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VARIETIES SATISFYING IDEAL EQUALITIES

IVAN CHAJDA, JAROMÍR, DUDA

Congruence permutability ($\Phi \cdot \Theta = \Theta \cdot \Phi$) as well as congruence distributivity ($\Psi \wedge (\Theta \vee \Phi) = (\Psi \wedge \Theta) \vee (\Psi \wedge \Phi)$) belong to the fundamental congruence equalities in universal algebra. Varieties of algebras with permutable congruences were described by A. I. Mal'cev in his pioneering paper [8], varieties satisfying the second equality were characterized by B. Jónsson, see [6, 7]. Weaker modifications of congruence permutability and/or distributivity were introduced for varieties with constant 0 as follows:

The equality $[0]\Psi \cdot \Theta = [0]\Theta \cdot \Psi$ is called *congruence permutability at* 0, see [3] and [5]; the equality $[0]\Psi \wedge (\Theta \vee \Phi) = [0](\Psi \wedge \Theta) \vee (\Psi \wedge \Phi)$ is named *distributivity* at 0, see [2].

Mal'cev's conditions for varieties permutable at 0 (distributive at 0) were already given in [5] ([2], respectively). Using the concept of an ideal in universal algebra, see [1], [5], [9], we can consider the following ideal equalities:

Let $\boldsymbol{K}$ be a class of algebras of the same type having a constant 0. Let I be an ideal in $A \in \boldsymbol{K}$ and $\Psi, \Theta, \Theta_1, \Theta_2$ be congruences on A. The equality $[I]\Psi \cdot \Theta = [I]\Theta \cdot \Psi$ is called *ideal permutability* and the equality

$$[I]\Psi \wedge (\Theta_1 \vee \Theta_2) = [I](\Psi \wedge \Theta_1) \vee (\Psi \wedge \Theta_2)$$

is called *ideal distributivity* in the sequel.

The aim of this paper is to describe varieties satisfying the last two ideal equalities. For this reason let us recall from [1], [5] and [9]:

Definition 0. Let $\boldsymbol{K}$ be a class of algebras of the same type with constant 0.

(i) A term $\boldsymbol{t}(\vec{x}, \vec{y}) = \boldsymbol{t}(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is called an *ideal term* in $\vec{y}$ if $\boldsymbol{t}(\vec{x}, \vec{0}) = 0$ is an identity in $\boldsymbol{K}$.

(ii) A nonempty subset I of $A \in \boldsymbol{K}$ is called an *ideal* if for every ideal term $\boldsymbol{t}(\vec{x}, \vec{y})$ in $\vec{y}$, $\vec{a} \in A^m$, $\vec{i} \in I^n$ it holds $\boldsymbol{t}(\vec{a}, \vec{i}) \in I$.

The following easy lemma justifies the title of the present paper:

Lemma 1. *Let $\boldsymbol{K}$ be a class of algebras of the same type with constant* 0. *For every ideal I of $A \in \boldsymbol{K}$ and each congruence $\Theta \in \operatorname{Con} A$, the subset*

$$[I]\Theta = \cup \{[i]\Theta;\ i \in I\}$$

of A is again an ideal of A.

Proof. Let $\boldsymbol{t}(\vec{x}, \vec{y}) = \boldsymbol{t}(x_1, \ldots, x_m, y_1, \ldots, y_n)$ be an ideal term in $\vec{y}$. Take $\vec{a} = \langle a_1, \ldots, a_m \rangle \in A^m$ and $\vec{b} = \langle b_1, \ldots, b_n \rangle \in ([I]\Theta)^n$. Evidently, $b_j \in [I]\Theta$ means that $b_j \Theta i_j$ for some element $i_j \in I$, $j = 1, \ldots, n$. Denote $\vec{i} = \langle i_1, \ldots, i_n \rangle \in I^n$. Then $\boldsymbol{t}(\vec{a}, \vec{b})\, \Theta\, \boldsymbol{t}(\vec{a}, \vec{i}) \in I$ or equivalently, $\boldsymbol{t}(\vec{a}, \vec{b}) \in [I]\Theta$, which was to be proved. □

Definition 1. Let $\boldsymbol{K}$ be a class of algebras of the same type with constant 0. An algebra $A \in \boldsymbol{K}$ is *ideal permutable* whenever $[I]\Theta \cdot \Psi = [I]\Psi \cdot \Theta$ holds for each ideal I of A and each $\Theta, \Psi \in \mathit{Con}\, A$. $\boldsymbol{K}$ is *ideal permutable* if each $A \in \boldsymbol{K}$ has this property.

Lemma 2. *Let A be an algebra with constant* 0 *and R, S, T be subalgebras of the direct product $A \times A$ (i.e. they are the compatible relations on A). Then*

(a) $[0]R \cdot S = [[0]R]S$;

(b) $[0]R \cdot S \cdot T = [0]S \cdot R \cdot T$ *whenever* $[0]R \cdot S = [0]S \cdot R$;

(c) $[0]R \cdot S = [0]R \vee S$ *whenever R, S are congruences and*

$$[0]R \cdot S = [0]S \cdot R.$$

For the proof, see Lemma 1 and Theorem 2 in [3]. Now, we are ready to prove

Theorem 1. *Let $\boldsymbol{V}$ be a variety with constant* 0. *The following conditions are equivalent*:

(1) $\boldsymbol{V}$ *is ideal permutable*;

(2) $\boldsymbol{V}$ *is permutable at* 0;

(3) *there is a binary term $\boldsymbol{s}$ such that*

$$\boldsymbol{s}(x, x) = 0 \quad \text{and} \quad \boldsymbol{s}(x, 0) = x$$

are identities in $\boldsymbol{V}$.

Proof. (1) ⇒ (2) is trivial. (2) ⇒ (3) can be found in the proof of Corollary 1.9 in [5], see also [3]. It remains to prove (3) ⇒ (1). Let I be an arbitrary ideal of an algebra $A \in \boldsymbol{V}$ and $\Theta, \Psi \in \mathit{Con}\, A$. The proof of Corollary 1.9 in [5] yields that $I = [0]\Phi$ for some $\Phi \in \mathit{Con}\, A$. By Theorem 1 in [3], (3) implies permutability at 0, hence

$$\begin{aligned}[I]\Theta \cdot \Psi &= [[0]\Phi]\Theta \cdot \Psi = [0]\Phi \cdot \Theta \cdot \Psi = [[0]\Phi \cdot \Theta]\Psi = \\ &= [[0]\Phi \vee \Theta]\Psi = [[0]\Psi]\Phi \vee \Theta \supseteq [[0]\Psi]\Phi \cdot \Theta = \\ &= [0]\Psi \cdot \Phi \cdot \Theta = [[0]\Psi \cdot \Phi] \cdot \Theta = [[0]\Phi \cdot \Psi]\Theta = \\ &= [[0]\Phi]\, \Psi \cdot \Theta = [I]\Psi \cdot \Theta.\end{aligned}$$

By symmetry the converse inclusion $[I]\Theta \cdot \Psi \subseteq [I]\Psi \cdot \Theta$ follows. □

Definition 2. Let $\boldsymbol{K}$ be a class of algebras of the same type with constant 0. An algebra $A \in \boldsymbol{K}$ is *ideal distributive* whenever

$$[I]\Psi \wedge (\Theta_1 \vee \Theta_2) = [I](\Psi \wedge \Theta_1) \vee (\Psi \wedge \Theta_2)$$

holds for each ideal I of A and every congruence $\Psi, \Theta_1, \Theta_2 \in Con\, A$. $\boldsymbol{K}$ is *ideal distributive* if each $A \in \boldsymbol{K}$ has this property.

Theorem 2. *Let $\boldsymbol{V}$ be a variety of algebras with constant* 0. *The following conditions are equivalent*:

(1) *$\boldsymbol{V}$ is ideal distributive*;

(2) *there exist an integer $n > 1$ and ternary terms $\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n$ such that*
$\boldsymbol{d}_0(x, y, z) = x$, $d_n(x, y, 0) = 0$
$\boldsymbol{d}_i(x, y, x) = x$ *for* $0 \leqslant i \leqslant n$
$\boldsymbol{d}_i(x, x, z) = \boldsymbol{d}_{i+1}(x, x, z)$ *for $i < n$ even*
$\boldsymbol{d}_i(x, z, z) = \boldsymbol{d}_{i+1}(x, z, z)$ *for $i < n$ odd.*

Proof. (1)$\Rightarrow$(2): Let $A = F_V(x, y, z)$ be the free algebra in $\boldsymbol{V}$ with free generators x, y, z. Let $I = I(z)$ be the principal ideal in A generated by z and put $\Psi = \Theta(x, z)$, $\Theta_1 = \Theta(x, y)$, $\Theta_2 = \Theta(y, z)$. Then $x \in [I]\Psi \wedge (\Theta_1 \vee \Theta_2)$ and so $x \in [I](\Psi \wedge \Theta_1) \vee (\Psi \wedge \Theta_2)$, by the hypothesis. Hence, there exist elements $d_0, \ldots, d_n \in A$ such that $x = d_0$, $d_n \in I(z)$ and

$$\langle d_i, d_{i+1} \rangle \in \Theta(x, z) \quad \text{for } 0 \leqslant i < n$$
$$\langle d_i, d_{i+1} \rangle \in \Theta(x, y) \quad \text{for } i < n \text{ even}$$
$$\langle d_i, d_{i+1} \rangle \in \Theta(y, z) \quad \text{for } i < n \text{ odd.}$$

The fact $\boldsymbol{d}_n(x, y, z) \in I(z)$ implies $\boldsymbol{d}_n(x, y, 0) = 0$. Other indentities of (2) can be proved by a standard procedure.

(2)$\Rightarrow$(1): Let $\Psi, \Theta_1, \Theta_2 \in Con\, A$ for some $A \in \boldsymbol{V}$. Suppose I is an ideal of A and $a \in [I]\Psi \wedge (\Theta_1 \vee \Theta_2)$. Then

$$\langle a, i \rangle \in \Psi \wedge (\Theta_1 \vee \Theta_2)$$

for some $i \in I$. From $\langle a, i \rangle \in \Theta_1 \vee \Theta_2$ we obtain elements $c_0, \ldots, c_k \in A$ such that

$$a = c_0 \Theta_1 c_1 \ldots c_{k-1} \Theta_2 c_k = i.$$

Consequently

$$\boldsymbol{d}_j(a, a, i) \Theta_1 \boldsymbol{d}_j(a, c_1, i) \ldots \boldsymbol{d}_j(a, c_{k-1}, i) \Theta_2 \boldsymbol{d}_j(a, i, i), \quad 0 \leqslant j \leqslant n.$$

Combine this with $\langle a, i \rangle \in \Psi$ and with identities $\boldsymbol{d}_j(x, y, x) = x$ from (2), we obtain

$$\boldsymbol{d}_j(a, a, i)(\Psi \wedge \Theta_1) \boldsymbol{d}_j(a, c_1, i) \ldots \boldsymbol{d}_j(a, c_{k-1}, i)(\Psi \wedge \Theta_2) \boldsymbol{d}_j(a, i, i)$$

for $0 \leqslant j \leqslant n$. Applying other identities from (2), we have

$$a(\Psi \wedge \Theta_1) \vee (\Psi \wedge \Theta_2) \boldsymbol{d}_n(a, i, i)$$

by transitivity. Since d_n is an ideal term in the third variable, $\boldsymbol{d}_n(a, i, i) \in I$ holds.

Thus

$$a \in [I](\Psi \wedge \Theta_1) \vee (\Psi \wedge \Theta_2). \quad \square$$

Remark 1. Contrary to Theorem 1, ideal distributive varieties do not coincide with varieties distributive at 0. The mentioned classes are separated by the following

Example. It is already known (see [2]) that $\wedge$*-semilattices with* 0 are distributive at 0. Now suppose the identities (2) of Theorem 2 are satisfied in a variety of $\wedge$-semilattices with 0. Then we state that $\boldsymbol{d}_i(x, y, z) = x$ for $0 \leqslant i \leqslant n$.

(a) For $i = 0$ the assertion holds trivial.

(b) Let $\boldsymbol{d}_k(x, y, z) = x$ for some k, $0 \leqslant k < n$.

Then $\boldsymbol{d}_{k+1}(x, x, z) = d_k(x, x, z) = x$ if k is even, or

$\boldsymbol{d}_{k+1}(x, z, z) = d_k(x, z, z) = x$ if k is odd.

In both cases $\boldsymbol{d}_{k+1}$ does not depend on the third variable. Further, the identity $\boldsymbol{d}_{k+1}(x, y, x) = x$ implies that $\boldsymbol{d}_{k+1}$ does not depend on the second variable, too. Hence $\boldsymbol{d}_{k+1}(x, y, z) = x$ (since there are no other unary terms in semilattices). Consequently, $\boldsymbol{d}_n(x, y, z) = x$. Combining this fact with $d_n(x, y, 0) = 0$, we find $x = 0$, which is a contradiction. $\square$

Corollary 1. *Let* $\boldsymbol{V}$ *be a variety permutable at* 0. *The following conditions are equivalent*:

(1) $\boldsymbol{V}$ *is ideal distributive*;

(2) $\boldsymbol{V}$ *is distributive at* 0.

Proof. Only (2) $\Rightarrow$ (1) is needed: let I be an ideal of $A \in \boldsymbol{V}$ and Ψ, Θ_1, $\Theta_2 \in$ $\in Con\,A$. The permutability at 0 yields a congruence Φ on A such that $I = [0]\Phi$, see [5]. Then

$$\begin{aligned}[I]\Psi \wedge (\Theta_1 \vee \Theta_2) &= [[0]\Phi]\Psi \wedge (\Theta_1 \vee \Theta_2) = [[0]\Psi \wedge (\Theta_1 \vee \Theta_2)]\Phi = \\ &= [[0](\Psi \wedge \Theta_1) \vee (\Psi \wedge \Theta_2)]\Phi = [[0]\Phi](\Psi \wedge \Theta_1) \vee (\Psi \wedge \Theta_2) = \\ &= [I](\Psi \wedge \Theta_1) \vee (\Psi \wedge \Theta_2),\end{aligned}$$

by Lemma 2. $\square$

One can easily verify that the ideal equality $[I]\Theta_1 \wedge \Theta_2 = [I]\Theta_1 \wedge [I]\Theta_2$ does not hold in every variety of algebras, see, e.g., the variety of $\wedge$-semilattices with 0.

Definition 3. Let $\boldsymbol{K}$ be a class of algebras of the same type with constant 0 and $\boldsymbol{u}$ be an n-ary ($n \geqslant 3$) term in $\boldsymbol{K}$. $\boldsymbol{u}$ is called an *ideal near unanimity term* whenever it satisfies the following identities:

$$\begin{gathered}\boldsymbol{u}(x, 0, \ldots, 0) = 0 \\ \boldsymbol{u}(x, y, x, \ldots, x) = x \\ \boldsymbol{u}(x, x, y, x, \ldots, x) = x \\ \vdots \\ \boldsymbol{u}(x, x, \ldots, x, y) = x.\end{gathered}$$

Theorem 3. *Let* V *be a variety with constant* 0. *The following conditions are equivalent*:

(1) *for each ideal* I *of every* $A \in V$ *and each* $\Theta_1, \ldots, \Theta_n \in Con\, A$,

$$[I] \bigwedge_{i \leqslant n} \Theta_i = \bigwedge_{i \leqslant n} [I] \Theta_i;$$

(2) *there is an* $(n+1)$*-ary ideal near unanimity term* u *in* V.

Proof. The equivalence (1)⇔(2) will be shown for $n = 2$ only. For $n > 2$ the argumentation can be modified in an evident way.

(1)⇒(2): Take $A = F_V(x, y, z) \in V$, $I = I(x, y)$ and $\Theta_1 = \Theta(x, z)$, $\Theta_2 = \Theta(y, z)$. Then $z \in [I]\Theta_1 \wedge [I]\Theta_2$ and so

$$z \in [I]\Theta_1 \wedge [I]\Theta_2 = [I(x, y)]\Theta(x, z) \wedge \Theta(x, z).$$

By Lemma 1.2 in [5], we have a 5-ary term p such that $p(x, y, z, 0, 0) = 0$ and

$$\langle z, p(x, y, z, x, y)\rangle \in \Theta(x, z) \wedge \Theta(y, z).$$

This statement implies

$$p(x, y, x, x, y) = x, \quad p(x, y, y, x, y) = y.$$

Introduce the ternary term u by

$$u(a, b, c) = p(b, c, a, b, c).$$

Then clearly

$$u(x, 0, 0) = p(0, 0, x, 0, 0) = 0$$
$$u(x, y, x) = p(y, x, x, y, x) = x$$
$$u(x, x, y) = p(x, y, x, x, y) = x$$

as required.

(2)⇒(1): Let I be an ideal of $A \in V$ and $\Theta_1, \Theta_2 \in Con\, A$. We have to prove the inclusion

$$[I]\Theta_1 \wedge [I]\Theta_2 \subseteq [I]\Theta_1 \wedge \Theta_2.$$

Suppose $a \in [I]\Theta_1 \wedge [I]\Theta_2$. Then $\langle i_1, a\rangle \in \Theta_1$ and $\langle i_2, a\rangle \in \Theta_2$ for some $i_1, i_2 \in I$. Consider the element $u(a, i_1, i_2)$. Using (2), we find that $u(a, i_1, i_2) \in I$ and

$$a = u(a, a, i_2)\Theta_1\, u(a, i_1, i_2)$$
$$a = u(a, i_1, a)\Theta_2\, u(a, i_1, i_2).$$

Hence $\langle a, u(a, i_1, i_2)\rangle \in \Theta_1 \wedge \Theta_2$, i.e. $a \in [I]\Theta_1 \in \Theta_2$. □

Theorem 4. *Let* V *be a variety with constant* 0 *and* u *be an n-ary near unanimity term in* V. *Then* V *is ideal distributive with* $2n - 1$ *characterizing ternary terms* $d_0, \ldots, d_{2n-2}$.

Proof. Put $d_0(x, y, z) = x$

$$d_1(x, y, z) = u(x, y, z, x, \ldots, x)$$

and for $0 < k \leqslant n - 1$ put

$$\boldsymbol{d}_{2k}(x, y, z) = \boldsymbol{u}(\boldsymbol{u}(x, y, z, x, \ldots, x), \ldots, \boldsymbol{u}(x, y, z, x, \ldots, x), \underbrace{z, \ldots, z}_{k \text{ times}})$$

$$\boldsymbol{d}_{2k+1}(x, y, z) =$$
$$= \boldsymbol{u}(\boldsymbol{u}(x, y, z, x, \ldots, x), \ldots, \boldsymbol{u}(x, y, z, x, \ldots, x), \boldsymbol{u}(y, \ldots, y, z), \underbrace{z, \ldots, z}_{k \text{ times}})$$

Then

(a) $\boldsymbol{d}_0(x, y, z) = x$ and
$\boldsymbol{d}_{2n-2}(x, y, 0) = \boldsymbol{u}(\boldsymbol{u}(x, y, 0, x, \ldots, x), 0, \ldots, 0) = 0.$

(b) $\boldsymbol{d}_0(x, y, x) = x$
$\boldsymbol{d}_1(x, y, x) = \boldsymbol{u}(x, y, x, \ldots, x) = x$

$$\boldsymbol{d}_{2k}(x, y, x) = \boldsymbol{u}(\boldsymbol{u}(x, y, x, \ldots, x), \ldots, \boldsymbol{u}(x, y, x, \ldots, x), x, \ldots, x) = \\ = \boldsymbol{u}(x, \ldots, x) = x$$

$$\boldsymbol{d}_{2k+1}(x, y, x) = \\ = \boldsymbol{u}(\boldsymbol{u}(x, y, x, \ldots, x), \ldots, \boldsymbol{u}(x, y, x, \ldots, x), \boldsymbol{u}(y, \ldots, y, x), x, \ldots, x) = \\ = \boldsymbol{u}(x, \ldots, x, \boldsymbol{u}(y, \ldots, y, x), x, \ldots, x) = x,$$

thus $\boldsymbol{d}_i(x, y, x) = x$ for all $i = 0, \ldots, 2n - 2$.

(c) Let i be even. If $i = 0$, then

$$\boldsymbol{d}_0(x, x, z) = x = \boldsymbol{u}(x, x, z, x, \ldots, x) = \boldsymbol{d}_1(x, x, z).$$

If $i > 0$, then $i = 2k$ for $k > 0$ and

$$\boldsymbol{d}_{2k}(x, x, z) = \boldsymbol{u}(\boldsymbol{u}(x, x, z, x, \ldots, x), \ldots, \boldsymbol{u}(x, x, z, x, \ldots, x), \underbrace{z, \ldots, z}_{k \text{ times}}) = \\ = \boldsymbol{u}(x, \ldots, x, \underbrace{z, \ldots, z}_{k \text{ times}});$$

$$\boldsymbol{d}_{2k+1}(x, x, z) = \\ = \boldsymbol{u}(\boldsymbol{u}(x, x, z, x, \ldots, x), \ldots, \boldsymbol{u}(x, x, z, x, \ldots, x), \boldsymbol{u}(x, \ldots, x, z), z, \ldots, z) = \\ = \boldsymbol{u}(x, \ldots, x, \underbrace{z, \ldots, z}_{k \text{ times}}),$$

thus $\boldsymbol{d}_i(x, x, z) = \boldsymbol{d}_{i+1}(x, x, z)$ for i even.

(d) Let i be odd. If $i = 1$, then

$$\boldsymbol{d}_1(x, z, z) = \boldsymbol{u}(x, z, z, x, \ldots, x)$$
$$\boldsymbol{d}_2(x, z, z) = \boldsymbol{u}(\boldsymbol{u}(x, z, z, x, \ldots, x), \ldots, \boldsymbol{u}(x, z, z, x, \ldots, x), z) = \\ = \boldsymbol{u}(x, z, z, x, \ldots, x),$$

i.e. $\boldsymbol{d}_1(x, z, z) = \boldsymbol{d}_2(x, z, z)$.
If $i > 1$, then $i = 2k + 1$ for $k > 0$ and

$$\begin{aligned}
&\boldsymbol{d}_{2k+1}(x, z, z) = \\
&= \boldsymbol{u}(\boldsymbol{u}(x, z, z, x, \ldots, x), \ldots, \boldsymbol{u}(x, z, z, x, \ldots, x), \boldsymbol{u}(z, \ldots, z), \underbrace{z, \ldots, z}_{k \text{ times}}) = \\
&= \boldsymbol{u}(\boldsymbol{u}(x, z, z, x, \ldots, x), \ldots, \boldsymbol{u}(x, z, z, x, \ldots, x), \underbrace{z, \ldots, z}_{k+1 \text{ times}}) = \\
&= \boldsymbol{d}_{2k+2}(x, z, z).
\end{aligned}$$

Hence

$$\boldsymbol{d}_i(x, z, z) = \boldsymbol{d}_{i+1}(x, z, z) \quad \text{for all } i \text{ odd.}$$

The proof is complete. □

It is evident that the existence of some n-ary ideal near unanimity term follows from the existence of a ternary ideal near unanimity term. Moreover, for varieties with a ternary ideal near unanimity term there holds the following

Theorem 5. *Let* $\boldsymbol{V}$ *be a variety with constant* 0. *The following conditions are equivalent*:

(1) $\boldsymbol{V}$ *is ideal distributive with three characterizing ternary terms* $\boldsymbol{d}_0$, $\boldsymbol{d}_1$, $\boldsymbol{d}_2$.

(2) $\boldsymbol{V}$ *has a ternary ideal near unanimity term.*

Proof. (1)⇒(2): By hypothesis, we have ternary terms $\boldsymbol{d}_0$, $\boldsymbol{d}_1$, $\boldsymbol{d}_2$ such that

$$\begin{aligned}
&\boldsymbol{d}_0(x, y, z) = x, \quad \boldsymbol{d}_2(x, y, 0) = 0, \\
&\boldsymbol{d}_1(x, y, x) = \boldsymbol{d}_2(x, y, x) = x \\
&\boldsymbol{d}_1(x, x, z) = x \\
&\boldsymbol{d}_1(x, z, z) = \boldsymbol{d}_2(x, z, z).
\end{aligned}$$

Apparently, $\boldsymbol{d}_1(x, y, z)$ is a ternary ideal near unanimity term in $\boldsymbol{V}$.

(2)⇒(1): Let $\boldsymbol{u}$ be a ternary ideal near unanimity term in $\boldsymbol{V}$. Put

$$\begin{aligned}
&\boldsymbol{d}_0(x, y, z) = x, \\
&\boldsymbol{d}_1(x, y, z) = \boldsymbol{u}(x, y, z), \\
&\boldsymbol{d}_2(x, y, z) = \boldsymbol{u}(\boldsymbol{u}(x, z, z), z, \boldsymbol{u}(x, y, y)).
\end{aligned}$$

Then

(a) $\boldsymbol{d}_0(x, y, z) = x$
$\boldsymbol{d}_2(x, y, 0) = \boldsymbol{u}(\boldsymbol{u}(x, 0, 0), 0, \boldsymbol{u}(x, y, y)) = \boldsymbol{u}(0, 0, \boldsymbol{u}(x, y, y)) = 0.$

(b) $\boldsymbol{d}_1(x, y, x) = \boldsymbol{u}(x, y, x) = x$
$\boldsymbol{d}_2(x, y, x) = \boldsymbol{u}(\boldsymbol{u}(x, x, x), x, \boldsymbol{u}(x, y, y)) = \boldsymbol{u}(x, x, \boldsymbol{u}(x, y, y)) = x.$

(c) for i even, we have only
$\boldsymbol{d}_0(x, x, z) = x = \boldsymbol{u}(x, x, z) = \boldsymbol{d}_1(x, x, z).$

(d) for i odd, we have only
$\boldsymbol{d}_1(x, z, z) = \boldsymbol{u}(x, z, z) = \boldsymbol{u}(\boldsymbol{u}(x, z, z), z, \boldsymbol{u}(x, z, z)) = \boldsymbol{d}_2(x, z, z).$ □

Assuming the permutability at 0, the foregoing theorem yields immediately:

Corollary 2. *Let* V *be a variety permutable at* 0. *The following conditions are equivalent*:

(1) V *has an ideal near unanimity term*;

(2) V *is distributive at* 0.

Proof. (1)$\Rightarrow$(2) follows directly from Theorem 5. Prove (2)$\Rightarrow$(1): Let I be an ideal of $A \in V$ and Θ_1, $\Theta_2 \in Con\, A$. By the permutability at 0, $I = [0]\Phi$ for some congruence $\Phi \in Con\, A$. Then

$$[I]\Theta_1 \wedge \Theta_2 = [[0]\Phi]\Theta_1 \wedge \Theta_2 = [0]\Phi \vee (\Theta_1 \wedge \Theta_2).$$

Using Lemma 2 of [3], we have

$$[0]\Phi \vee (\Theta_1 \wedge \Theta_2) = [0](\Phi \vee \Theta_1) \wedge (\Phi \vee \Theta_2),$$

so

$$[I]\Theta_1 \wedge \Theta_2 = [0](\Phi \vee \Theta_1) \wedge [0](\Phi \vee \Theta_2) = $$
$$= [[0]\Phi]\Theta_1 \wedge [[0]\Phi]\Theta_2 = [I]\Theta_1 \wedge [I]\Theta_2 .$$

Theorem 3 completes the proof. □

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МНОГООБРАЗИЯ, ВЫПОЛНЯЮЩИЕ ИДЕАЛНЫЕ ЭКВИВАЛЕНТНОСТИ

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Резюме

Гумм и Урсини ввели концепт идеала универзальной алгебры. Мы даем концепт идеално перестановочных и идеално дистрибутивных конгруэнций и характеризируем многообразия таких алгебер условиями Мальцева.