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ON THE STRUCTURE OF SOLUTION SETS OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT. In this paper, we investigate topological structure of solution sets of the Cauchy problem in Banach spaces which are defined on an unbounded interval.

The main condition in our results is formulated in terms of the Kuratowski or an axiomatic measure of noncompactness.

1. Introduction

The topological structure of solution sets of differential equations in Banach spaces where the argument is defined on a bounded domain was investigated by many authors (see [9] and references given there). But recently there have appeared papers in which a bounded domain is replaced by an unbounded one (see e.g. [4], [5], [8], [14], [15], [16], [17], [18]). In this article, we consider the following initial value problem

$$x' = f(t,x), \qquad x(0) = x_0, \tag{1}$$

where $f\colon I \times E \to E$, $I = [0,+\infty)$ and E is a Banach space. We shall give sufficient conditions which guarantee that the set of all solutions of (1) has the Aronszajn property, i.e. it is an R_δ.

The proofs of our theorems are based on the following theorem.

THEOREM 1. ([14], [17]). *Let K be a convex unbounded subset of a normed space, E be a Banach space and let $C = C(K,E)$ denote the Frechét space of all continuous locally bounded functions $K \to E$ with the topology of locally uniform convergence. Assume that $F\colon C \to C$ is a continuous mapping such that*

1° *there exists $t_0 \in K$, $x_0 \in C$ such that*

$$F(x)(t_0) = x_0 \qquad \textit{for every} \quad x \in C,$$

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2° *the family $F(C)$ is locally equiuniformly continuous,*
3° *for every $\varepsilon > 0$ the following implication holds*

$$x|_{K_\varepsilon} = y|_{K_\varepsilon} \implies F(x)|_{K_\varepsilon} = F(y)|_{K_\varepsilon}, \qquad x, y \in C.$$

where $K_\varepsilon = K \cap B(t_0, \varepsilon)$ and $B(t_0, \varepsilon)$ denotes the closed ball of center t_0 and radius ε,
4° *every sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \in C$, such that*

$$\lim_{n\to\infty} \big(x_n - F(x_n)\big) = 0$$

has a limit point.

Then the set of all fixed points of F is an R_δ, i.e. it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.

Moreover, in what follows, we apply the following result from [10].

THEOREM 2. *Suppose that $\phi, \omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\varphi(t)$ is not identically equal to zero. Then the solutions of the problem*

$$r' = \phi(t)\omega(r)$$

are defined in the future if and only if $\omega \in \mathcal{R}_0$, where $\mathcal{R}_0$ is the family of continuous functions $v\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$v(r) > 0\,, \quad r \geq \delta\,, \qquad \int_\delta^{+\infty} \frac{\mathrm{d}s}{v(s)} = +\infty \quad \textit{for some } \delta \geq 0\,.$$

The main condition in our theorems is the Cellina type assumption (cf. [7]) formulated in terms of the Kuratowski or axiomatic measure of noncompactness. Recall that for any bounded subset A of a metric space the Kuratowski measure of noncompactness α is defined as the infimum of positive numbers ε such that A can be covered by a finite number of sets of diameter $< \varepsilon$ (for examples and the basic properties see e.g. [2] or [6]). The definition and some properties of an axiomatic measure of noncompactness we recall in Section 3.

Finally, for the completeness, let us recall Krasnoselski-Krein's lemma.

LEMMA. *Let $J = [0, d] \subset \mathbb{R}$ be a compact interval and let $f\colon J \times E \to E$ be a continuous mapping. Then for any $u \in C(J, E)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sup_{t\in J} \big\|f\big(t, x(t)\big) - f\big(t, u(t)\big)\big\| < \varepsilon$$

whenever $x \in C(J, E)$ and $\sup_{t\in J} \|x(t) - u(t)\| < \delta$.

2. Main result

Consider the problem (1). Assume that

1) $f : I \times E \to E$ is a continuous function;
2) there exists $\omega \in \mathcal{R}_0$ and a continuous function $\varphi \colon I \to \mathbb{R}_+$ such that
$$\|f(t,x)\| \le \varphi(t)\omega(\|x\|) \qquad \text{for all} \quad t \in I \quad \text{and} \quad x \in E\,;$$
3) for each bounded subset $B \subset E$ and for each interval $[0,a]$, $a > 0$, there exists a continuous nondecreasing function $h_{B,a} \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that the inequality
$$u(t) \le \int_0^t h_{B,a}\big(u(s)\big)\, \mathrm{d}s\,, \qquad t \in [0,a]\,,$$
has only the trivial solution $u \equiv 0$ and
$$\alpha\big(f(A \times X)\big) \le h_{B,a}\big(\alpha(X)\big)$$
for $A \subset [0,a]$ and $X \subset B$.

Now, we prove the following theorem.

THEOREM 3. *Under the above assumptions the set S of all solutions of* (1), *defined on I, is an R_δ.*

P r o o f. In view of 2) it is clear that each solution of (1) satisfies the inequalities
$$\limsup_{h \to 0^+} \frac{\|x(t+h)\| - \|x(t)\|}{h} \le \|x'(t)\| = \big\|f\big(t,x(t)\big)\big\| \le \varphi(t)\omega\big(\|x(t)\|\big)\,,$$
$$t \in I\,.$$
By Theorem 2, we infer that the maximal solution $r_0(t)$ of the problem
$$r' = \varphi(t)\omega(r)\,, \qquad r(0) = \|x_0\|\,,$$
is defined on I. Thus by the theorem on differential inequalities (see [11]) we obtain
$$\|x(t)\| \le r_0(t) \qquad \text{for} \quad t \in I\,.$$
Let $\psi \colon E \to [0,1]$ be a continuous function such that $\psi(u) = 1$ if $\|u\| \le 1$ and $\psi(u) = 0$ if $\|u\| > 2$. Put $p(s,x) = \psi\Big(\frac{x}{1+r_0(s)}\Big) f(s,x)$, $s \in I$, $x \in E$. Define the mapping
$$F(x)(t) = x_0 + \int_0^t p\big(s,x(s)\big)\, \mathrm{d}s\,, \qquad t \in I\,, \;\; x \in C\,,$$

where $C = C(I,E)$. It can be easily verified that $x \in C$ is a solution of (1) if and only if $x = F(x)$. Thus it is enough to prove that the set of all fixed points of F is an R_δ. Obviously, $F(C) \subset C$. Further, in view of the inequality

$$\begin{aligned} \|F(x)(t_1) - F(x)(t_2)\| &= \left\| \int_0^{t_1} p(s,x(s))\,\mathrm{d}s - \int_0^{t_2} p(s,x(s))\,\mathrm{d}s \right\| \\ &= \left\| \int_{t_1}^{t_2} p(s,x(s))\,\mathrm{d}s \right\| \\ &\le \int_{t_1}^{t_2} \varphi(s) \max_{t\in[0,2(r_0(a)+1)]} \omega(t)\,\mathrm{d}s\,, \qquad x \in C\,,\ \ 0 \le t_1 < t_2 \le a\,, \end{aligned}$$

where a is any positive number, we deduce that F satisfies 2°. Moreover, by Krasnoselskii-Krein lemma ([13]) and 2° we infer that F is continuous. Now we show that F satisfies 4°. Let (x_n) be a sequence satisfying the condition in 4°. Set $V = \{x_n : n \in \mathbb{N}\}$, $V(t) = \{x_n(t) : n \in \mathbb{N}\}$, $t \in I$. Since $V \subset (\mathrm{Id} - F)(V) + F(V)$, from 2° and condition in 4° it is clear that V is equiuniformly continuous on $[0,a]$ for every $a > 0$. Fix $a > 0$ and let B be the closed ball in E of center 0 and radius $2(r_0(a)+1)$. In view of the properties of α we have

$$\begin{aligned} \alpha\big(p(A\times X)\big) &\le \alpha\left(\psi\left(\frac{X}{1+r_0(A)}\right) f(A\times X)\right) \le \alpha\left(\bigcup_{0\le\lambda\le 1} \lambda f(A\times X)\right) \\ &= \alpha\big(f(A\times X)\big) \le h_{B,a}\big(\alpha(X)\big)\,, \end{aligned}$$

for any subset $X \subset B$ and $A \subset [0,a]$.

Now let $W = F(V)$, $v(t) = \alpha\big(V(t)\big)$ and $w(t) = \alpha\big(W(t)\big)$ for $t \in I$. From the basic properties of the index α we obtain

$$v(t) \le \alpha\big((\mathrm{Id} - F)(V)(t)\big) + \alpha\big(W(t)\big) = w(t) \tag{2}$$

and, similarly,

$$\alpha\big(V(T)\big) \le \alpha\big(W(T)\big) \qquad \text{for each compact subset} \quad T \subset I\,.$$

Further, we have

$$\begin{aligned} |w(t_1) - w(t_2)| &= \big|\alpha\big(F(V)(t_1)\big) - \alpha\big(F(V)(t_2)\big)\big| \\ &\le \sup_{u,v} \|F(u)(t_1) - F(u)(t_2) - F(v)(t_1) + F(v)(t_2)\| \\ &\le 2\sup_{u\in V} \|F(u)(t_1) - F(u)(t_2)\|\,, \qquad t_1, t_2 \in I\,. \end{aligned}$$

By the above inequality and equiuniformly continuity of W on every bounded subset of I, we deduce that w is continuous on I.

Divide the interval $[0,t]$, $t < a$, into n parts: $0 = t_0 < t_1 < \cdots < t_n = t$ in such a way that $t_i - t_{i-1} = \frac{t}{n}$ for $i = 1,\dots,n$. Put $T_i = [t_{i-1}, t_i]$, $i = 1,\dots,n$.

Since W is equiuniformly continuous and uniformly bounded on every bounded subset of I, by Ambrosetti's lemma ([1]) and the continuity of w there exists $p_i \in T_i$ such that

$$\alpha\big(W(T_i)\big) = \sup_{t\in T_i} \alpha\big(W(t)\big) = \sup_{t\in T_i} w(t) = w(p_i)\,.$$

In view of the mean value theorem, for every $x \in V$ we obtain

$$\begin{aligned} F(x)(t) &= x_0 + \int_0^t p\big(s, x(s)\big)\,\mathrm{d}s \\ &= x_0 + \sum_{i=1}^n \int_{T_i} p\big(s, x(s)\big)\,\mathrm{d}s \in x_0 + \sum_{i=1}^n \mu(T_i)\overline{\mathrm{conv}}\,p\big(T_i \times x(T_i)\big)\,, \end{aligned}$$

where μ denotes here the Lebesgue measure.

Thus

$$F(V)(t) \subset x_0 + \sum_{i=1}^n \mu(T_i)\overline{\mathrm{conv}}\,p\big(T_i \times V(T_i)\big)\,.$$

Hence, by the properties of index α, we obtain

$$\begin{aligned} w(t) &\le \sum_{i=1}^n \mu(T_i)\alpha\big(p(T_i \times V(T_i))\big) \le \sum_{i=1}^n \mu(T_i) h_{B,a}\big(\alpha(V(T_i))\big) \\ &\le \sum_{i=1}^n \mu(T_i) h_{B,a}\big(\alpha(W(T_i))\big) = \sum_{i=1}^n \mu(T_i) h_{B,a}\big(w(p_i)\big)\,. \end{aligned}$$

If $n \to \infty$, in view of the continuity of h and w we obtain

$$w(t) \le \int_0^t h_{B,a}\big(w(s)\big)\,\mathrm{d}s\,, \qquad t \in [0,a]\,.$$

Thus, by the assumption 3), $w(t) = 0$ and therefore by (2), $v(t) = 0$ for every $t \in [0,a]$. Hence $V(t)$ is relatively compact for $t \in [0,a]$. Since a is arbitrary, in view of Ascoli's theorem we infer that V is relatively compact. Hence the sequence (x_n) has a limit point.

We see that F satisfies all assumptions of Theorem 1 and therefore the set of all its fixed points is an R_δ. This completes the proof. □

Remark 1. An analogous result to Theorem 3 in the case of a bounded interval was proved in [3; Theorem 3], but their proofs are different.

3. Axiomatic measure of noncompactness

In this section we extend Theorem 3 applying an axiomatic measure of noncompactness. Recall that this notion was introduced by Banaś and Goebel [1] and found many applications in the theory of differential and integral equations in Banach spaces or in the fixed point theory (see [2], [5], [12]).

Denote by $\mathcal{M}_E$ the family of all bounded subsets of a given Banach space E, and by $\mathcal{N}_E$ the family of all relatively compact subsets of E (briefly: $\mathcal{M}$ and $\mathcal{N}$, respectively).

Definition 1. ([2]) A function $\mu\colon \mathcal{M} \to [0,+\infty)$ is said to be an *axiomatic measure of noncompactness* if it satisfies the following conditions:

5° the family $\ker\mu = \{X \in \mathcal{M} : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathcal{N}$;
6° $X \subset Y \implies \mu(X) \le \mu(Y)$;
7° $\mu(\overline{\operatorname{conv}} X) = \mu(X)$;
8° $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.

If additionally a measure μ satisfies the condition

9° $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$ for any $X, Y \in \mathcal{M}$,

then we say that μ has the *maximum property*.

Our next result is given by the following theorem.

Theorem 4. *Assume that the function f satisfies* 1), 2). *Moreover, let the measure μ have the maximum property and satisfy the following conditions:*

10° $\mu(\{x\}) = 0$ *for any* $x \in E$,
11° $\mu(A + X) \le \mu(X)$ *for any bounded subsets* $X \subset E$ *and* $A \in \ker\mu$.

If for each bounded subset B of E and for each interval $(n-1.n]$, $n \in \mathbb{N}$, there exists a continuous nondecreasing function $h_{B,n}\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that the inequality

$$u(t) \le \int_{n-1}^{t} h_{B,n}(u(s))\,\mathrm{d}s, \qquad t \in (n-1,n],$$

has only the trivial solution $u \equiv 0$ on $[n-1,n]$ and

$$\mu(f(A \times X)) \le h_{B,n}(\mu(X)) \tag{3}$$

for $X \subset B$ and $A \subset (n-1,n]$, then the set of all solutions of (1), *defined on I, is an R_δ.*

P r o o f . Analogously as in the proof of Theorem 3 we define the mapping F and verify that it satisfies 2° and it is continuous.

Now, we verify that the mapping p satisfies (3) (in which we replace f by p). Let $p(s,x) \in \psi\left(\frac{X}{1+r_0(A)}\right) f(A \times X)$, where $A \subset (n-1,n]$, $X \subset B$. Then $p(s,x) = \lambda y$, where $y \in f(A \times X)$, $\lambda \in [0,1]$. Thus $p(s,x) \in \operatorname{conv}(\{0\} \cup f(A \times X))$, so $\psi\left(\frac{X}{1+r_0(A)}\right) f(A \times X) \subset \operatorname{conv}(\{0\} \cup f(A \times X))$. Hence, in view of 6°, 7°, 9°, 10° we have

$$\begin{aligned}\mu\left(\psi\left(\frac{X}{1+r_0(A)}\right) f(A \times X)\right) &\le \mu(\operatorname{conv}(\{0\} \cup f(A \times X)))\\ &= \mu(\{0\} \cup f(A \times X))\\ &= \max\{\mu(\{0\}), \mu(f(A \times X))\} = \mu(f(A \times X))\\ &\le h_{B,n}(\mu(X)).\end{aligned}$$

Now arguing similarly as in the proof of Theorem 3 and developing techniques suitable for axiomatic measures of noncompactness (see [19] for details) we obtain

$$w(t) \le \int_0^t h_{B,1}(w(s))\, \mathrm{d}s, \qquad t \in [0,1].$$

Thus, by the assumption on $h_{B,1}$, $w(t) = 0$ and therefore $v(t) = 0$ for $t \in [0,1]$. Now, let $t \in (1,2]$. By 11° we have

$$\begin{aligned}w(t) = \mu(F(V)(t)) &= \mu\left(\left\{x_0 + \int_0^t p(s,x(s))\, \mathrm{d}s :\ x \in V\right\}\right)\\ &= \mu\left(\left\{x_0 + \int_0^1 p(s,x(s))\, \mathrm{d}s + \int_1^t p(s,x(s))\, \mathrm{d}s :\ x \in V\right\}\right)\\ &\le \mu\left(\left\{\int_1^t p(s,x(s)) ds :\ x \in V\right\}\right).\end{aligned}$$

Arguing similarly as in the case when $t \in [0,1]$ we can show that $v(t) = w(t) = 0$ for $t \in (1,2]$. Hence $v(t) = 0$ for every $t \in I$. In view of Ascoli's theorem we infer that V is relatively compact, and therefore the sequence (x_n) has a limit point.

Hence it is clear that the operator F satisfies all assumptions of Theorem 1, and therefore the set of all its fixed points is an R_δ, what completes our proof. □

Remark 2. Theorem 4 extends also Theorem 5 from [5].

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