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LOCAL CHARACTERIZATIONS OF DARBOUX BAIRE 1 FUNCTIONS

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In [1] there are given several characterizations of Darboux Baire 1 functions. Some of them can be considered as local characterizations. In this paper we are concerned with mutual relations between them, including a local characterization of Darboux functions given in [2].

Let f denote a real valued function defined on an interval I. Let us consider the following properties of f at $x \in I$:

(1) *For each real number* $h>0$ *we have* $I_+(x) \subset f(\langle x, x+h))$ *and* $I_-(x) \subset f((x-h, x\rangle)$, *where* $I_+(x) = \liminf_{h\to x^+} f(\langle x, x+h)), \limsup_{h\to x^+} f(\langle x, x+h)))$ *and* $I_-(x) = (\liminf_{h\to x^+} f((x-h, x\rangle), \limsup_{h\to x^+} f((x-h, x\rangle))$.

(2) $f(x) \in \langle \liminf_{t\to x^-} f(t), \limsup_{t\to x^-} f(t)\rangle \cap \langle \liminf_{t\to x^+} f(t), \limsup_{t\to x^+} f(t)\rangle$.

(3) $f(x) \in K^+(f, x) \cap K^-(f, x)$, *where* $K^+(f, x) = \{y$; *for each neighbourhood* N *of* y *and for each real number* $h>0$ *is* $f^{-1}(N) \cap (x, x+h) \neq \emptyset\}$, *similarly* $K^-(f, x)$.

(4) *For each* $a>f(x)$ *and* $h>0$ *we have* $\{t; f(t)<a\} \cap (x, x+h) \neq \emptyset$ *and* $\{t; f(t)<a\} \cap (x-h, x) \neq \emptyset$, *and for each* $b<f(x)$ *and* $h>0$ *we have* $\{t; f(t)>b\} \cap (x-h, x) \neq \emptyset$ *and* $\{t; f(t)>b\} \cap (x, x+h) \neq \emptyset$.

(5) *For each* $a>f(x)$ *and* $h>0$, *cardinalities of* $\{t; f(t)<a\} \cap (x, x+h)$ *and* $\{t; f(t)<a\} \cap (x-h, x)$ *are* c. *For each* $b<f(x)$ *and* $h>0$, *cardinalities of* $\{t; f(t)>b\} \cap (x, x+h)$ *and* $\{t; f(t)>b\} \cap (x-h, x)$ *are* c.

(6) *There is a perfect set* P *such, that* $x \in P$, *is a bilateral limit point of* P *and* f/P *is continuous at* x.

In [5] and [7] it is proved that for Baire 1 functions condition (2) is a local condition for Darboux functions. In [8] it is proved that for Baire 1 functions property (3) fulfilled in each $x \in I$ is equivalent to Darboux property.

It is obvious that f has the property (4), resp. (5), in each $x \in I$ iff f is from the class $\mathcal{M}_0$, resp. $\mathcal{M}_1$, defined in [9]. In [9] it is also proved that for functions of Baire class 1 the class of Darboux functions is equivalent to the class $\mathcal{M}_0$ and $\mathcal{M}_1$.

In [6] it is proved that a function is of Baire class 1 with the Darboux property in I iff for each $x \in I$ the property (6) is fulfilled.

In the following we prove the relations between the properties (1)—(6), for any real functions defined on the interval I.

Theorem 1. *The properties* (2) *a* (4) *are equivalent.*

Proof, (2) implies (4). If $f(x)<a$, then $a>\liminf_{t\to x^-} f(t)$ and $a>\liminf_{t\to x^+} f(t)$. Hence x is a bilateral accumulation point *of* $\{t; f(t)<a\}$. If $f(x)>b$, then $b<\limsup_{t\to x^-} f(t)$ and $b<\limsup_{t\to x^+} f(t)$, hence x is a bilateral accumulation point of $\{t; f(t)>b\}$.

(4) implies (2). Let $f(x)\notin\langle \liminf_{t\to x^-} f(t), \limsup_{t\to x^-} f(t)\rangle \cap \langle \liminf_{t\to x^-} f(t), \limsup_{t\to x^+} f(t)\rangle$. Then the following cases can occur: $f(x)<\liminf_{t\to x^-} f(t)$, $f(x)<\liminf_{t\to x^+} f(t)$, $f(x)>\limsup_{t\to x^-} f(t)$, and $f(x)>\limsup_{t\to x^+} f(t)$.

Let $f(x)<\liminf_{t\to x^-} f(t)$. Put $a\in(f(x), \liminf_{t\to x^-} f(t))$. Then from the definition of lim inf it follows that there is $h>0$ such that $\{t; f(t)<a\}\cap(x-h, x)$ is an empty set, which contradicts (4).

The proof in other cases is similar.

Theorem 2. *If a function f has the property* (1) *at* $x\in I$, *then it also has the properties* (2)—(5) *at* x.

Proof. (1) implies (2). Let $f(x)\notin\langle \liminf_{t\to x^-} f(t), \limsup_{t\to x^-} f(t)\rangle \cap \langle \liminf_{t\to x^+} f(t), \limsup_{t\to x^+} f(t)\rangle$. Then one of the four inequalities: $f(x)<\liminf_{t\to x^-} f(t)$, $f(x)>\limsup_{t\to x^-} f(t)$, $f(x)<\liminf_{t\to x^+} f(t)$ and $f(x)>\limsup_{t\to x^+} f(t)$ holds. Let $f(x)<\liminf_{t\to x^-} f(t)$. Then there are $h>0$ and c such that $f(x)<c<\liminf_{t\to x^-} f(t)$ and $\langle f(x), c\rangle \cap f((x-h, x))=\emptyset$, which is a contradiction to the assumption because $\langle f(x), c\rangle \subset I_-(x)$. The proof in the other cases is similar.

(1) implies (3). Let $f(x)\notin K^+(f, x)\cap K^-(f, x)$. Then there exists $h>0$ and N such that $f^{-1}(N)\cap(x, x+h)=\emptyset$ or $f^{-1}(N)\cap(x-h, x)=\emptyset$, where N is a neigbourhood of $f(x)$. Thus we have a contradiction.

(1) implies (4). We have already proved that (1) implies (2) and according to Theorem 1 (2) and (4) are equivalent.

(1) implies (5). If $I_+(x)=\emptyset$, then the function f is continuous at x from the right and the sets $\{t; f(t)<a\} \cap (x, x+h)$ and $\{t; f(t)>b\} \cap (x, x+h)$, for $a>f(x)$ and $b<f(x)$, have cardinalities c for every $h>0$.

Similarly in the case $I_-(x)=\emptyset$ the sets $\{t; f(t)<a\} \cap (x-h, x)$ and $\{t; f(t)>b\} \cap (x-h, x)$ have cardinalities c for every $h>0$, $a>f(x)$ and $b<f(x)$.

Let $I_+(x)\neq\emptyset$. Let $a>f(x)$. Denote $a_1=\min(a, \limsup_{k\to 0^+} f(\langle x, x+k)))$. Then we have $(x, x+h) \cap \{t; f(t)<a\} \supset (x, x+h) \cap f^{-1}((\liminf_{k\to 0^+} f(\langle x, x+k)), a_1))$, which has cardinality c for every $h>0$.

For $b<f(x)$ let us denote $b_1=\max(b, \liminf_{k\to 0^+} f(\langle x, x+k)))$. Then we have $(x, x+h) \cap \{t; f(t)>b\} \supset (x, x+h) \cap f^{-1}((b_1, \limsup_{k\to 0^+} f(\langle x, x+k))))$, which has cardinality c for every $h>0$.

Similarly for $I_-(x)\neq\emptyset$ we have that $(x-h, x) \cap \{t; f(t)<a\}$ and $(x-h, x) \cap \{t; f(t)>b\}$ for $f(x)<a$ and $f(x)>b$, resp., have cardinalities c for every $h>0$.

Remark 1. For functions of Baire class 1 none of conditions (2)—(6) implies (1).

Example 1. Define a function f on $\langle -1, 1\rangle$ as follows:

$$f(x)=\begin{cases} 1 \text{ for } x\in\{1/n\}_{n=1}^{\infty}\cup\{-1/n\}_{n=1}^{\infty}, \\ 0 \text{ for } x\in\langle -1, 1\rangle-(\{1/n\}_{n=1}^{\infty}\cup\{-1/n\}_{n=1}^{\infty}). \end{cases}$$

For $x=0$ the function f fulfils conditions (2)—(6) at x, but it does not fulfil condition (1); moreover it is approximately continuous at $x=0$.

Remark 2. It is known that approximately continuous functions have the Darboux property. Example 1 shows that an approximately continuity of a function f at a point x does not imply local property for Darboux function at x.

Theorem 3. *Property* (3) *implies properties* (2) *and* (4). *Property* (5) *implies* (4) *and* (2).

The proof is evident.

Remark 3. The reverse implications of Theorem 3 are not valid.

Example 2. Let us define a function as follows:

$$f(x)=\begin{cases} 1 \text{ for } x\in\{1/n\}_{n=1}^{\infty}\cup\{-1/n\}_{n=1}^{\infty}, \\ 1/2 \text{ for } x=0, \\ 0 \text{ for } x\in\langle -1, 1\rangle-(\{1/n\}_{n=1}^{\infty}\cup\{-1/n\}_{n=1}^{\infty}\cup\{0\}). \end{cases}$$

The function f is of Baire class 1 and at the point $x=0$ it fulfils conditions (2) and (4) and it does not fulfil any conditions of (3), (5) and (6).

Remark 4. Properties (3) and (5) are independent.

Example 3. Let f be a characteristic function of a set $A = \{1, 1/2, 1/3, 1/4, \ldots, -1, -1/2, -1/3, \ldots, 0\}$. The function f fulfils condition (3) at $x=0$, but does not fulfil (5) and (6).

Example 4. Put

$$f(x)=\begin{cases} 1 \text{ for } x \in \bigcup_{i=1}^{\infty} (2^{-2i-1}, 2^{-2i}\rangle, \\ 0 \text{ for } x \in \langle -1, 0\rangle, \\ -1 \text{ for } x \in \bigcup_{i=1}^{\infty} (2^{-2i}, 2^{-2i+1}\rangle. \end{cases}$$

The function f is of Baire class 1, it fulfils condition (5) at $x=0$ but it does not fulfil conditions (3) and (6).

Remark 5. The above mentioned examples demonstrate that any of the properties (2)—(5) does not imply (6).

Theorem 5. *If a function f has the property* (6) *at $x \in I$, then f has also the properties* (2)—(5) *at x.*

Proof. (6) implies (3). Let there exist a perfect set P such that $x \in P$, x is a bilateral point of accumulation of P and f/P, is continuous at x. Hence $f(x) \in K^+(f, x) \cap K^-(f, x)$.

(6) implies (2) and (4). According to Theorem 3, (3) implies (2) and (4) and from the foregoing it follows that (6) implies (2) and (4).

(6) implies (5). Let there exist a perfect set P such that $x \in P$, x is a bilateral point of accumulation and f/P is continuous at x. Let $a > f(x)$. Then $\{t; f(t) < a\} \cap (x-h, x) \supset \{t; f(t) < a\} \cap (x-h, x) \cap P$, which is a set of cardinality c, because f/P is continuous at x.

Similarly $\{t; f(t) < a\} \cap (x, x+h)$ has cardinality c and for $b < f(x)$ the sets $\{t; f(t) > b\} \cap (x-h, x)$ and $\{t; f(t) > b\} \cap (x, x+h)$ have cardinalities c.

In [3] there is given an example of a function which takes on every real number as value a continuum number of times on every perfect set. It is obvious that such a function is Darboux but it does not fulfil condition (6).

We can prove the following theorem.

Theorem 6. *If f is a Borel function, then* (1) *implies* (6).

Lemma. *Let f satisfy* (1) *at $x \in I$, then for each $h > 0$ and $\varepsilon > 0$ we have $f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon)) \cap \langle x, x+h)$ and $f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon)) \cap (x-h, x\rangle$ are sets of cardinalities c.*

Proof of Lemma. Let $x \in I$ and $I_+(x) \subset f(\langle x, x+h))$ for every $h>0$. We show that for every $\varepsilon>0$, $f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon)) \cap \langle x, x+h)$ has cardinality c.

Let us consider the following cases: (a) $I_+(x)=\emptyset$ and (b) $\overline{I_+(x)} \in f(x)$.

(a) Let $I_+(x)=\emptyset$. Then $\lim_{h\to 0^+} \sup f(\langle x, x+h)) = \lim_{h\to 0^+} \inf f(\langle x, x+h)) = f(x)$. Hence f is continuous from the right at x and we have that card $f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon)) \cap \langle x, x+h)=c$ for every $h>0$ and $\varepsilon>0$.

(b). If $f(x) \in \overline{I_+(x)}$, then $I_\varepsilon = I_+(x) \cap (f(x)-\varepsilon, f(x)+\varepsilon)$ is a subinterval of $f(<x, x+h))$. Clearly card $f^{-1}(I_\varepsilon) \cap \langle x, x+h)=c$ and $f^{-1}(I_\varepsilon) \cap \langle x, x+h) \subset f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon)) \cap \langle x, x+h)$.

Similarly it can be proved that the condition $I_-(x) \subset f((x-h, x\rangle)$ for every $h>0$ implies that $f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon)) \cap (x-h, x\rangle$ has cardinality c for every $h>0$ and $\varepsilon>0$.

Proof of Theorem 6. According to Lemma, the set $f^{-1}(f(x)-\varepsilon, f(x)+\varepsilon) \cap \langle x, x+h)$ has cardinality c for every $h>0$ and $\varepsilon>0$. Then there is $h_1>0$, $h_1<h$ such that $f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon)) \cap \langle x+h_1, x+h)$ has cardinality c. This follows from the fact that $f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon)) \cap (x, x+h) = \bigcup_{n=n_0}^{\infty} (\langle x+1/n, x+h)) \cap f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon))$, $n_0 = \min\{n; 1/n<h\}$. Since f is a Borel function, $f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon)) \cap \langle x+h_1, x+h)$ is a Borel set and according to the Alexandroff—Hausdorff's theorem [4, p. 355], there is a perfect set $P_1 \subset f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon)) \cap \langle x+h_1, x+h)$.

Similarly, according to Lemma the set $f^{-1}((f(x)-\varepsilon/2, f(x)+\varepsilon/2)) \cap \langle x, x+h_1)$ has cardinality c and there is $h_2>0$, $h_2<h_1$ such that $f^{-1}((f(x)-\varepsilon/2, f(x)+\varepsilon/2)) \cap \langle x+h_2, x+h_1)$ is a Borel set of cardinality c. Therefore there is a perfect set $P_2 \subset f^{-1}((f(x)-\varepsilon/2, f(x)+\varepsilon/2)) \cap \langle x+h_2, x+h_1)$. By induction there is a decreasing sequence of positive numbers $\{h_i\}_{i=1}^{\infty}$ converging to 0 such that $f^{-1}((f(x)-\varepsilon/2^{i-1}, f(x)+\varepsilon/2^{i-1})) \cap \langle x+h_i, x+h_{i-1})$ has cardinality c. Therefore there is a sequence of perfect sets $\{P_i\}_{i=1}^{\infty}$,

$$f^{-1}((f(x)-\varepsilon/2^{i-1}, f(x)+\varepsilon/2^{i-1})) \cap \langle x+h_i, x+h_{i-1}) \supset P_i .$$

Put $P_+ = \overline{\bigcup_{n=1}^{\infty} P_n}$. Since $\bigcup_{n=1}^{\infty} P_n$ is dense in itself, P_+ is a perfect set. From the construction of P_+ it follows that f/P_+ is continuous from the right at x.

Similarly from the condition that $I_-(x) \subset f((x-h, x\rangle)$ for every $h>0$ it follows that there is a perfect set P_- containing x and f/P_- is continuous from the left at x.

Put $P=P_+ \cup P_-$. P is a perfect set and f/P is continuous at x.

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ЛОКАЛЬНЫЕ СВОЙСТВА ФУНКЦИИ ДАРБУ ПЕРВОГО КЛАССА ВЭРА

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Резюме

В статье исследуются взаимоотношения между свойствами (1)—(6), кторые являются локальными свойствами функции Дарбу первого класса Бэра, для функции одного переменного.