K(a, b)\cup f^{-1}(\{a\})]': a, b\in Z\}.$$

Let $m': A_f\cup A_g\cup A_{f+g}\to\langle 0, 1\rangle$ be such a σ-additive function that is determined by the following requirements:

$$m'[f^{-1}(\{0\})] = m'[f^{-1}(\{1\})] = m'[g^{-1}(\{0\})] =$$
$$= m'[g^{-1}(\{1\})] = m'[(f+g)^{-1}(\{0\})] = m'[(f+g)^{-1}(\{1\})] = \tfrac{1}{2}.$$

The extension of m' from $A_f \cup A_g \cup A_{f+g}$ to $C_{f,g}$ is unique and results to a measure m. One can easily see that

$$m[K(a, b)] = m[f^{-1}((-\infty, a))] + m[g^{-1}((-\infty, b))] - m[(f+g)^{-1}((-\infty, a+b))].$$

This yields that

$$\int f \, dm + \int g \, dm = 1 \neq \int (f+g) \, dm = \tfrac{1}{2},$$

which we wanted to show.

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СОСТОЯНИЯ НА КОНКРЕТНОЙ ЛОГИКЕ, ПРИНИМАЮЩИЕ ТОЛЬКО ДВА ЗНАЧЕНИЯ И ПРОБЛЕМА АДДИТИВНОСТИ

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Резюме

В статье рассматривается вопрос об аддитивности интеграла на σ-классах, поставленный С. Гаддером. Эта проблема решается для мер, принимающих только два значения.