Mathematica Slovaca

Michal Krupka

Orientability of higher order Grassmannians

Mathematica Slovaca, Vol. 44 (1994), No. 1, 107--115

Persistent URL: http://dml.cz/dmlcz/128620

Math. Slovaca, **44** (1994), No. 1, 107–115

Mathematica Slovaca

ORIENTABILITY OF HIGHER ORDER GRASSMANNIANS

MICHAL KRUPKA

(*Communicated by Július Korbaš*)

ABSTRACT. Let $\operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m)$, $n \leq m$, be the set of r-jets of immersions with source $0 \in \mathbb{R}^n$ and target $0 \in \mathbb{R}^m$. The r-order Grassmannian with indices m, n is the quotient space $G^r_{m,n} = \operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m)/L^r_n$, where L^r_n is the rth differential group of $\mathbb{R}^n$ which acts on $\operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m)$ to the right. We prove that $G^r_{m,n}$ is orientable if and only if the number $\binom{n+r}{r} + (m-n)\binom{n+r}{r-1}$ is odd.

1. Introduction

The aim of this short remark is to study the orientability of the higher order Grassmann manifolds $G^r_{m,n}$, which generalize the classical notion of a (first order) Grassmann manifold $G_{m,n}$. The geometric structures of this type have been introduced by E h r e s m a n n [2] and are also used as underlying structures for the geometric theory of partial differential equations (see [3]; the *manifold* N^k_m *of* k*-jets of* n*-dimensional submanifolds of a manifold* N from [3; 7.1] is a fibre bundle with base N and type fibre $G^k_{n+m,n}$).

The Grassmann manifold $G_{m,n}$ consists of n-dimensional vector subspaces of $\mathbb{R}^m$; these subspaces can be canonically identified with some equivalence classes of 1-jets of immersions from $\mathbb{R}^n$ to $\mathbb{R}^m$ with source and target at the origin 0. We understand $G_{m,n}$ as a manifold of such equivalence classes. The rth order Grassmann manifold $G^r_{m,n}$ is then defined as a manifold of equivalence classes of r-jets of immersions from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Using the methods of algebraic topology one can easily see that $G_{m,n}$ is orientable if and only if m is even. In this paper, we find by an elementary method a condition of orientability of $G^r_{m,n}$ for arbitrary r.

AMS Subject Classification (1991): Primary 53C42, 58A20.
Key words: Immersion, r-jet, Differential group, Higher order Grassmannian.

2. Higher order Grassmannians

In this section, we define the manifold $G^r_{m,n}$. Our method is analogous to a method used in [1; 16.11.10] in the special case $r = 1$.

Let r, n, and m be positive integers, $n \leq m$. Denote by L^r_n the rth differential group of $\mathbb{R}^n$, i.e. the group of invertible r-jets with source and target at $0 \in \mathbb{R}^n$. Consider the manifold $\operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m)$ of regular r-jets with source $0 \in \mathbb{R}^n$ and target at $0 \in \mathbb{R}^m$ and the following canonical right action of L^r_n on $\operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m)$:

$$\operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m) \times L^r_n \ni (J^r_0 g, J^r_0 \alpha) \to J^r_0 g \circ \alpha \in \operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m). \quad (2.1)$$

An orbit of this action containing an r-jet $J^r_0 g$ will be denoted by $[J^r_0 g]$, the orbit space $\operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m)/L^r_n$ by $G^r_{m,n}$, and the canonical projection of $\operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m)$ onto $G^r_{m,n}$ by π.

For fixed m and n we shall denote by I, J, K, etc., multi-indices of the form $\{i_1, i_2, \dots, i_n\}$, where $1 \leq i_1 < i_2 < \dots < i_n \leq m$. For a multi-index $I = \{i_1, i_2, \dots, i_n\}$ we denote $\{i_{n+1}, i_{n+2}, \dots, i_m\} = \{1, 2, \dots, m\} - I$, where $i_{n+1} < i_{n+2} < \dots < i_m$, and define mappings $\tau_I \colon \mathbb{R}^m \to \mathbb{R}^n$ and $\kappa_I \colon \mathbb{R}^m \to \mathbb{R}^{m-n}$ by

$$\begin{aligned} \tau_I(x^1, \dots, x^m) &= (x^{i_1}, \dots, x^{i_n}), \\ \kappa_I(x^1, \dots, x^m) &= (x^{i_{n+1}}, \dots, x^{i_m}). \end{aligned} \quad (2.2)$$

Further we set

$$\begin{aligned} \rho_I(J^r_0 g) &= (J^r_0 \tau_I g, J^r_0 \kappa_I g), \\ T_I &= \{J^r_0 g \in \operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m) \mid J^r_0 \tau_I g \in L^r_n\}. \end{aligned} \quad (2.3)$$

ρ_I is a diffeomorphism from $J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m)$ to $J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^n) \times J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^{m-n})$, and the restriction $\rho_I|_{T_I}$ is a diffeomorphism from T_I to $L^r_n \times J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^{m-n})$. Then T_I is an open (obviously L^r_n-invariant) submanifold of $\operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m)$.

LEMMA. *The canonical action of the differential group L^r_n defines on $\operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m)$ the structure of a principal L^r_n-bundle.*

Proof. We have to show that the graph $\operatorname{Graph} \mathcal{R}$ of the equivalence relation $\mathcal{R}$ on $\operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m)$ associated with the group action (2.1) is a closed submanifold of $\operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m) \times \operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m)$, and that the action (2.1) is free (see [1]).

Consider for any multi-index I the graph $\operatorname{Graph}\Gamma_I$ of the mapping

$$\Gamma_I : T_I \times L^r_n \ni (J^r_0 g, J^r_0 \alpha) \to J^r_0 \kappa_I \circ J^r_0 g \circ (J^r_0 \tau_I \circ J^r_0 g)^{-1} \circ J^r_0 \alpha \in J^r_{(0,0)}(\mathbb{R}^m, \mathbb{R}^{m-n}) .$$

Since this mapping is smooth, $\operatorname{Graph}\Gamma_I$ is a closed submanifold of $T_I \times L^r_n \times J^r_{(0,0)}(\mathbb{R}^m, \mathbb{R}^{m-n})$. But

$$\begin{aligned} &\operatorname{Graph}\mathcal{R} \cap \big(T_I \times \operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m)\big) \\ &\qquad = (\operatorname{id}_{T_I} \times \rho_I^{-1})(\operatorname{Graph}\Gamma_I) \cap \big(T_I \times \operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m)\big) . \end{aligned}$$

Since $\bigcup T_I = \operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m)$, the set $\operatorname{Graph}\mathcal{R}$ is a closed submanifold of $\operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m) \times \operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m)$.

To complete the proof, we have to show that the action (2.1) is free. Choose for any multi-index I two jets, $J^r_0 g_1 \in T_I$ and $J^r_0 g_2 \in \operatorname{Imm} J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^m)$, and suppose that there exists $J^r_0 \alpha \in L^r_n$ such that $J^r_0 g_2 = J^r_0 g_1 \circ J^r_0 \alpha$. Since $J^r_0 \tau_I \circ J^r_0 g_2 = J^r_0 \tau_I \circ J^r_0 g_1 \circ J^r_0 \alpha$, we have $J^r_0 \alpha = (J^r_0 \tau_I \circ J^r_0 g_1)^{-1} \circ (J^r_0 \tau_I \circ J^r_0 g_2)$, which completes the proof.

From Lemma it follows that there exists a unique smooth structure on $G^r_{m,n}$ such that the mapping π is a smooth surjective submersion. Considered with this smooth structure, $G^r_{m,n}$ is called the *rth order Grassmannian* (with indices m, n).

We shall introduce an important example of a smooth atlas on the manifold $G^r_{m,n}$. Set for any multi-index I

$$U_I = \pi(T_I) \tag{2.4}$$

and consider the mapping

$$\Phi_I : U_I \ni [J^r_0 g] \to J^r_0 \kappa_I \circ J^r_0 g \circ (J^r_0 \tau_I \circ J^r_0 g)^{-1} \in J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^{m-n}) . \tag{2.5}$$

Since $\Phi_I \circ \pi$ is smooth, and

$$(\Phi_I^{-1})(J^r_0 h) = \pi\big(J^r_0(\tau_I, \kappa_I)^{-1} \circ J^r_0(\operatorname{id}_{\mathbb{R}^n}, h)\big) , \tag{2.6}$$

Φ_I is a diffeomorphism. We set

$$\varphi_I = \chi \circ \Phi_I , \tag{2.7}$$

where χ is the canonical global system of coordinates on $J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^{m-n})$. If $J^r_0 h \in J^r_{(0,0)}(\mathbb{R}^n, \mathbb{R}^{m-n})$, $h = (h^{n+1}, \dots, h^m)$, then

$$\chi(J^r_0 h) = \left(\frac{\partial^s \bar{h}^\sigma}{\partial x^{k_1} \dots \partial x^{k_s}}(0) \right), \tag{2.8}$$

where $1 \le s \le r$, $n+1 \le \sigma \le m$, and $1 \le k_1 \le \cdots \le k_s \le n$. The pair (U_I, φ_I) is a chart on $G^r_{m,n}$ and the system $\big((U_I, \varphi_I)\big)$ is a smooth atlas.

In the next paragraph we shall use the mapping

$$\Psi\colon L^r_m \times G^r_{m,n} \ni \big(J^r_0\alpha, [J^r_0 g]\big) \to [J^r_0 \alpha \circ g] \in G^r_{m,n}\,. \tag{2.9}$$

It is easily seen that this mapping is defined correctly, and defines a smooth left action of L^r_m on $G^r_{m,n}$.

3. Higher order Grassmannians orientability theorem

The following theorem clarifies the orientability of the higher order Grassmannians.

THEOREM. *The rth order Grassmannian $G^r_{m,n}$ is orientable if and only if the number $\binom{n+r}{r} + (m-n)\binom{n+r}{r-1}$ is odd.*

P r o o f . We shall use indices σ, μ, k, t, s, and $k_1, \dots, k_s$, where $n+1 \le \sigma \le m$, $n+2 \le \mu \le m$, $1 \le k \le n$, $1 \le t \le n-1$, $1 \le s \le r$, and $1 \le k_1 \le \cdots \le k_s \le n$.

The proof can be divided into three steps. In the first step, we derive the transformation formula (3.7) between charts (U_I, φ_I), (U_J, φ_J), where $I = \{1, \dots, n\}$, and $J = \{1, \dots, n-1, n+1\}$. We note that if the manifold $G^r_{m,n}$ is orientable, then for any two points $x, \bar{x} \in U_I \cap U_J$ it holds $\operatorname{sgn}\det D(\varphi_I \circ \varphi_J^{-1})(x) = \operatorname{sgn}\det D(\varphi_I \circ \varphi_J^{-1})(\bar{x})$. In the second step, we show that this formula considered for specially chosen points $x, \bar{x} \in U_I \cap U_J$ is equivalent to saying that the number $\binom{n+r}{r} + (m-n)\binom{n+r}{r-1}$ is odd. This will prove the first implication of the theorem. In the third step, we prove that from the same formula it follows that the manifold $G^r_{m,n}$ is orientable.

Set

$$I = \{1, \dots, n\}\,, \qquad J = \{1, \dots, n-1, n+1\}\,. \tag{3.1}$$

and denote $\varphi_I = (x^\sigma_{k_1\dots k_s})$, and $\varphi_J = (\bar{x}^\sigma_{k_1\dots k_s})$. For fixed indices s, σ, and $k_1, \dots, k_s$ define

$$\alpha(\sigma, s, k_1, \dots, k_s) = \begin{cases} 1 & \text{if } \sigma = n+1\,, \\ 0 & \text{if } \sigma > n+1 \end{cases}$$

and denote by $\beta(\sigma, s, k_1, \dots, k_s)$ the number of indices $k_1, \dots, k_s$ which are equal to n, in the special case of $\sigma = n+1$, $s = 1$, and $k_1 = n$ set

$$\alpha(n+1, 1, n) = 0\,,$$
$$\beta(n+1, 1, n) = 2\,.$$

If there is no danger of confusion, we write α, β instead of $\alpha(\sigma, s, k_1, \dots, k_s)$, $\beta(\sigma, s, k_1, \dots, k_s)$.

The set of polynomials in the variables $(x^{\nu}_{t_1})$, $(x^{\nu}_{t_1 t_2})$, $\dots$, $(x^{\nu}_{t_1 \dots t_{s-1}})$, $(x^{\nu}_{t_1 \dots t_{s-1} n})$ ($\nu \in \{\sigma, n+1\}$, $\{t_1, \dots, t_{s-1}\} \subset \{k_1, \dots, k_s, n\}$, $t_1 \leq \dots \leq t_{s-1}$), each non-zero member of which is independent of the variable x^{n+1}_n and is at least of second degree, will be denoted by $P^{\sigma}_{k_1 \dots k_s}$. The set of functions of the form

$$q = p_0 + \frac{p_1}{x^{n+1}_n} + \frac{p_2}{(x^{n+1}_n)^2} + \dots + \frac{p_{s+1}}{(x^{n+1}_n)^{s+1}},$$

where $p_0, p_1, \dots, p_{s-1} \in P^{\sigma}_{k_1 \dots k_s}$, will be denoted by $Q^{\sigma}_{k_1 \dots k_s}$.

Let $x \in U_I \cap U_J$, $\Phi_I(x) = J^r_0 h$. Since, by our choice of I and J,

$$\frac{\partial h^{n+1}}{\partial x^n} \neq 0,$$

then there exists a mapping $\bar{h}$ such that, on a neighbourhood of $0 \in \mathbb{R}^n$, we have

$$\begin{aligned} &h^{n+1}(x^1, \dots, x^{n-1}, \bar{h}^{n+1}(x^1, \dots, x^n)) = x^n, \\ &\bar{h}^{\mu}(x^1, \dots, x^n) = h^{\mu}(x^1, \dots, x^{n-1}, \bar{h}^{n+1}(x^1, \dots, x^n)) \end{aligned} \tag{3.2}$$

(inverse function theorem). Then $\Phi_J(x) = J^r_0 \bar{h}$.

There is the following relation between the mappings h, and $\bar{h}$:

$$\begin{aligned} \frac{\partial^s \bar{h}^{\sigma}}{\partial x^{k_1} \dots \partial x^{k_s}} &= (-1)^{\alpha} \frac{\dfrac{\partial^s h^{\sigma}}{\partial x^{k_1} \dots \partial x^{k_s}}}{\left(\dfrac{\partial h^{n+1}}{\partial x^n}\right)^{\alpha+\beta}} \\ &\quad + q\left(\frac{\partial h^{\nu}}{\partial x^{t_1}}, \frac{\partial^2 h^{\nu}}{\partial x^{t_1} \partial x^{t_2}}, \dots, \frac{\partial^{s-1} h^{\nu}}{\partial x^{t_1} \dots \partial x^{t_{s-1}}}, \frac{\partial^s h^{\nu}}{\partial x^{t_1} \dots \partial x^{t_{s-1}} \partial x^n}\right), \end{aligned} \tag{3.3}$$

where $q \in Q^{\sigma}_{k_1 \dots k_s}$, $\nu \in \{\sigma, n+1\}$, $\{t_1, \dots, t_{s-1}\} \subset \{k_1, \dots, k_s, n\}$, $t_1 \leq \dots \dots \leq t_{s-1}$. This formula can be verified by induction; by a direct calculation with the help of (3.2), we obtain

$$\frac{\partial \bar{h}^{n+1}}{\partial x^n} = \frac{1}{\dfrac{\partial h^{n+1}}{\partial x^n}}, \qquad \frac{\partial \bar{h}^{n+1}}{\partial x^t} = -\frac{\dfrac{\partial h^{n+1}}{\partial x^t}}{\dfrac{\partial h^{n+1}}{\partial x^n}},$$

$$\frac{\partial \bar{h}^{\mu}}{\partial x^n} = \frac{\dfrac{\partial h^{\mu}}{\partial x^n}}{\dfrac{\partial h^{n+1}}{\partial x^n}}, \qquad \frac{\partial \bar{h}^{\mu}}{\partial x^t} = \frac{\partial h^{\mu}}{\partial x^t} - \frac{\dfrac{\partial h^{\mu}}{\partial x^n}\dfrac{\partial h^{n+1}}{\partial x^t}}{\dfrac{\partial h^{n+1}}{\partial x^n}}, \tag{3.4}$$

$$\frac{\partial^2 \bar{h}^{n+1}}{\partial (x^n)^2} = -\frac{\dfrac{\partial^2 h^{n+1}}{\partial (x^n)^2}}{\left(\dfrac{\partial h^{n+1}}{\partial x^n}\right)^3}, \qquad \frac{\partial^2 \bar{h}^{n+1}}{\partial x^t \partial x^n} = -\frac{\dfrac{\partial^2 h^{n+1}}{\partial x^t \partial x^n}}{\left(\dfrac{\partial h^{n+1}}{\partial x^n}\right)^2} + \frac{\dfrac{\partial h^{n+1}}{\partial x^t}\dfrac{\partial^2 h^{n+1}}{\partial (x^n)^2}}{\left(\dfrac{\partial h^{n+1}}{\partial x^n}\right)^3}.$$

which satisfies (3.3), and by differentiation of the $(s-1)$-order formula

$$\frac{\partial^{s-1}\bar{h}^{\sigma}}{\partial x^{k_1}\dots\partial x^{k_{s-1}}} = (-1)^{\alpha}\frac{\dfrac{\partial^{s-1}h^{\sigma}}{\partial x^{k_1}\dots\partial x^{k_{s-1}}}}{\left(\dfrac{\partial h^{n+1}}{\partial x^n}\right)^{\alpha+\gamma}} + q\left(\frac{\partial h^{\nu}}{\partial x^{t_1}}, \frac{\partial^2 h^{\nu}}{\partial x^{t_1}\partial x^{t_2}}, \dots, \frac{\partial^{s-2}h^{\nu}}{\partial x^{t_1}\dots\partial x^{t_{s-2}}}, \frac{\partial^{s-1}h^{\nu}}{\partial x^{t_1}\dots\partial x^{t_{s-2}}\partial x^{n}}\right) \tag{3.5}$$

($q_1 \in Q^{\sigma}_{k_1\dots k_{s-1}}$, $\nu \in \{\sigma, n+1\}$, $\{t_1,\dots,t_{s-2}\} \subset \{k_1,\dots,k_{s-1},n\}$, $t_1 \le \dots \le t_{s-2}$, and $\gamma = \beta(\sigma, s-1, k_1,\dots,k_{s-1})$) with respect to x^{k_s}, we obtain (3.3).

Since

$$x^{\sigma}_{k_1\dots k_s}(x) = \frac{\partial^s h^{\sigma}}{\partial x^{k_1}\dots\partial x^{k_s}}(0), \qquad \bar{x}^{\sigma}_{k_1\dots k_s}(x) = \frac{\partial^s \bar{h}^{\sigma}}{\partial x^{k_1}\dots\partial x^{k_s}}(0) \tag{3.6}$$

(see (2.8)), formula (3.3) has in $0 \in \mathbb{R}^n$ the form

$$\bar{x}^{\sigma}_{k_1\dots k_s} = (-1)^{\alpha}\frac{x^{\sigma}_{k_1\dots k_s}}{\left(x^{n+1}_n\right)^{\alpha+\beta}} + q\left(x^{\nu}_{t_1}, x^{\nu}_{t_1 t_2}, \dots, x^{\nu}_{t_1\dots t_{s-1}}, x^{\nu}_{t_1\dots t_{s-1}n}\right), \tag{3.7}$$

which is the transformation formula between the charts (U_I, φ_I), (U_J, φ_J).

Now let us consider two specially chosen points $x, \bar{x} \in U_I \cap U_J$, $x = [J_0^r g]$, $\bar{x} = [J_0^r \bar{g}]$, where $g(x^1, \dots, x^n) = (x^1, \dots, x^n, x^n, 0, \dots, 0)$, and $\bar{g}(x^1, \dots, x^n) = (x^1, \dots, x^n, -x^n, 0, \dots, 0)$. According to (2.5), there holds $\Phi_I(x) = \Phi_J(x) = J_0^r h$, and $\Phi_I(\bar{x}) = \Phi_J(\bar{x}) = J_0^r \bar{h}$, where $h(x^1, \dots, x^n) = (x^n, 0, \dots, 0)$, and $\bar{h}(x^1, \dots, x^n) = (-x^n, 0, \dots, 0)$. From (2.7) and (2.8) it immediately follows that

$$\begin{aligned} x^\sigma_{k_1 \dots k_s}(x) = \bar{x}^\sigma_{k_1 \dots k_s}(x) &= \begin{cases} 1 & \text{for } \sigma = n+1, \ s = 1, \ k_1 = n, \\ 0 & \text{in all other cases}, \end{cases} \\ x^\sigma_{k_1 \dots k_s}(\bar{x}) = \bar{x}^\sigma_{k_1 \dots k_s}(\bar{x}) &= \begin{cases} -1 & \text{for } \sigma = n+1, \ s = 1, \ k_1 = n, \\ 0 & \text{in all other cases}. \end{cases} \end{aligned} \tag{3.8}$$

Using (3.7) we get

$$\begin{aligned} \det D(\varphi_J^{-1} \circ \varphi_I)(\varphi_I(x)) &= \prod_{\sigma, s, k_1, \dots, k_s} (-1)^\alpha, \\ \det D(\varphi_J^{-1} \circ \varphi_I)(\varphi_I(\bar{x})) &= \prod_{\sigma, s, k_1, \dots, k_s} (-1)^\alpha (-1)^{\alpha+\beta}, \end{aligned} \tag{3.9}$$

which means that if the manifold $G^r_{m,n}$ is orientable, then

$$\prod_{\sigma, s, k_1, \dots, k_s} (-1)^\alpha = \prod_{\sigma, s, k_1, \dots, k_s} (-1)^\alpha (-1)^{\alpha+\beta}, \tag{3.10}$$

which is equivalent to saying that the number

$$\sum_{\sigma, s, k_1, \dots, k_s} (\alpha + \beta)$$

is even. After some combinatorical calculations we get

$$\sum_{\sigma, s, k_1, \dots, k_s} (\alpha + \beta) = \binom{n+r}{r} + (m-n)\binom{n+r}{r-1} - 1. \tag{3.11}$$

In the last part of the proof, we shall show that the condition (3.10) is sufficient for the manifold $G^r_{m,n}$ to be orientable. We shall need the following simple assertion, which can be proved by induction.

Let G be a smooth manifold, $((U_\iota,\varphi_\iota))$, $\iota=1,\dots,N$, a smooth atlas on G. Suppose that there is a point $x_0\in\bigcap U_\iota$, and that for any indices $\iota_1,\iota_2\in\{1,\dots,N\}$ the mapping $\det D(\varphi_{\iota_1},\varphi_{\iota_2}^{-1})$ has a constant sign on all its domain. Then the manifold G is orientable.

Let $I=\{i_1,i_2,\dots,i_n\}$, $J=\{j_1,j_2,\dots,j_n\}$ be two arbitrary multi-indices. Denote again $\varphi_I=(x^\sigma_{k_1\dots k_s})$ and $\varphi_J=(\bar{x}^\sigma_{k_1\dots k_s})$. The set U_I-U_J is given by

$$U_I-U_J=\left\{x\in U_I \mid \det\left(A_t^{j_k}(x)\right)=0,\ k,t\in\{1,2,\dots,n\}\right\}. \tag{3.12}$$

where $A\colon U_I\to\mathbb{R}^n\times\mathbb{R}^m$ is a matrix such that $A_t^{i_k}(x)=\delta_t^k$, and $A_t^{i_\sigma}(x)=x_t^\sigma(x)$. Then the set $U_I\cap U_J$ is non-connected and has two components. The set U_I-U_J, and therefore the set $G^r_{m,n}-U_J$ is of measure zero. Then there exists a point $x_0\in\bigcap U_I$.

If the function $\det D(\varphi_I\circ\varphi_J^{-1})$ has a constant sign on all its domain, we write $(U_I,\varphi_I)\sim(U_J,\varphi_J)$. We shall prove that the relation $\sim$ is transitive. Suppose $(U_I,\varphi_I)\sim(U_J,\varphi_J)$ and $(U_J,\varphi_J)\sim(U_K,\varphi_K)$ and choose two elements $x_1,x_2\in U_I\cap U_J\cap U_K$ belonging to different components of $U_I\cap U_K$. Now

$$\begin{aligned}
&\operatorname{sgn}\det D(\varphi_I\circ\varphi_K^{-1})\big(\varphi_K(x_1)\big)\\
&=\operatorname{sgn}\det D(\varphi_I\circ\varphi_J^{-1})\big(\varphi_J(x_1)\big)\cdot\operatorname{sgn}\det D(\varphi_J\circ\varphi_K^{-1})\big(\varphi_K(x_1)\big)\\
&=\operatorname{sgn}\det D(\varphi_I\circ\varphi_J^{-1})\big(\varphi_J(x_2)\big)\cdot\operatorname{sgn}\det D(\varphi_J\circ\varphi_K^{-1})\big(\varphi_K(x_2)\big)\\
&=\operatorname{sgn}\det D(\varphi_I\circ\varphi_K^{-1})\big(\varphi_K(x_2)\big).
\end{aligned}$$

Thus, $(U_I,\varphi_I)\sim(U_K,\varphi_K)$.

Let S_m be the permutation group of m elements. Define a group homomorphism $F\colon S_m\to L^r_m$ by $F(\pi)=J_0^r\alpha_\pi$, where $\alpha_\pi\colon\mathbb{R}^m\to\mathbb{R}^m$ is given by $\alpha_\pi(x^1,\dots,x^m)=\left(x^{\pi(1)},\dots,x^{\pi(m)}\right)$. F is obviously injective. We denote $P^r_m=F(S_m)$. The mapping Ψ_p, $\Psi_p(x)=\Psi(p,x)$, (see (2.9)) is a diffeomorphism of $G^r_{m,n}$.

Let I, J, K, L be multi-indices such that the sets $I-J$ and $K-L$ have just one element. There evidently exists an element $p\in P^r_m$ such that $\left(\Psi_p(U_I),\varphi_I\circ\Psi_p\right)=(U_K,\varphi_K)$, and $\left(\Psi_p(U_J),\varphi_J\circ\Psi_p\right)=(U_L,\varphi_L)$. Hence, from $(U_I,\varphi_I)\sim(U_J,\varphi_J)$ it follows $(U_K,\varphi_K)\sim(U_L,\varphi_L)$. Finally, if I and J are arbitrary multi-indices, then there exists a sequence $I=K_1,\dots,K_N=J$ such that the set $K_{\iota+1}-K_\iota$ has for any $\iota<N$ just one element. From the transitivity of the relation $\sim$ and from the above assertion it follows that, if there exists a pair of charts (U_I,φ_I) and (U_J,φ_J) such that $(U_I,\varphi_I)\sim(U_J,\varphi_J)$, then

the manifold $G^r_{m,n}$ is orientable. As we have proved, if the number $\binom{n+r}{r} + (m-n)\binom{n+r}{r-1}$ is odd, then for $I = \{1,\dots,n\}$ and $J = \{1,\dots,n-1,n+1\}$, $(U_I,\varphi_I) \sim (U_J,\varphi_J)$. This proves our theorem.

REFERENCES

[1] DIEUDONNÉ, J.: *Treatise on Analysis. Vol. III*, Academic Press, New York-London, 1972.

[2] EHRESMANN, C.: *Les prolongements d'une variété différentiable*, C.R. Acad. Sci. Paris Sér. I Math. **233** (1951), 598–600.

[3] VINOGRADOV, A. M.—KRASILŠČIK, I. S—LYČAGIN, V. V: *Introduction to the Geometry of Nonlinear Differential Equations* (Russian), Nauka, Moskva, 1986.

Received March 2, 1992

Revised December 7, 1992

Department of Mathematics
Silesian University at Opava
Bezručovo nám. 13
CZ-746 01 Opava
Czech Republic
E-mail: krul1um@decsu.fpf.slu.cz