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EXISTENCE CRITERIONS FOR GENERALIZED SOLUTIONS OF FUNCTIONAL BOUNDARY VALUE PROBLEMS WITHOUT GROWTH RESTRICTIONS

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ABSTRACT. Existence results are given for the functional differential equation $(g(x'))' = f(t, x, x', x_t, x'_t)$ with nonlinear boundary conditions. Sufficient conditions are formulated only in terms of sign conditions. Solutions are considered in the generalized sense.

1. Introduction

For $r > 0$, let C_r be the Banach space of continuous functions on $[-r, 0]$ with the norm $\|\cdot\|_0$ and let $\mathbf{X}$ be the Banach space of continuous functions on $J = [0, 1]$ endowed with the norm $\|\cdot\|$. For each $x \in C^0([-r, 1])$ and $t \in J$, $x_t \in C_r$ is defined by

$$x_t(s) = x(t+s), \qquad s \in [-r, 0]. \tag{1}$$

We say that $F \colon J \times \mathbb{R}^2 \times C_r^2 \to \mathbb{R}$ satisfies the local Carathéodory conditions on $J \times \mathbb{R}^2 \times C_r^2$ ($F \in \operatorname{Car}(J \times \mathbb{R}^2 \times C_r^2)$ for short) if

(i) $F(\cdot, x, y, u, v)$ is measurable on J for each $(x, y, u, v) \in \mathbb{R}^2 \times C_r^2$,
(ii) $F(t, \cdot, \cdot, \cdot, \cdot)$ is continuous on $\mathbb{R}^2 \times C_r^2$ for a.e. $t \in J$, and
(iii) for arbitrary $a > 0$, there exists an $h_a \in L_1(J)$ such that

$$|x| + |y| + \|u\|_0 + \|v\|_0 \le a \implies |F(t, x, y, u, v)| \le h_a(t) \quad \text{for a.e. } t \in J.$$

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Consider the functional differential equation

$$x'' = F(t, x, x', x_t, x'_t) \tag{2}$$

with $F \in \mathrm{Car}(J \times \mathbb{R}^2 \times C_r^2)$. If $\varphi, \varphi' \in C_r$, then the initial values $x_0 = \varphi$, $x'_0 = \varphi'$ (i.e., $x(t) = \varphi(t)$, $x'(t) = \varphi'(t)$ for $t \in [-r, 0]$) determine — under proper assumptions on F — the unique "classical" solution $x \in C^1([-r, 1]) \cap AC^1(J)$ of the initial problem

$$(2), \qquad x_0 = \varphi, \qquad x'_0 = \varphi', \tag{3}$$

and, consequently, there is no sensible way to give boundary conditions for solutions of (3). Here $AC^1(J)$ denotes the set of functions having absolutely continuous derivative on J. Boundary conditions of the type

$$l_1(x_0) + l_2(x_1) = \varphi, \qquad l_3(x'_0) + l_4(x'_1) = \psi \tag{4}$$

for (2) were considered, for example, in [2], [5] and [8] with the linear bounded operators $l_i \colon C_r \to C_r$ $(i = 1, 2, 3, 4)$ and $\varphi, \psi \in C_r$. The special case of (4) are periodic boundary conditions $x_0 = x_1$, $x'_0 = x'_1$. For F independent of x'_t, the boundary value problem for (2) with boundary conditions (for $\varphi \in C_r$, $A \in \mathbb{R}$)

$$x_0 = \varphi, \qquad x(1) = A$$

was considered, for example, in [2] and [9]. Another approach to "classical" solutions of BVPs for (2) was given by Haščák [6] who considered the nth order linear differential equations with delays. Let $\varphi, \psi \in C_r$ and $\mathcal{D}(\varphi, \psi)$ be the set of all maximal solutions $x \in C^0([-r, 0] \cup J_x) \cap AC^1(J_x)$ of (2) satisfying $x_0 = \varphi + c_1$, $x'_0 = \psi + c_2$ and $\lim_{t \to 0} x'(t) = \psi(0) + c_2$, where $c_1, c_2 \in \mathbb{R}$ and J_x is an interval, $0 \in J_x \subset J$. Then the set $\mathcal{D}(\varphi, \psi)$ depends on two parameters c_1 and c_2. To obtain an $x \in \mathcal{D}(\varphi, \psi)$ we can give two boundary conditions (generally nonlinear) as is shown in [12]. Here the existence results are proved by Leray-Schauder degree theory and Borsuk's theorem.

There is another approach to BVPs for functional differential equations which is connected with the conception of "generalized" solutions (see, e.g., [1]). The principle difference between "classical" solutions and "generalized" ones of BVPs for functional differential equations consists in the continuity of "classical" solution at the point $t = 0$ while this condition is not (generally) claimed for "generalized" ones. Moreover, for our second order equation (2) the initial values for solutions and their first derivatives (that is x_0 and x'_0) can be arbitrary points of the Banach space C_r. This paper considers functional BVPs from the point of view of "generalized" solutions.

2. Formulation of BVP, notation

Let $r > 0$ be a positive number. We say that $x\colon [-r,0] \to \mathbb{R}$ is a D-function if either x is continuous on $[-r,0]$ or there exists exactly one point of discontinuity $t_x \in (-r,0]$ for x such that $\lim\limits_{t\to t_x-} x(t)$ exists and is finite and $\lim\limits_{t\to t_x+} x(t) = x(t_x)$. For $t_x = 0$ define $\lim\limits_{t\to 0+} x(t) = x(0)$. Denote by D_r the topological space of D-functions (on $[-r,0]$) with the topology of pointwise convergence on $[-r,0]$. We say that $f \in \mathrm{Car}(J \times \mathbb{R}^2 \times D_r^2)$ (i.e., f satisfies the local Carathéodory conditions on $J \times \mathbb{R}^2 \times D_r^2$) if (cf. (i)–(iii))

(i') $f(\cdot, x, y, u, v)$ is measurable on J for each $(x,y,u,v) \in \mathbb{R}^2 \times D_r^2$,

(ii') $f(t, \cdot, \cdot, \cdot, \cdot)$ is continuous on $\mathbb{R}^2 \times D_r^2$ for a.e. $t \in J$, and

(iii') for arbitrary $a > 0$, there exists an $h_a \in L_1(J)$ such that

$$|x| + |y| + \sup\bigl\{|u(s)|;\ s \in [-r,0]\bigr\} + \sup\bigl\{|v(s)|;\ s \in [-r,0]\bigr\} \le a$$
$$\implies |f(t,x,y,u,v)| \le h_a(t) \quad \text{for a.e. } t \in J\,.$$

Before we formulate our BVP, we define the sets $\mathcal{A}$ and $\mathcal{B}$ which are connected with boundary conditions. Let $\mathcal{A}$ be the set of all functionals $\gamma\colon \mathbf{X} \to \mathbb{R}$ that are

(a) continuous, $\gamma(0) = 0$,

(b) increasing (i.e., $x, y \in \mathbf{X}$, $x(t) < y(t)$ for $t \in J \implies \gamma(x) < \gamma(y)$)

and $\mathcal{B}$ be the set of all continuous functionals $\Phi\colon \mathbf{X} \times \mathbf{X} \to \mathbb{R}$.

Some examples of functionals belonging to $\mathcal{A}$ are given below:

$$\max\bigl\{x(t);\ t \in J_1\bigr\}, \qquad \min\bigl\{x(t);\ t \in J_1\bigr\}, \qquad q\bigl(x(t_0)\bigr), \qquad \int_{t_1}^{t_2} q\bigl(x(s)\bigr)\,\mathrm{d}s\,,$$

where $J_1 \subset J$ is a compact interval, $t_0 \in J$, $0 \le t_1 < t_2 \le 1$ and $q\colon \mathbb{R} \to \mathbb{R}$ is continuous increasing, $q(0) = 0$, while the following functionals (for $0 \le t_1 < t_2 \le 1$, $a, b \in J$, J_1, J_2 compact subintervals of J, $q, p \in C^0(\mathbb{R})$)

$$\int_{t_1}^{t_2} \sqrt{1 + x^2(t)}\,\mathrm{d}t\,, \qquad \int_{t_1}^{t_2} x(t)\sqrt{1 + y^2(t)}\,\mathrm{d}t\,,$$
$$\max\bigl\{q\bigl(x(t)\bigr);\ t \in J_1\bigr\} + \min\bigl\{p\bigl(y(t)\bigr);\ t \in J_2\bigr\}$$

belong to the set $\mathcal{B}$.

Let $\varphi, \psi \in C_r$, $\alpha, \beta \in \mathcal{A}$, $\Lambda \in \mathcal{B}$ and $f \in \mathrm{Car}(J \times \mathbb{R}^2 \times D_r^2)$. Consider BVP

$$\bigl(g(x')\bigr)' = f(t, x, x', x_t, x_t')\,, \tag{5}$$

$$\begin{aligned} \alpha(x) = 0\,, \qquad & \beta(x') = \Lambda(x, x')\,, \\ x_0(s) = \varphi(s)\,, \quad x_0'(s) = \psi(s) \qquad & \text{for} \quad s \in [-r, 0)\,, \end{aligned} \tag{6}$$

where $g\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with inverse g^{-1} and such that $g(0) = 0$.

By a *solution of BVP* (5), (6) we mean a function $x \in C^1(J)$ such that $g(x') \in AC(J)$, $\alpha(x) = 0$, $\beta(x') = \Lambda(x, x')$ and (5) is satisfied for a.e. $t \in J$ where

$$x_t(s) = \begin{cases} \varphi(t+s) & \text{for } t+s \in [-r, 0)\,, \\ x(t+s) & \text{for } t+s \in J\,, \end{cases}$$

$$x'_t(s) = \begin{cases} \psi(t+s) & \text{for } t+s \in [-r, 0)\,, \\ x'(t+s) & \text{for } t+s \in J\,. \end{cases}$$

Here $AC(J)$ denotes the set of absolutely continuous functions on J.

We present sufficient conditions for the existence of BVP (5), (6). The conditions are formulated only in terms of sign conditions. The proofs of existence results are based on the topological degree method and Borsuk's theorem (see e.g. [4]).

The special case of BVP (5), (6) (with $\Lambda = 0$) is BVP

$$\big(g(x')\big)' = h(t, x, x')\,, \tag{7}$$

$$\alpha(x) = 0\,, \qquad \beta(x') = 0\,, \tag{8}$$

where $\alpha, \beta \in \mathcal{A}$ and $h \in \operatorname{Car}(J \times \mathbb{R}^2)$. Setting $\alpha(x) = x(0)$, $\beta(x) = \int_0^1 x(s)\,\mathrm{d}s$ for $x \in \mathbf{X}$, the boundary conditions (8) have the form of the Dirichlet conditions

$$x(0) = 0\,, \qquad x(1) = 0\,. \tag{9}$$

Our existence results for BVP (5), (6) generalize those for BVP (7), (8) with $g(z) \equiv z$ and $h \in C^0(J \times \mathbb{R}^2)$ in [7] and are closely related to results in [10] and [11]. We observe that BVP (7), (9) with h independent of x' was considered in [3] from the point of view of existence results for multiple solutions.

Next we use the following notation.

For each $K > 0$,

$$[K]_{\mathcal{D}} = \Big\{(x, u, v)\,;\ (x, u, v) \in \mathbb{R} \times D_r^2\,,\ |x| \le K\,,$$
$$\sup\big\{|u(t)|\,;\ t \in [-r, 0]\big\} \le K\,,\ \sup\big\{|v(t)|\,;\ t \in [-r, 0]\big\} \le K\Big\}\,,$$

and for each $L_3 < L_1 < 0 < L_2 < L_4$,

$$[L_3, L_1, L_2, L_4]_{\mathcal{AB}} =$$
$$= \Big\{(\gamma, \Phi)\,;\ (\gamma, \Phi) \in \mathcal{A} \times \mathcal{B}\,,$$
$$\sup\big\{|\Phi(x, y)|\,;\ (x, y) \in \mathbf{X}^2,\ \|x\| \le L\,,\ \|y\| \le L\big\} \le \tfrac{1}{2} \min\big\{-\gamma(L_1), \gamma(L_2)\big\}\Big\}\,,$$

where $L = \max\{-L_3, L_4\}$.

Throughout this paper, we shall assume that $f \in \mathrm{Car}(J \times \mathbb{R}^2 \times D_r^2)$ satisfies the assumption:

(H) There exist constants $L_i \in \mathbb{R}$ $(i = 1, \dots, 4)$ and $\mu, \nu \in \{-1, 1\}$ such that $L_3 < L_1 < 0 < L_2 < L_4$ and

$$\nu f(t, x, L_3, u, v) \geq 0, \qquad \nu f(t, x, L_1, u, v) \leq 0,$$
$$\mu f(t, x, L_2, u, v) \geq 0, \qquad \mu f(t, x, L_4, u, v) \leq 0$$

for a.e. $t \in J$ and each $(x, u, v) \in [L]_{\mathcal{D}}$ with $L = \max\{-L_3, L_4\}$.

3. Auxiliary results

Lemma 1. *Let $f \in \mathrm{Car}(J \times \mathbb{R}^2 \times D_r^2)$, $\varphi, \psi \in C_r$ and $u, v \in \mathbf{X}$. Then $f(t, u(t), v(t), u_t, v_t) \in L_1(J)$, where*

$$u_t(s) = \begin{cases} \varphi(t+s) & \text{for } t+s \in [-r, 0), \\ u(t+s) & \text{for } t+s \in J, \end{cases}$$

$$v_t(s) = \begin{cases} \psi(t+s) & \text{for } t+s \in [-r, 0), \\ v(t+s) & \text{for } t+s \in J. \end{cases}$$

Proof. Evidently, $u_t, v_t \in D_r$ for $t \in J$. Set $a = 2(\|u\| + \|v\|) + \|\varphi\|_0 + \|\psi\|_0$. By (iii'), there exists an $h_a \in L_1(J)$ such that

$$|f(t, u(t), v(t), u_t, v_t)| \leq h_a(t) \qquad \text{for a.e.} \quad t \in J;$$

hence to prove our lemma it is sufficient to show that $f(t, u(t), v(t), u_t, v_t)$ is measurable on J. For $n \in \mathbb{N}$, $\frac{1}{n} < r$, define $u_n, v_n \in C^0([-r, 1])$ by

$$u_n(t) = \begin{cases} \varphi(t) & \text{for } t \in [-r, -\frac{1}{n}), \\ -n(\varphi(-\frac{1}{n}) - u(0))t + u(0) & \text{for } t \in [-\frac{1}{n}, 0), \\ u(t) & \text{for } t \in J, \end{cases}$$

$$v_n(t) = \begin{cases} \psi(t) & \text{for } t \in [-r, -\frac{1}{n}), \\ -n(\psi(-\frac{1}{n}) - v(0))t + v(0) & \text{for } t \in [-\frac{1}{n}, 0), \\ v(t) & \text{for } t \in J. \end{cases}$$

We first prove that $f(t, u(t), v(t), u_{nt}, v_{nt})$ is measurable. Fix $n \in \mathbb{N}$, $\frac{1}{n} < r$. Let us set $\xi_{ik} = \frac{i}{k}$ for $i = 0, 1, \dots, k$, $k = 2, 3, \dots$ and $p_k(t) = u(\xi_{1k})$, $q_k(t) = v(\xi_{1k})$, $z_{kt} = u_{n\xi_{1k}}$, $w_{kt} = v_{n\xi_{1k}}$ for $t \in [\xi_{0k}, \xi_{2k}]$, $p_k(t) = u(\xi_{ik})$, $q_k(t) =$

$v(\xi_{ik})$, $z_{kt} = u_{n\xi_{ik}}$, $w_{kt} = v_{n\xi_{ik}}$ for $t \in (\xi_{ik}, \xi_{i+1,k}]$, $i = 2,3,\dots,k-1$. Then the functions $f\big(t, p_k(t), q_k(t), z_{kt}, w_{kt}\big)$ are measurable for each $k \in \mathbb{N}$, $k \geq 2$, $\lim_{k\to\infty} p_k(t) = u(t)$, $\lim_{k\to\infty} q_k(t) = v(t)$ for $t \in J$ and $\lim_{k\to\infty} z_{kt} = u_{nt}$, $\lim_{k\to\infty} w_{kt} = v_{nt}$ in D_r for each $t \in J$. Therefore (cf. (ii')) $\lim_{k\to\infty} f\big(t, p_k(t), q_k(t), z_{kt}, w_{kt}\big) = f\big(t, u(t), v(t), u_{nt}, v_{nt}\big)$ for a.e. $t \in J$ which implies that $f\big(t, u(t), v(t), u_{nt}, v_{nt}\big)$ is measurable for all $n \in \mathbb{N}$, $\frac{1}{n} < r$. Since $\lim_{n\to\infty} u_{nt} = u_t$, $\lim_{n\to\infty} v_{nt} = v_t$ in D_r for each $t \in J$, $\lim_{k\to\infty} f\big(t, u(t), v(t), u_{nt}, v_{nt}\big) = f\big(t, u(t), v(t), u_t, v_t\big)$ for a.e. $t \in J$ by (ii'), and consequently $f\big(t, u(t), v(t), u_t, v_t\big)$ is measurable. □

LEMMA 2. *Let $\gamma \in \mathcal{A}$, $\mu \in [0,\infty)$ and let the equality $\gamma(x) - \mu\gamma(-x) = 0$ be satisfied for some $x \in \mathbf{X}$. Then there exists a $\xi \in J$ such that $x(\xi) = 0$.*

P r o o f. Define $\varrho \in \mathcal{A}$ by $\varrho(w) = \gamma(w) - \mu\gamma(-w)$ for $w \in \mathbf{X}$. Then $\varrho(x) = 0$. If $x(t) \neq 0$ on J, then $\varrho(x) > 0$ provided $x(t) > 0$ on J and $\varrho(x) < 0$ provided $x(t) < 0$ on J, and so $\varrho(x) \neq \varrho(0) = 0$, which is a contradiction. □

LEMMA 3. *Let $(\gamma, \Phi) \in [L_3, L_1, L_2, L_4]_{AB}$, $\mu \in [0,1]$ and let the equality*

$$\gamma(y) - \mu\gamma(-y) = \Phi(x,y) - \mu\Phi(-x,-y)$$

be satisfied for some $x, y \in \mathbf{X}$, $\|x\| \leq L$, $\|y\| \leq L$, where $L = \max\{-L_3, L_4\}$. Then there exists a $\tau \in J$ such that

$$L_1 \leq y(\tau) \leq L_2 .$$

P r o o f. By the definition of the set $[L_3, L_1, L_2, L_4]_{AB}$,

$$|\Phi(x,y)| \leq \frac{1}{2}\min\{-\gamma(L_1), \gamma(L_2)\}, \qquad |\Phi(-x,-y)| \leq \frac{1}{2}\min\{-\gamma(L_1), \gamma(L_2)\},$$

and consequently

$$|\gamma(y) - \mu\gamma(-y)| = |\Phi(x,y) - \mu\Phi(-x,-y)| \leq \min\{-\gamma(L_1), \gamma(L_2)\}. \tag{10}$$

If $y(t) < L_1$ on J, then $\gamma(y) - \mu\gamma(-y) < \gamma(L_1) - \mu\gamma(-L_1) \leq \gamma(L_1) < 0$; hence $|\gamma(y) - \mu\gamma(-y)| > -\gamma(L_1)$ which contradicts (10). If $y(t) > L_2$ on J, then $\gamma(y) - \mu\gamma(-y) > \gamma(L_2) - \mu\gamma(-L_2) \geq \gamma(L_2) > 0$, a contradiction. Therefore $L_1 \leq y(\tau) \leq L_2$ for a $\tau \in J$. □

Let $L_3 < L_1 < 0 < L_2 < L_4$ be constants (see assumption (H)), $L = \max\{-L_3, L_4\}$ and let $n_0 \in \mathbb{N}$ be a positive integer such that

$$L_1 - L_3 > \frac{2}{n_0}, \qquad L_4 - L_2 > \frac{2}{n_0}.$$

For each $n \in \mathbb{N}$, $n \geq n_0$, define f_n by f as follows:

$$f_n(t,x,y,u,v) = f\big(t, \bar{x}, h_n(y), \tilde{u}, \tilde{v}\big) \qquad \text{for} \quad (t,x,y,u,v) \in J \times \mathbb{R}^2 \times D_r^2, \tag{11}$$

where

$$\bar{x} = \begin{cases} x & \text{for } |x| \le L, \\ L\operatorname{sign}(x) & \text{for } |x| > L, \end{cases}$$

$$\tilde{u}(s) = \begin{cases} u(s) & \text{for } |u(s)| \le L, \\ L\operatorname{sign}\big(u(s)\big) & \text{for } |u(s)| > L, \end{cases} \tag{12}$$

(similarly for $\tilde{v}$) and

$$h_n(y) = \begin{cases} L_4 & \text{for } L_4 < y, \\ y & \text{for } L_2 + \frac{2}{n} < y \le L_4, \\ 2y - L_2 - \frac{2}{n} & \text{for } L_2 + \frac{1}{n} < y \le L_2 + \frac{2}{n}, \\ L_2 & \text{for } L_2 < y \le L_2 + \frac{1}{n}, \\ y & \text{for } L_1 \le y \le L_2, \\ L_1 & \text{for } L_1 - \frac{1}{n} \le y < L_1, \\ 2y - L_1 + \frac{2}{n} & \text{for } L_1 - \frac{2}{n} \le y < L_1 - \frac{1}{n}, \\ y & \text{for } L_3 \le y < L_1 - \frac{2}{n}, \\ L_3 & \text{for } y < L_3. \end{cases} \tag{13}$$

Then $f_n \in \operatorname{Car}(J \times \mathbb{R}^2 \times D_r^2)$. Since $f \in \operatorname{Car}(J \times \mathbb{R}^2 \times D_r^2)$, there exists a $p \in L_1(J)$ such that

$$|f(t,x,y,u,v)| \le p(t) \quad \text{for a.e. } t \in J, \text{ each } (x,u,v) \in [L]_{\mathcal{D}} \text{ and } L_3 \le y \le L_4.$$

Clearly (cf. (11)),

$$|f_n(t,x,y,u,v)| \le p(t) \tag{14}$$

for a.e. $t \in J$, each $(x,y,u,v) \in \mathbb{R}^2 \times D_r^2$ and $n \ge n_0$.

Let $\mathbf{Y} = C^1(J)$ be the Banach space with the usual norm. For using the topological degree argument and Borsuk's theorem to prove an existence result for BVP (5), (6) we investigate an auxiliary operator equation and an auxiliary BVP which are defined below.

Let $\varphi, \psi \in C_r$, $\alpha \in \mathcal{A}$, $(\beta, \Lambda) \in [L_3, L_1, L_2, L_4]_{\mathcal{AB}}$. For each $c \in [0,1]$ and $n \in \mathbb{N}$, $n \ge n_0$, we define the operator

$$T_{nc} \colon \mathbf{Y} \times \mathbb{R}^2 \to \mathbf{Y} \times \mathbb{R}^2$$

by

$$T_{nc}(x,A,B) =$$
$$= \Bigg(A + \int_0^t g^{-1}\Big(B + c\int_0^s f_n\big(\tau, x(\tau), x'(\tau), x_\tau, x'_\tau\big)\,\mathrm{d}\tau\Big)\,\mathrm{d}s + (c-1)g^{-1}(-B)t,$$
$$A + \alpha(x) + (c-1)\alpha(-x),$$
$$B + \beta(x') + (c-1)\beta(-x') - \tilde\Lambda(x,x') - (c-1)\tilde\Lambda(-x,-x')\Bigg)\,, \tag{15}$$

where

$$x_\tau(s) = \begin{cases} \varphi(\tau+s) & \text{for } \tau+s\in[-r,0)\,,\\ x(\tau+s) & \text{for } \tau+s\in J\,,\end{cases}$$
$$x'_\tau(s) = \begin{cases} \psi(\tau+s) & \text{for } \tau+s\in[-r,0)\,,\\ x'(\tau+s) & \text{for } \tau+s\in J\end{cases} \tag{16}$$

and $\tilde\Lambda\colon \mathbf{X}\times\mathbf{X}\to\mathbb{R}$ is given by the formula

$$\tilde\Lambda(x,y) = \Lambda(\hat x,\hat y) \tag{17}$$

with $\hat x\in\mathbf{X}$ defined by

$$\hat x(t) = \begin{cases} x(t) & \text{for } |x(t)|\le L\,,\\ L\operatorname{sign}\big(x(t)\big) & \text{for } |x(t)|>L\end{cases} \tag{18}$$

(similarly for $\hat y$). The operator T_{nc} is well-defined because, by Lemma 1,

$$f_n\big(t,x(t),x'(t),x_t,x'_t\big)\in L_1(J)\,.$$

Moreover, $(\beta,\tilde\Lambda)\in[L_3,L_1,L_2,L_4]_{AB}$. We next consider the auxiliary BVP (cf. (6), (11) and (17))

$$\big(g(x')\big)' = f_n(t,x,x',x_t,x'_t)\,,\qquad n\ge n_0\,, \tag{19_n}$$

$$\alpha(x)=0\,,\qquad \beta(x') = \tilde\Lambda(x,x')\,,$$
$$x_0(s)=\varphi(s)\,,\quad x'_0(s)=\psi(s)\qquad \text{for}\quad s\in[-r,0)\,, \tag{20}$$

together with the operator equation (cf. (15))

$$T_{nc}(x,A,B) = (x,A,B)\,,\qquad c\in[0,1]\,,\ n\ge n_0\,. \tag{$(21_n)_c$}$$

We see that x is a solution on BVP (19_n), (20) if and only if $\big(x,x(0),g(x'(0))\big)$ is a solution of the operator equation $(21_n)_1$. Thus to prove an existence result for BVP (19_n), (20) it is sufficient to show that there exists a solution of $(21_n)_1$.

We denote by $M_{g^{-1}}$ the modulus of continuity of g^{-1} on the interval $I = \left\{t;\ t \in \mathbb{R},\ |t| \le \int_0^1 p(s)\ \mathrm{d}s + \max\{-g(L_3-1), g(L_4+1)\}\right\}$, i.e.,

$$M_{g^{-1}}(\varepsilon) = \sup\left\{|g^{-1}(t_1) - g^{-1}(t_2)|;\ t_1, t_2 \in I,\ |t_1 - t_2| \le \varepsilon\right\} \quad \text{for} \quad \varepsilon \in [0, \infty).$$

LEMMA 4. *Let f satisfy* (H). *Let (u, A_0, B_0) be a solution of $(21_n)_c$ for some $c \in [0,1]$ and $n \ge n_0$. Then the inequalities*

$$\|u\| < \max\{-L_3, L_4\} + \frac{1}{n}, \qquad L_3 - \frac{1}{n} < u'(t) < L_4 + \frac{1}{n}, \qquad t \in J, \tag{22}$$

$$|A_0| < \max\{-L_3, L_4\} + \frac{1}{n}, \qquad |B_0| < \max\{-g(L_3 - \tfrac{1}{n}), g(L_4 + \tfrac{1}{n})\} \tag{23}$$

are satisfied and, moreover,

$$|u'(t_1) - u'(t_2)| \le M_{g^{-1}}\left(\left|\int_{t_1}^{t_2} p(s)\ \mathrm{d}s\right|\right), \qquad t_1, t_2 \in J. \tag{24}$$

Proof. First we assume that $c = 0$, that is (u, A_0, B_0) is a solution of $(21_n)_0$. Then $u(t) = A_0 + \big(g^{-1}(B_0) - g^{-1}(-B_0)\big)t$, $\alpha(u) - \alpha(-u) = 0$, $\beta(u') - \beta(-u') = \tilde{\Lambda}(u, u') - \tilde{\Lambda}(-u, -u')$, and consequently $u(\xi) = 0$ for a $\xi \in J$ by Lemma 2 (with $\gamma = \alpha$, $\mu = 1$) and $L_1 \le u'(\tau) \le L_2$ for a $\tau \in J$ by Lemma 3 (with $\gamma = \beta$, $\Phi = \tilde{\Lambda}$, $\mu = 1$). Hence $A_0 = -\big(g^{-1}(B_0) - g^{-1}(-B_0)\big)\xi$, $L_1 \le g^{-1}(B_0) - g^{-1}(-B_0) \le L_2$ and then

$$\begin{aligned} |u(t)| &= \left|\big(g^{-1}(B_0) - g^{-1}(-B_0)\big)(t - \xi)\right| \le |g^{-1}(B_0) - g^{-1}(-B_0)| \\ &\le \max\{-L_1, L_2\} < \max\{-L_3, L_4\}, \\ &\qquad L_3 < L_1 \le u'(t) \le L_2 < L_4 \end{aligned}$$

for $t \in J$ and $|u'(t_1) - u'(t_2)| = 0$ for $t_1, t_2 \in J$. Thus inequalities (22)–(24) are satisfied.

Let $c \in (0,1]$. Then the equalities

$$\begin{aligned} u(t) = A_0 + \int_0^t g^{-1}\left(B_0 + c\int_0^s f_n\big(\tau, u(\tau), u'(\tau), u_\tau, u'_\tau\big)\ \mathrm{d}\tau\right) \mathrm{d}s& \\ +(c-1)g^{-1}(-B_0)t, \qquad t \in J& \end{aligned} \tag{25}$$

and

$$\begin{aligned} &\alpha(u) + (c-1)\alpha(-u) = 0, \\ &\beta(u') + (c-1)\beta(-u') = \tilde{\Lambda}(u, u') + (c-1)\tilde{\Lambda}(-u, -u') \end{aligned} \tag{26}$$

are satisfied. By Lemma 2 (with $\gamma = \alpha$, $\mu = 1 - c$), there exists a $\xi \in J$ such that

$$u(\xi) = 0 \tag{27'}$$

and, by Lemma 3 (with $\gamma = \beta$, $\Phi = \tilde{\Lambda}$, $\mu = 1 - c$), there exists an $\eta \in J$ such that

$$L_1 \le u'(\eta) \le L_2 \,. \tag{27''}$$

From (25) we deduce

$$u(0) = A_0 \,, \qquad u'(0) = g^{-1}(B_0) + (c-1)g^{-1}(-B_0) \tag{28}$$

and

$$u'(t) = g^{-1}\left(B_0 + c\int_0^t f_n\big(s, u(s), u'(s), u_s, u'_s\big)\, \mathrm{d}s\right) + (c-1)g^{-1}(-B_0)\,, \qquad t \in J\,. \tag{29}$$

Using the second equality in (28) we shall prove that

$$|B_0| \le \big|g\big(u'(0)\big)\big|\,. \tag{30}$$

Indeed, if $u'(0) \ge 0$, then necessarily $B_0 \ge 0$ and therefore $u'(0) \ge g^{-1}(B_0)$ since $(c-1)g^{-1}(-B_0) \ge 0$. Hence $B_0 \le g\big(u'(0)\big)$ and (30) is satisfied. If $u'(0) < 0$, then $B_0 < 0$, and consequently $u'(0) \le g^{-1}(B_0) < 0$, which implies (30).

We now show that inequalities (22) are satisfied. Assume (cf. (27'')) $0 < \eta < 1$. Let $\max\{u'(t)\,;\ \eta \le t \le 1\} = u'(t_0) \ge L_{3+\mu} + \frac{1}{n}$ for a $t_0 \in (\eta, 1]$ where $\mu \in \{-1, 1\}$ (for μ see assumption (H)). Then there are $\eta \le t_1 < t_2 \le t_0$ such that $u'(t_1) = L_{3+\mu}$, $u'(t_2) = L_{3+\mu} + \frac{1}{n}$ and $L_{3+\mu} \le u'(t) \le L_{3+\mu} + \frac{1}{n}$ for $t \in [t_1, t_2]$; hence $u'(t_2) - u'(t_1) = \frac{1}{n} > 0$. On the other hand (cf. (11)-(13) and (29)),

$$\begin{aligned} &u'(t_2) - u'(t_1) \\ &= g^{-1}\left(B_0 + c\int_0^{t_1} f_n\big(s, u(s), u'(s), u_s, u'_s\big)\, \mathrm{d}s + c\int_{t_1}^{t_2} f\big(s, \overline{u(s)}, L_{3+\mu}, \tilde{u}_s, \widetilde{u'_s}\,\big)\, \mathrm{d}s\right) \\ &\qquad\qquad - g^{-1}\left(B_0 + c\int_0^{t_1} f_n\big(s, u(s), u'(s), u_s, u'_s\big)\, \mathrm{d}s\right) \le 0 \end{aligned}$$

since g^{-1} is increasing and $c\int_{t_1}^{t_2} f\big(s, \overline{u(s)}, L_{3+\mu}, \tilde{u}_s, \widetilde{u'_s}\,\big)\, \mathrm{d}s \le 0$ by (H), a contradiction.

Let $\max\{u'(t);\ 0 \le t \le \eta\} = u'(t^*) \ge L_{3-\mu} + \frac{1}{n}$ for a $t^* \in [0,\eta)$. Then there are $t^* \le t_3 < t_4 \le \eta$ such that $u'(t_3) = L_{3-\mu} + \frac{1}{n}$, $u'(t_4) = L_{3-\mu}$ and $L_{3-\mu} \le u'(t) \le L_{3-\mu} + \frac{1}{n}$ for $t \in [t_3, t_4]$; hence $u'(t_3) - u'(t_4) = \frac{1}{n} > 0$. On the other hand (cf. (11)–(13) and (29))

$$
\begin{aligned}
&u'(t_3) - u'(t_4)\\
&= g^{-1}\left(B_0 + c\int_0^{t_3} f_n\big(s, u(s), u'(s), u_s, u'_s\big)\,\mathrm{d}s\right)\\
&\quad - g^{-1}\left(B_0 + c\int_0^{t_3} f_n\big(s, u(s), u'(s), u_s, u'_s\big)\,\mathrm{d}s + c\int_{t_3}^{t_4} f\big(s, \overline{u(s)}, L_{3-\mu}, \tilde{u}_s, \widetilde{u'_s}\big)\,\mathrm{d}s\right)\\
&\le 0
\end{aligned}
$$

since g^{-1} is increasing and $c\int_{t_3}^{t_4} f\big(s, \overline{u(s)}, L_{3-\mu}, \tilde{u}_s, \widetilde{u'_s}\big)\,\mathrm{d}s \ge 0$ by (H), a contradiction. If $\eta = 0$ (resp. $\eta = 1$) we can similarly prove that $\max\{u'(t);\ t \in J\} < L_{3+\mu} + \frac{1}{n}$ (resp. $\max\{u'(t);\ t \in J\} < L_{3-\mu} + \frac{1}{n}$). This proves $u'(t) < L_4 + \frac{1}{n}$ for $t \in J$. The proof for $u'(t) > L_3 - \frac{1}{n}$ on J is similar. The inequalities $L_3 - \frac{1}{n} < u'(t) < L_4 + \frac{1}{n}$ for $t \in J$ and (27′) show that the first inequality in (22) is satisfied. Then (23) follows from (22), (28) and (30).

Finally, we verify (24). Fix $t_1, t_2 \in J$. Then (cf. (14), (23) and (29))

$$
\begin{aligned}
|u'(t_1) - u'(t_2) &= \Bigg| g^{-1}\left(B_0 + c\int_0^{t_1} f_n\big(s, u(s), u'(s), u_s, u'_s\big)\,\mathrm{d}s\right)\\
&\qquad - g^{-1}\left(B_0 + c\int_0^{t_2} f_n\big(s, u(s), u'(s), u_s, u'_s\big)\,\mathrm{d}s\right)\Bigg|\\
&\le M_{g^{-1}}\left(\left|\int_{t_1}^{t_2} \big|f_n\big(s, u(s), u'(s), u_s, u'_s\big)\big|\,\mathrm{d}s\right|\right)\\
&\le M_{g^{-1}}\left(\left|\int_{t_1}^{t_2} p(s)\,\mathrm{d}s\right|\right).
\end{aligned}
$$

□

4. Existence results

LEMMA 5. *Let f satisfy* (H) *and $\varphi, \psi \in C_r$. Then for each $n \in \mathbb{N}$, $n \geq n_0$, the operator equation $(21_n)_1$ has a solution (u, A_0, B_0) satisfying* (22) – (24).

Proof. Fix $n \in \mathbb{N}$, $n \geq n_0$. Set ($L = \max\{-L_3, L_4\}$)

$$\Omega_n = \Big\{(x, A, B)\,;\ (x, A, B) \in \mathbf{Y} \times \mathbb{R}^2\,,\ \|x\| < L + \tfrac{1}{n}\,,\ \|x'\| < L + \tfrac{1}{n}\,,$$
$$|A| \leq L + \tfrac{1}{n}\,,\ |B| < \max\big\{-g\big(L_3 - \tfrac{1}{n}\big),\, g\big(L_4 + \tfrac{1}{n}\big)\big\}\Big\}.$$

Then Ω_n is an open bounded subset of $\mathbf{Y} \times \mathbb{R}^2$ and is symmetric with respect to $0 \in \Omega_n$. Define the operator $W_n\colon [0,1] \times \bar{\Omega}_n \to \mathbf{Y} \times \mathbb{R}^2$ by

$$W_n(c, x, A, B) = T_{nc}(x, A, B)\,.$$

Clearly, W_n is continuous and we show that W_n is a compact operator. Let $\{(c_j, x_j, A_j, B_j)\} \subset [0,1] \times \bar{\Omega}_n$ be a sequence and set

$$(z_j, R_j, V_j) = W_n(c_j, x_j, A_j, B_j)$$

for $j \in \mathbb{N}$. Then

$$z_j(t) = A_j + \int_0^t g^{-1}\Big(B_j + c_j \int_0^s f_n(\tau, x_j(\tau), x_j'(\tau), x_{j\tau}, x_{j\tau}')\,\mathrm{d}\tau\Big)\,\mathrm{d}s$$
$$+ (c_j - 1) g^{-1}(-B_j) t\,,$$

$$R_j = A_j + \alpha(x_j) + (c_j - 1)\alpha(-x_j)\,,$$
$$V_j = B_j + \beta(x_j') + (c_j - 1)\beta(-x_j') - \tilde{\Lambda}(x_j, x_j') - (c_j - 1)\tilde{\Lambda}(-x_j, -x_j')\,,$$

and so (cf. (14))

$$|z_j(t)| \leq L + \frac{1}{n} + g^{-1}\Big(S_n + \int_0^1 p(s)\,\mathrm{d}s\Big) + \max\big\{-g^{-1}(-S_n),\, g^{-1}(S_n)\big\}\,,$$

$$|z_j'(t)| \leq g^{-1}\Big(S_n + \int_0^1 p(s)\,\mathrm{d}s\Big) + \max\big\{-g^{-1}(-S_n),\, g^{-1}(S_n)\big\}$$

for $t \in J$ and $j \in \mathbb{N}$, where

$$S_n = \max\big\{-g\big(L_3 - \tfrac{1}{n}\big),\, g\big(L_4 + \tfrac{1}{n}\big)\big\}\,.$$

Moreover,

$$\begin{aligned}
&|z_j'(t_1) - z_j'(t_2)| \\
&\leq \left| g^{-1}\left(B_j + c_j \int_0^{t_1} f_n\big(s, x_j(s), x_j'(s), x_{js}, x_{js}'\big)\, \mathrm{d}s\right)\right. \\
&\qquad \left. - g^{-1}\left(B_j + c_j \int_0^{t_2} f_n\big(s, x_j(s), x_j'(s), x_{js}, x_{js}'\big)\, \mathrm{d}s\right)\right| \\
&\leq M_{g^{-1}}\left(\left|\int_{t_1}^{t_2} \left|f_n\big(s, x_j(s), x_j'(s), x_{js}, x_{js}'\big)\right|\, \mathrm{d}s\right|\right) \\
&\leq M_{g^{-1}}\left(\left|\int_{t_1}^{t_2} p(s)\, \mathrm{d}s\right|\right)
\end{aligned}$$

for $t_1, t_2 \in J$ and $j \in \mathbb{N}$. This proves that $\{z_j\}$ is bounded in $\mathbf{Y}$ and $\{z_j'(t)\}$ is equicontinuous on J. Furthermore, $\{R_j\}$ and $\{V_j\}$ are bounded in $\mathbb{R}$ because of $\{c_j\}$, $\{A_j\}$, $\{B_j\}$ are bounded in $\mathbb{R}$, $|\alpha(\pm x_j)| \leq \max\{-\alpha(-L-\frac{1}{n}),\ \alpha(L+\frac{1}{n})\}$, $|\beta(\pm x_j')| \leq \max\{-\beta(-L-\frac{1}{n}),\ \beta(L+\frac{1}{n})\}$ and

$$|\tilde{\Lambda}(\pm x_j, \pm x_j')| \leq \frac{1}{2}\min\{-\beta(L_1), \beta(L_2)\}\,.$$

By the Arzelà-Ascoli theorem and the Bolzano-Weierstrass theorem, $\{(z_j, R_j, V_j)\}$ is compact in $\mathbf{Y} \times \mathbb{R}^2$; hence W_n is a compact operator.

By Lemma 4, $W_n(c, x, A, B) \neq (x, A, B)$ for each $(x, A, B) \in \partial\Omega_n$ and $c \in [0,1]$. Thus $\mathrm{D}\big(W_n(1,\cdot,\cdot,\cdot) - I, \Omega_n, 0\big) = \mathrm{D}\big(W_n(0,\cdot,\cdot,\cdot) - I, \Omega_n, 0\big)$, where "D" denotes the Leray-Schauder degree and I is the identity operator on $\mathbf{Y} \times \mathbb{R}^2$. Since

$$\begin{aligned}
W_n(0, -x, -A, -B) &= T_{n0}(-x, -A, -B) \\
&= \Big(-A + \big(g^{-1}(-B) - g^{-1}(B)\big)t,\ -A + \alpha(-x) - \alpha(x), \\
&\qquad - B + \beta(-x') - \beta(x') - \tilde{\Lambda}(-x, -x') + \tilde{\Lambda}(x, x')\Big) \\
&= -T_{n0}(x, A, B) = -W_n(0, x, A, B)
\end{aligned}$$

for each $(x, A, B) \in \bar{\Omega}_n$, $W_n(0,\cdot,\cdot,\cdot)$ is an odd operator and then $\mathrm{D}\big(W_n(0,\cdot,\cdot,\cdot) - I, \Omega_n, 0\big) \neq 0$ by the Borsuk theorem. Thus $\mathrm{D}\big(W_n(1,\cdot,\cdot,\cdot) - I, \Omega_n, 0\big) \neq 0$, and consequently $(21_n)_1$ has a solution $(u, A_0, B_0) \in \Omega_n$. This solution satisfies (22) – (24) by Lemma 4. □

THEOREM 1. *Let f satisfy* (H), $\alpha \in \mathcal{A}$, $(\beta, \Lambda) \in [L_3, L_1, L_2, L_4]_{\mathcal{AB}}$ *and* $\varphi, \psi \in C_r$, $\|\varphi\|_0 \le L$, $\|\psi\|_0 \le L$ *with* $L = \max\{-L_3, L_4\}$. *Then BVP* (5), (6) *has a solution x satisfying the inequalities*

$$\|x\| \le L, \qquad L_3 \le x'(t) \le L_4 \quad \textit{for } t \in J. \tag{31}$$

P r o o f . For each $n \in \mathbb{N}$, $n \ge n_0$, there exists a solution (u_n, A_n, B_n) of $(21_n)_1$ satisfying (22)–(24) (with $(u, A_0, B_0) = (u_n, A_n, B_n)$) by Lemma 5, and consequently u_n is a solution of BVP (19_n), (20). The Arzelà-Ascoli theorem and the Bolzano-Weierstrass theorem show that there exists a convergent subsequence $\{(u_{k_n}, A_{k_n}, B_{k_n})\}$ of $\{(u_n, A_n, B_n)\}$ in $\mathbf{Y} \times \mathbb{R}^2$ and let $(u_{k_n}, A_{k_n}, B_{k_n}) \to (u, A, B)$ as $n \to \infty$. Then (cf. (16) with $x = u_n$ and $x' = u'_n$) $\{u_{k_n t}\}$, $\{u'_{k_n t}\}$ are convergent in D_r and $u_{k_n t} \to u_t$, $u'_{k_n t} \to u'_t$ as $n \to \infty$, where

$$u_t(s) = \begin{cases} \varphi(t+s) & \text{for } t+s \in [-r, 0), \\ u(t+s) & \text{for } t+s \in J, \end{cases}$$

$$u'_t(s) = \begin{cases} \psi(t+s) & \text{for } t+s \in [-r, 0), \\ u'(t+s) & \text{for } t+s \in J. \end{cases}$$

Evidently, $\|u\| \le L$, $-L_3 \le u'(t) \le L_4$ for $t \in J$, and consequently

$$\sup\{|u_t(s)|;\ s \in [-r, 0]\} \le L, \qquad \sup\{|u'_t(s)|;\ s \in [-r, 0]\} \le L$$

for $t \in J$ and (cf. (13)) $\lim\limits_{n\to\infty} h_{k_n}(u'_{k_n}(t)) = u'(t)$ uniformly on J. Taking the limit in the equalities

$$g(u'_{k_n}(t)) = g(u'_{k_n}(0)) + \int_0^t f_{k_n}(s, u_{k_n}(s), u'_{k_n}(s), u_{k_n s}, u'_{k_n s})\, \mathrm{d}s, \qquad t \in J,$$

$$\alpha(u_{k_n}) = 0, \qquad \beta(u'_{k_n}) = \tilde{\Lambda}(u_{k_n}, u'_{k_n})$$

as $n \to \infty$, we obtain (cf. (11) and (17))

$$g(u'(t)) = g(u'(0)) + \int_0^t f(s, u(s), u'(s), u_s, u'_s)\, \mathrm{d}s, \qquad t \in J,$$

$$\alpha(u) = 0, \qquad \beta(u') = \Lambda(u, u'),$$
$$u_0(s) = \varphi(s), \quad u'_0(s) = \psi(s) \qquad \text{for} \quad s \in [-r, 0).$$

Hence u is a solution of BVP (5), (6) satisfying (31). □

COROLLARY 1. *Let $h \in \mathrm{Car}(J \times \mathbb{R}^2 \times D_r^2)$, $q \in C^0(\mathbb{R})$ and there exist constants $L_3 < L_1 < 0 < L_2 < L_4$ such that $q(L_i) = 0$ for $i = 1,2,3,4$. If $\alpha \in \mathcal{A}$, $(\beta, \Lambda) \in [L_3, L_1, L_2, L_4]_{\mathcal{AB}}$, then BVP*

$$\big(g(x')\big)' = q(x')h(t,x,x',x_t,x_t'), \qquad (6) \tag{32}$$

has a solution x satisfying (31) *provided $\varphi, \psi \in C_r$, $\|\varphi\|_0 \le L$, $\|\psi\|_0 \le L$ with $L = \max\{-L_3, L_4\}$.*

Proof. Let us set $f(t,x,y,u,v) = q(y)h(t,x,y,u,v)$ for $(t,x,y,u,v) \in J \times \mathbb{R}^2 \times D_r^2$. Then f satisfies the assumptions of Theorem 1. The proof is completed by applying Theorem 1. □

Applying Theorem 1 to (7) we give the following corollary.

COROLLARY 2. *Suppose that there exist constants $L_1 < L_3 < 0 < L_2 < L_4$ and $\mu, \nu \in \{-1, 1\}$ such that*

$$\nu h(t,x,L_3) \ge 0\,, \qquad \nu h(t,x,L_1) \le 0\,,$$
$$\mu h(t,x,L_2) \ge 0\,, \qquad \mu h(t,x,L_4) \le 0$$

for a.e. $t \in J$ and each $x \in [-L, L]$, $L = \max\{-L_3, L_4\}$. Then BVP

$$(7), \qquad \alpha(x) = 0\,, \qquad \beta(x') = \Lambda(x,x') \tag{33}$$

has a solution x satisfying

$$\|x\| \le L\,, \qquad L_3 \le x'(t) \le L_4 \quad \textit{for } t \in J$$

provided $\alpha \in \mathcal{A}$, $(\beta, \Lambda) \in [L_3, L_1, L_2, L_4]_{\mathcal{AB}}$.

EXAMPLE 1. Consider the functional differential equation

$$\big(g_p(x')\big)' = q(x)\sin(x') + h(t,x,x',x_t,x_t')\,, \tag{34}$$

where $g_p(u) = |u|^{p-2}u$, $p > 1$, $g_p(0) = 0$, $q \in C^0(\mathbb{R})$, $h \in \mathrm{Car}(J \times \mathbb{R}^2 \times D_r^2)$, subject to the boundary conditions

$$\begin{gathered} \max\{x(t);\ t \in J\} = 0\,, \qquad x'(t_0) = \lambda \int_0^1 \sqrt{1 + \big(x'(t)\big)^2}\,\mathrm{d}t\,, \\ x_0(s) = \varphi(s)\,, \quad x_0'(s) = \psi(s) \qquad \text{for} \quad s \in [-r, 0)\,, \end{gathered} \tag{35}$$

where $t_0 \in J$, $\varphi, \psi \in C_r$ and $\lambda \in \mathbb{R}$.

Assume that there exists a positive constant K such that $q(z) \ge K$ for $z \in \left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$ and $|h(t,x,y,u,v)| \le K$ for a.e. $t \in J$ and each $(x,u,v) \in \left[\frac{3\pi}{2}\right]_{\mathcal{D}}$, $|y| \le \frac{3\pi}{2}$. Then the function $f\colon J \times \mathbb{R}^2 \times D_r^2 \to \mathbb{R}$, $f(t,x,y,u,v) = q(x)\sin(y) +$

$h(t,x,y,u,v)$ satisfies assumption (H) with $-L_3 = L_4 = \frac{3\pi}{2}$, $-L_1 = L_2 = \frac{\pi}{2}$ and $\nu = \mu = 1$. Boundary conditions (35) are the special case of those for (6) with $\alpha(x) = \max\{x(t);\ t \in J\}$, $\beta(x) = x(t_0)$ and $\Lambda(x,y) = \lambda \int_0^1 \sqrt{1 + (y(t))^2}\, dt$ for $x, y \in \mathbf{X}$. Clearly, $\alpha \in \mathcal{A}$ and $(\beta, \Lambda) \in \left[-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}\right]_{\mathcal{AB}}$ for $|\lambda| \le \frac{3\pi^2}{2(2+9\pi^2)}$ since $\beta \in \mathcal{A}$, $\Lambda \in \mathcal{B}$,

$$\sup\left\{|\Lambda(x,y)|;\ (x,y) \in \mathbf{X}^2,\ \|x\| \le \frac{3\pi}{2},\ \|y\| \le \frac{3\pi}{2}\right\} = |\lambda| \int_0^1 \sqrt{1 + \left(\frac{3\pi}{2}\right)^2}\, dt$$
$$< \frac{2 + 9\pi^2}{6\pi} |\lambda| \le \frac{\pi}{4}$$

and $\min\{-\beta(-\frac{\pi}{2}), \beta(\frac{\pi}{2})\} = \frac{\pi}{2}$. By Theorem 1, for each $\varphi, \psi \in C_r$ and $\lambda \in \mathbb{R}$ such that $\|\varphi\|_0 \le \frac{3\pi}{2}$, $\|\psi\|_0 \le \frac{3\pi}{2}$, $|\lambda| \le \frac{3\pi^2}{2(2+9\pi^2)}$, there exists a solution x of BVP (34), (35) satisfying the inequalities

$$|x(t)| \le \frac{3\pi}{2}, \quad |x'(t)| \le \frac{3\pi}{2} \qquad \text{for} \quad t \in J.$$

EXAMPLE 2. Consider BVP

$$\left(sh(x')\right)' = p(t,x) + kx'^2(3 - x'^4), \tag{36}$$

$$\int_0^1 \arctan x(t)\, dt = 0, \qquad \min\{x'(t);\ t \in J\} = \mu \int_0^1 x(t)\sqrt{1 + (x'(t))^2}\, dt, \tag{37}$$

where $p \in \mathrm{Car}(J \times \mathbb{R})$ and $k, \mu \in \mathbb{R}$, $k \ne 0$. Assume $|p(t,x)| \le 2|k|$ for $|x| \le 2$ and a.e. $t \in J$. Then the function $h\colon J \times \mathbb{R}^2 \to \mathbb{R}$, $h(t,x,y) = p(t,x) + ky^2(3 - y^4)$ satisfies the assumptions of Corollary 2 with $-L_3 = L_4 = 2$, $-L_1 = L_2 = 1$ and $\mu = -\nu = \operatorname{sign} k$. Boundary conditions (37) are the special case of those for BVP (33) with $\alpha(x) = \int_0^1 \arctan x(t)\, dt$, $\beta(x) = \min\{x(t);\ t \in J\}$ and $\Lambda(x,y) = \mu \int_0^1 x(t)\sqrt{1 + (y(t))^2}\, dt$ for $x, y \in \mathbf{X}$. Evidently, $\alpha \in \mathcal{A}$ and $(\beta, \Lambda) \in [-2, -1, 1, 2]_{\mathcal{AB}}$ for $|\mu| \le \frac{1}{4\sqrt{5}}$ since $\beta \in \mathcal{A}$, $\Lambda \in \mathcal{B}$, $\sup\{|\Lambda(x,y)|;\ (x,y) \in \mathbf{X}^2$, $\|x\| \le 2$, $\|y\| \le 2\} = |\mu| 2\sqrt{5} \le \frac{1}{2}$ and $\min\{-\beta(-1), \beta(1)\} = 1$. Thus, by Corollary 2, there exists a solution x of BVP (36), (37) for any $|\mu| \le \frac{1}{4\sqrt{5}}$ satisfying the inequalities $\|x\| \le 2$, $\|x'\| \le 2$.

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