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A THEOREM CONCERNING THE RESTRICTION OF THE $\mathscr{D}$-STRUCTURE OF A SEMIGROUP S TO A SUBSEMIGROUP OF S

FRANCIS PASTIJN

Notations. By $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$ we mean Green's relations for a semigroup S and $\mathscr{L}'$, $\mathscr{R}'$, $\mathscr{H}'$, $\mathscr{D}'$ will be the relations of Green for the subsemigroup S' of S. The $\mathscr{L}$-class [resp. $\mathscr{L}'$-class] containing a will be denoted by L_a [resp. L'_a] and analogously for what concerns the other relations of Green (1).

λ_a [resp. ϱ_a] denotes an inner left [resp. right] translation (2).

$\Gamma_{S'}(H'_a)$ means the Schützenberger group of H'_a in the semigroup S' and $\Gamma'_{S'}(H'_a)$ its dual Schützenberger group (3). $T_{S'}(H'_a)$ denotes the set $\{t \in S' \,\|\, H'_a t \subseteq H'_a\}$ and $T'_{S'}(H'_a)$ denotes the set $\{t \in S' \,\|\, tH'_a \subseteq H'_a\}$.

We shall use the following lemmas, which are mentioned in (4) and (5). The first one is a direct consequence of Green's lemma.

Lemma. *Let a and b be elements of S' such that $a\mathscr{R}b$; then S^1 contains elements x and x' such that $ax = b$ and $bx' = a$. The mappings $\varrho_x \mid L'_a$ and $\varrho_{x'} \mid L'_b$ are mutually inverse $\mathscr{R}$-class preserving one-to-one mappings between L'_a and L'_b. If $a\mathscr{R}'b$, then $\varrho_x \mid L_a \cap S'$ and $\varrho_{x'} \mid L_b \cap S'$ are mutually inverse $\mathscr{R}'$-class preserving one-to-one mappings between $L_a \cap S'$ and $L_b \cap S'$; in this case $\varrho_{x'} \mid L_a \cap S'$ and $\varrho_{x'} \mid L_b \cap S'$ map $\mathscr{L}'$-classes onto $\mathscr{L}'$-classes and $\mathscr{H}'$-classes onto $\mathscr{H}'$-classes.*

Lemma. *If a is any regular element of S', then $H'_a = D'_a \cap H_a$.*

Now we prove our main theorem.

Theorem. *Let D be a regular $\mathscr{D}$-class such that the $\mathscr{L}$-classes and $\mathscr{R}$-classes which have a non-void intersection with $D \cap S'$, contain at least one idempotent in $D \cap S'$. Then the following conditions are equivalent:*

(i) *If $a \in D \cap S'$, then $D'_a \cap H_a = H'_a$.*

(ii) *If e, f are idempotents in $D \cap S'$, $a \in L_e \cap R_f \cap S'$, and $a' \in R_e \cap L_f$ is an inverse of a in S, then the mappings*

$$\Theta : H'_e \to H'_f,\ x \to axa'$$

and

$$\Theta' : H'_f \to H'_e,\ y \to a'ya$$

are mutually inverse isomorphisms.

(iii) *If $a, ag \in R_a \cap S'$, with $g \in S$, then $H'_a g = H'_{ag}$, and, if $b, qb \in L_b \cap S'$ with $q \in S$, then $qH'_b = H'_{qb}$.*

(iv) *If e is an idempotent of $D \cap S'$, and $a \in L_e \cap S'$, $b \in R_e \cap S'$, then $aH'_b = H'_a b = H'_a H'_b = H'_{ab}$.*

(v) *If e, f are idempotents in $D \cap S'$, and $a \in L_e \cap R_f \cap S'$, then $T_{S'}(H'_e) = T_{S'}(H'_a)$ and $T'_{S'}(H'_f) = T'_{S'}(H'_a)$.*

(vi) *If e, f are idempotents in $D \cap S'$, $a \in L_e \cap R_f \cap S'$, and $a' \in R_e \cap L_f$ is an inverse of a in S, then the mappings*

$$\varphi : \Gamma'_{S'}(H'_a) \to \Gamma'_{S'}(H'_e),\quad \lambda_t \mid H'_a \to \lambda_{a'ta} \mid H'_e,$$

and

$$\varphi' : \Gamma'_{S'}(H'_e) \to \Gamma'_{S'}(H'_a),\quad \lambda_s \mid H'_e \to \lambda_{asa'} \mid H'_a,$$

$$\psi : \Gamma_{S'}(H'_a) \to \Gamma_{S'}(H'_f),\quad \varrho_v \mid H'_a \to \varrho_{ava'} \mid H'_f,$$

and

$$\psi' : \Gamma_{S'}(H'_f) \to \Gamma_{S'}(H'_a),\quad \varrho_w \mid H'_f \to \varrho_{a'wa} \mid H'_a,$$

are pairs of mutually inverse isomorphisms.

Proof.

(i) implies (ii). Since $a\mathscr{L}e$, with $ae = a$ and $a'a = e$, the left inner translation $\lambda_a \mid R'_e$ is a one-to-one mapping of R'_e upon R'_a. Moreover, this mapping $\lambda_a \mid R'_e$ is $\mathscr{L}$-class preserving (4). $D'_a \cap H_a = H'_a$ implies $R'_a \cap L_a = H'_a$, and $D'_e \cap H_e = H'_e$ implies $R'_e \cap L_a = H'_e$. Thus $\lambda_a \mid H'_e$ is a one-to-one mapping of H'_e upon H'_a. Since $a\mathscr{R}f$, with $aa' = f$ and $fa = a$, the inner right translation $\varrho_{a'} \mid L'_a$ is a one-to-one mapping of L'_a upon L'_f. Moreover, this mapping is $\mathscr{R}$-class preserving (4). $D'_a \cap H_a = H'_a$ implies $L'_a \cap R_a = H'_a$, and $D'_f \cap H_f = H'_f$ implies $L'_f \cap R_a = H'_f$. Thus $\varrho_{a'} \mid H'_a$ is a one-to-one mapping of H'_a upon H'_f. We conclude that $\Theta = (\lambda_a \mid H'_e) \circ (\varrho_{a'} \mid H'_a)$ is a one-to-one mapping of H'_e upon H'_f. Dually, $\Theta' = (\varrho_a \mid H'_f) \circ (\lambda_{a'} \mid H'_a)$ is a one-to-one mapping of H'_f upon H'_e. Clearly Θ' is the inverse of Θ. If x and y are elements of H'_e then $(xy)\Theta = axya' = axeya' = axa'aya' = (x)\Theta(y)\Theta$. We conclude that Θ and Θ' are mutually inverse isomorphisms.

(ii) implies (iii). Since $a\mathscr{R}ag$, the right inner translation $\varrho_g \mid L'_a$ is a one-to-one mapping of L'_a upon L'_{ag}, and since this mapping is $\mathscr{R}$-class preserving,

$(L'_a \cap \boldsymbol{R}_x)g = (L'_{ag} \cap \boldsymbol{R}_a)$. $D \cap S'$ contains the idempotents e and f such that $a \in L_e \cap \boldsymbol{R}_f \cap S'$. We know that $H_e \cap D'_e = H'_e$ and $H_f \cap D'_f = H'_f$ (4). Let a' be the inverse of a contained in $\boldsymbol{R}_e \cap L_f$. Let b be an element of $L'_a \cap \boldsymbol{R}_a$. The inner right translation $\varrho_a \mid L'_f$ is a one-to-one mapping of L'_f upon L'_a. More precisely, $\varrho_a \mid L'_f$ will map $L'_f \cap \boldsymbol{R}_a$ upon $L'_a \cap \boldsymbol{R}_a$. We can put $L'_f \cap \boldsymbol{R}_a = H'_f$, since $H_f \cap D'_f = H'_f$. Thus $b = xa$ for some $x \in H'_f$. By (ii) $a'b = a'xa \in H'_e$. The inner left translation $\lambda_a \mid \boldsymbol{R}'_e$ is a one-to-one mapping of $\boldsymbol{R}'_e$ upon $\boldsymbol{R}'_a$. More precisely, $\lambda_a \mid \boldsymbol{R}'_e$ maps $\boldsymbol{R}'_e \cap L_a$ upon $\boldsymbol{R}'_a \cap L_a$. Since $a'xa \in \boldsymbol{R}'_e \cap L_a$, we have $a(a'xa) \in \boldsymbol{R}'_a \cap L_a$, or, $fxa \in \boldsymbol{R}'_a \cap L_a$, or $b = xa \in \boldsymbol{R}'_a$. We conclude that $L'_a \cap \boldsymbol{R}_a = H'_a$. In a similar way we can prove $L'_{ag} \cap \boldsymbol{R}_a = H'_{ag}$. Hence $H'_a g = H'_{ag}$. The rest follows dually.

(iii) implies (iv). If e is an idempotent of $D \cap S'$, and $a \in L_e \cap S'$, $b \in \boldsymbol{R}_e \cap S'$, then $aH_b = H_a b = H_a H_b = H_{ab}$ (6). Evidently $ab \in \boldsymbol{R}_a \cap L_b \cap S'$, and therefore, by (iii) $H'_a b = aH'_b = H'_{ab}$. Let $c \in H'_a$, then $cH'_b = H'_{cb}$ by the same argument. Since $cb \in H'_a b = H'_{ab}$ we must have $H'_{cb} = H'_{ab}$, and so $cH'_b = H'_{ab}$ for any $c \in H'_a$. We conclude that $H'_a H'_b = \bigcup_{c \in H'_a} cH'_b = H'_{ab}$.

(iv) implies (v). Let $t \in T_{S'}(H'_a)$. Then $\varrho_t \mid L_a \cap S'$ is a $\mathscr{R}'$-class preserving one-to-one mapping of $L_a \cap S'$ upon itself (4). If e is an idempotent contained in $L_a \cap S'$, we must have $\boldsymbol{R}'_e \cap L_a = H'_e$, and hence $et \in H'_e$. The element t therefore belongs to $T_{S'}(H'_e)$, and we can put $T_{S'}(H'_a) \subseteq T_{S'}(H'_e)$.

If in (iv) $b = e$, then $aH'_e = H'_a$, This implies $H'_a T_{S'}(H'_e) = aH'_e T_{S'}(H'_e) = aH'_e = H'_a$, and so $T_{S'}(H'_e) \subseteq T_{S'}(H'_a)$.

We conclude that $T_{S'}(H'_e) = T_{S'}(H'_a)$. The rest follows dually.

(v) implies (vi). If $a \in L_e \cap S'$ and $e \in D \cap S'$, a' is any inverse of a in $\boldsymbol{R}_e$, we know that $\lambda_a \mid H_e$ is a one-to-one mapping of H_e upon H_a, and $\lambda_{a'} \mid H_a$ is its inverse. Furthermore, $ae = a$. By (v) $aeT_{S'}(H'_e) = aT_{S'}(H'_a)$, or, $aH'_e = H'_a$. Hence, $\lambda_a \mid H'_e$ and $\lambda_{a'} \mid H'_a$ are mutually inverse one-to-one mappings between H'_e and H'_a. If $\lambda_t \mid H'_a \in \Gamma'_{S'}(H'_a)$ and $x \in H'_e$, then

$$\begin{aligned}(x)\lambda_a \circ (\lambda_t \mid H'_a) \circ \lambda_{a'} &= (ax)\,(\lambda_t \mid H'_a) \circ \lambda_{a'} \\ &= (tax)\lambda_{a'} \\ &= a'tax \in H'_e\,.\end{aligned}$$

Thus, $(\lambda_a \mid H'_e) \circ (\lambda_t \mid H'_a) \circ (\lambda_{a'} \mid H'_a) = \lambda_{a'ta} \mid H'_e \in \Gamma'_{S'}(H'_e)$. This implies $\Gamma'_{S'}(H'_a)\varphi \subseteq \Gamma'_{S'}(H'_e)$. Analogously $\Gamma'_{S'}(H'_e)\varphi' \subseteq \Gamma'_{S'}(H'_a)$. It should be clear that $(\lambda_t \mid H'_a)\varphi\varphi' = \lambda_{aa'taa'} \mid H'_a = \lambda_t \mid H'_a$, and consequently $\varphi\varphi'$ is the identity mapping of $\Gamma'_{S'}(H'_a)$. Similarly, $\varphi'\varphi$ is the identity mapping of $\Gamma'_{S'}(H'_e)$, and so φ and φ' are mutually inverse one-to-one mappings between $\Gamma'_{S'}(H'_a)$ and $\Gamma'_{S'}(H'_e)$. Let us now assume that $\lambda_{t_1} \mid H'_a$ and $\lambda_{t_2} \mid H'_a$ are elements of $\Gamma'_{S'}(H'_a)$. Then

$$((\lambda_{t_1} \mid H'_a) \circ (\lambda_{t_2} \mid H'_a))\varphi = (\lambda_a \mid H'_e) \circ (\lambda_{t_1} \mid H'_a) \circ (\lambda_{t_2} \mid H'_a) \circ (\lambda_{a'} \mid H'_a)$$

$$= (\lambda_a \mid H'_e) \circ (\lambda_{t_1} \mid H'_a) \circ (\lambda'_{a'} \mid H'_a) \circ (\lambda_a \mid H'_e) \circ (\lambda_{t_1} \mid H'_a) \circ (\lambda_{a'} \mid H'_a)$$

$$= (\lambda_{t_1} \mid H'_a)\varphi \circ (\lambda_{t_2} \mid H'_a)\varphi .$$

Therefore φ and φ' are group morphisms. We conclude that φ and φ' are mutually inverse isomorphisms. The rest follows dually.

(vi) implies (i). If e is an idempotent of $D \cap S'$, and $a \in L_e \cap S'$, then S^1 contains an element a' such that $a'a = e$. Then $\lambda_a \mid R'_e \cap L_a = \lambda_a \mid H'_e$ and $\lambda_{a'} \mid R'_a \cap L_a$ are mutually inverse one-to-one mappings between H'_e and $R'_a \cap L_a$ (4). Let c be an element of $R'_a \cap L_a$, then S^1 contains a t such that $ta = c$. Consequently $a'ta \in H'_e$ and $\lambda_{a'ta} \mid H'_e \in \Gamma'_{S'}(H'_e)$. By (vi)

$$(\lambda_{a'ta} \mid H'_e)\varphi' = (\lambda_{a'} \mid H'_a) \circ (\lambda_{a'ta} \mid H'_e) \circ (\lambda_a \mid H'_e) \in \Gamma'_{S'}(H'_a) ,$$

or,

$$\lambda_{aa'taa'} \mid H'_a \in \Gamma'_{S'}(H'_a) .$$

Therefore

$$(aa'taa')a \in H'_a ,$$

or

$$aa'ce \in H'_a ,$$

$$aa'c \in H'_a .$$

Since $\lambda_{a'} \mid R'_a \cap L_a$ and $\lambda_a \mid H'_e$ are mutually inverse one-to-one mappings between $R'_a \cap L_a$ and H'_e, we must have $aa'c = c \in H'_a$. We have $R'_a \cap L_a = H'_a$. Dually we can prove that $L'_a \cap R_a = H'_a$. We conclude that $D'_a \cap H_a = H'_a$.

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