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GENERAL COMPARABILITY OF PSEUDO MV-ALGEBRAS

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ABSTRACT. The conditions of general comparability and weak general comparability have been used by investing the states on pseudo MV-algebras. In the present note we show that these conditions are equivalent.

1. Introduction

Pseudo MV-algebras have been introduced independently by Georgescu and Iorgulescu [5] and by Rachůnek [7] (in [7], they were called generalized MV-algebras).

States on pseudo MV-algebras have been studied by Dvurečenskij [1], [2] and by Dvurečenskij and Kalmbach [4].

In [2], the conditions of general comparability and weak general comparability of a pseudo MV-algebra M have been used.

In [2] there is remarked that the general comparability of M implies the weak general comparability, and that the converse implication seems to be open.

In the present note we prove that the mentioned conditions are equivalent.

2. Preliminaries

For the sake of completeness, we recall the definition of a pseudo MV-algebra.

Let $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$ be an algebra of type $(2,1,1,0,0)$. For $x, y \in A$ we put

$$x \odot y = \sim(\neg x \oplus \neg y)\,.$$

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Assume that the following identities are valid in $\mathcal{A}$:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 = 1 \oplus x$;
(A4) $\neg 1 = 0$; $\sim 1 = 0$;
(A5) $\neg(\sim x \oplus \sim y) = \sim(\neg x \oplus \neg y)$;
(A6) $x \oplus (y \odot \sim x) = y \oplus (x \odot \sim y) = (\neg y \odot x) \oplus y = (\neg x \odot y) \oplus x$;
(A7) $x \odot (\neg x \oplus y) = (x \oplus \sim y) \odot y$;
(A8) $\sim \neg x = x$.

Then the algebraic structure $\mathcal{A}$ is said to be a *pseudo MV-algebra.*

Let $\mathcal{A}$ be a pseudo MV-algebra. For $x, y \in A$ we put $x \leqq y$ if and only if $\neg x \oplus y = 1$. Then $(A; \leqq)$ turns out to be a *distributive lattice* with the least element 0 and the greatest element 1; we denote $(A; \leqq) = \ell(\mathcal{A})$.

Let G be a lattice ordered group with a strong unit u. We put $A = [0, u]$ and for $x, y \in A$ we set

$$x \oplus y = (x + y) \wedge u\,, \quad \neg x = u - x\,, \quad \sim x = -x + u\,, \quad 1 = u\,.$$

Then the structure $\big([0, u]; \oplus, \neg, \sim, 0, u\big)$ is a *pseudo MV-algebra*; we denote it by $\Gamma(G, u)$.

Dvurečenskij [3] proved that for each pseudo MV-algebra $\mathcal{A}$ there exists a lattice ordered group G with a strong unit u such that

$$\mathcal{A} = \Gamma(G, u)\,. \tag{1}$$

Moreover, $\ell(\mathcal{A})$ coincides with the interval $[0, u]$ of G.

Throughout the present paper we assume that $\mathcal{A}$, G and u are as in (1).

3. Comparability conditions

An element $e \in A$ is called *Boolean* if $e \oplus e = e$. We denote by $B(\mathcal{A})$ the set of all Boolean elements of $\mathcal{A}$. If $e \in B(\mathcal{A})$, then $\neg e = \sim e$ and $\neg e$ is the complement of e in the lattice $\ell(\mathcal{A})$; further, $\neg e \in B(\mathcal{A})$. (Cf. [5].) We put $\neg e = e'$.

Let $e \in B(\mathcal{A})$ and $x \in A$. We set

$$p_e^A(x) = x \wedge e\,.$$

The interval $[0, e]$ of the lattice $\ell(\mathcal{A})$ will be denoted by A_e. In [6] it has been proved that the set A_e is closed with respect to the operation $\oplus$ and that we can define unary operations $\neg_e$, $\sim_e$ in A_e such that $\mathcal{A}_e = (A; \oplus, \neg_e, \sim_e, 0, e)$

turns out to be a pseudo MV-algebra; moreover, $\mathcal{A}_e$ is a direct factor of $\mathcal{A}$ and for each $x \in A$, $p_e^A(x)$ is the component of x in $\mathcal{A}_e$.

We denote by G_e the convex ℓ-subgroup of G generated by the element e. In view of [2; Proposition 3.2], G_e is a direct factor of the lattice ordered group G and the corresponding complementary direct factor of G is $G_{e'}$; for $g \in G$ we denote by $p_e(g)$ the component of g in G_e. If $g \in G^+$, then by [2; Proposition 3.3],

$$p_e(g) = g \wedge ne\,,$$

where n is a positive integer with $g \leqq nu$.

The *lattice ordered group G* is said to *satisfy the general comparability* if for each $x, y \in G$ there exists $e \in B(\mathcal{A})$ such that

$$p_e(x) \leqq p_e(y)\,, \qquad p_{e'}(x) \geqq p_{e'}(y)\,.$$

We say that the *pseudo MV-algebra $\mathcal{A}$ satisfies the general comparibility* if G satisfies this condition.

Finally, $\mathcal{A}$ is said to *satisfy the weak general comparability* if for each $x, y \in A$ there exists $e \in B(\mathcal{A})$ such that

$$p_e^A(x) \leqq p_e^A(y)\,, \qquad p_{e'}^A(x) \geqq p_{e'}^A(y)\,.$$

PROPOSITION. *Let $\mathcal{A}$ be a pseudo MV-algebra. Then $\mathcal{A}$ satisfies the weak general comparability if and only if it satisfies the general comparability.*

Proof. It suffices to deal with the condition "only if". Assume that $\mathcal{A}$ satisfies the weak general comparability. Let $x, y \in G$. Put

$$x \wedge y = z\,, \quad x' = x - z\,, \quad y' = y - z\,.$$

Then $x' \wedge y' = 0$. We denote

$$x' \wedge u = x_1\,, \qquad y' \wedge u = y_1\,.$$

We have

$$x_1 \wedge y_1 = 0\,. \tag{1}$$

There exists a positive integer n such that $x' \vee y' \leqq nu$. Then in view of the Riesz decomposition theorem there exist

$$x_i^1, y_i^1 \in [0, u] \qquad (i = 1, 2, \dots, n)$$

such that

$$x' = x_1^1 + x_2^1 + \dots + x_n^1\,, \qquad y' = y_1^1 + y_2^1 + \dots + y_n^1\,.$$

According to the assumption, there exists $e \in B(\mathcal{A})$ such that

$$p_e^A(x_1) \leqq p_e^A(y_1)\,, \qquad p_{e'}^A(x_1) \geqq p_{e'}^A(y_1)\,.$$

Therefore we also have

$$p_e(x_1) \leqq p_e(y_1)\,, \tag{2}$$
$$p_{e'}(x_1) \geqq p_e(y_1)\,. \tag{3}$$

From (1) we obtain

$$p_e(x_1) \wedge p_e(y_1) = 0\,, \qquad p_{e'}(x_1) \wedge p_{e'}(y_1) = 0\,.$$

Hence (2) and (3) yield

$$p_e(x_1) = 0 = p_{e'}(y_1)\,. \tag{4}$$

Let $i \in \{1, 2, \dots, n\}$. We have $x_i^1 \leqq x'$ and $x_i' \leqq u$, whence $x_i^1 \leqq x' \wedge u = x_1$. Then from (4) we obtain $p_e(x_i^1) = 0$. Thus $p_e(x') = 0$. Therefore

$$p_e(x') \leqq p_e(y')\,. \tag{5}$$

Analogously we obtain

$$p_{e'}(x') \geqq p_{e'}(y')\,. \tag{6}$$

Since $x = x' + z$, $y = y' + z$, we have

$$p_e(x) = p_e(x') + p_e(z)\,, \qquad p_e(y) = p_e(y') + p_e(z)\,.$$

Thus from (5) we get

$$p_e(x) \leqq p_e(y)\,.$$

Similarly, in view of (6),

$$p_{e'}(x) \geqq p_{e'}(y)\,.$$

Therefore G satisfies the general comparability. □

We conclude that in [2; 4.4–4.8] we can take weak general comparability instead of strong general comparability.

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