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ISOPERIMETRIC NUMBERS AND SPECTRAL RADIUS OF SOME INFINITE PLANAR GRAPHS

BOJAN MOHAR[1)]

ABSTRACT. Let N be a triangulation of a non-compact, open subset of the 2-sphere, projective plane, torus, or Klein bottle, and let G be its (geometric) dual graph. If every 0-simplex of N is contained in at least k 2-simplices, where $k \geqq 7$, then the isoperimetric number $i(G)$ of G is at least $i(G) \geqq 3(k-6)/(5k-18)$. If G has at most m ends then, if $(k-3)m \geqq k\chi(S)$, $i(G) \geqq 3(k-6)/[(5-2\chi(S)/m)k-18]$, and $i(G) \geqq (k-6)/(k-4)$ otherwise. These bounds, except the last one, are shown to be the best possible. Even better bounds are obtained, assuming G is cyclically t-edge connected ($3 < t \leqq k$). Also nontrivial bounds on the spectral radius of G are derived from the above results.

1. Introduction

The *isoperimetric number* $i(G)$ of a locally finite infinite graph G is equal to

$$i(G) = \inf_X \frac{|\partial X|}{|X|}, \tag{1.1}$$

where X runs over all finite non-empty subsets of $V(G)$, and ∂X denotes the set of all edges of G which have one end in X and the other end in $V(G)\setminus X$. This number is a combinatorial analogue (cf. [2, 4]) of the geometric isoperimetric numbers which are equal to the minimum quotient between the area of a cutset and the volume of a part which the cutset separates.

Considering triangulations of surfaces we shall be using the standard terminology of combinatorial topology [8]. By a triangle, edge, or vertex of a triangulation, we mean a 2-, 1-, or 0-simplex, respectively.

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[1)] This paper was written in 1986 while the author was visiting the Simon Fraser University, Burnaby, B.C., Canada

Let N be a triangulation of an open subset of the 2-sphere (i.e., a planar surface), the projective plane, Klein bottle, or the torus, such that every vertex of N is contained in at least k ($k \geqq 7$) triangles. Let L be a finite subcomplex of N with $r(L)$ triangles and with $q_0(L)$ edges belonging to less than two triangles of L. In [4] Dodziuk proved that, if $k \geqq 7$, then $r(L)/q_0(L) \leqq 26$ (only the planar case). In our paper we extend his results to obtain the best possible bounds of this form by showing that

$$\frac{r(L)}{q_0(L)} \leqq \frac{5k-18}{3(k-6)} - \frac{2k\chi(S)}{(k-6)q_0(L)}, \tag{1.2}$$

where $\chi(S)$ is the Euler characteristic of the surface containing N, and this bound is sharp. The dual version of this result states that the cubic graph G, the dual of N, has the isoperimetric number at least

$$i(G) \geqq \frac{3(k-6)}{5k-18} \tag{1.3}$$

(and this bound is the best possible).

Let K be an infinite graph (or a simplicial complex). The maximal number m of infinite components of $K \backslash X$, where X is a finite subgraph (resp. subcomplex) of K, is the *number of ends* of K [5]. Note that m may be infinite as well. For triangulations with at most m ends and for their dual graphs (they also have at most m ends), the inequalities (1.2) and (1.3) are improved in Theorem 4.2 and 6.2, respectively. For the isoperimetric number of a cubic planar, projective planar, Klein bottle, or toroidal graph with all faces of length k or more, with duals being triangulations and with at most m ends the following bounds hold. If $(k-3)m \geqq k\chi(S)$, then

$$i(G) \geqq 3(k-6)/\left[(5-2\chi(S)/m)k-18\right] \tag{1.4}$$

and otherwise

$$i(G) \geqq (k-6)/(k-4). \tag{1.5}$$

Let G be a cubic graph, and let $A(G)$ denote its adjacency matrix. Then $A(G)$ naturally acts (as a matrix on column vectors) on the "vectors" $(x_v\,;\ v \in V(G))$ which are sequences of complex numbers x_v indexed by the vertex set of G. For any two vectors $(x_v\,;\ v \in V(G))$ and $(y_v\,;\ v \in V(G))$ their inner product is

$$(x,y) := \sum_{v \in V(G)} x_v \bar{y}_v$$

and it is well defined at least for those vectors which have the finite norm:

$$\|x\| = (x,x)^{1/2} < \infty$$

(these vectors form the Hilbert space $l^2(V(G))$). The *spectral radius* $\rho(G)$ of G is given by

$$\rho(G) = \sup\big(|(Ax,x)|\,;\ \|x\| = 1\big)$$

which is also the maximum of the spectrum of $A(G)$, viewed as a linear operator acting on $l^2(V(G))$. See, e.g., [1, 6].

From the bounds (1.3), (1.4), and (1.5) for the isoperimetric numbers of cubic graphs we derive interesting lower bounds on the spectral radius of these graphs (Corollaries 6.3 and 6.4).

In the last Section we remark that for cyclically t-edge connected graphs ($3 < t \leqq k$), the bound (1.3) can be improved to

$$i(G) \geqq t(k-6)/\big[(t+2)k - 4t - 6\big]\,.$$

2. Class $\mathcal{T}_k^S$ of triangulations

From now on, and up to the end of the paper, k will denote a fixed integer greater than or equal to 7. Let $\mathcal{T}_k^S$ be the class of all triangulated surfaces without boundary, which are homeomorphic to an open subset of the surface S, with the property that each vertex (i.e. 0-simplex) is contained in at least k triangles. In other words, a pure 2-dimensional simplical complex K is in $\mathcal{T}_k^S$ if and only if $|K|$ is homeomorphic to an open subset of S and for each vertex $v \in V(K)$

$$\deg(v) \geqq k\,. \tag{2.1}$$

We will confine ourselves to cases where S is either the 2-sphere, projective plane, torus, or the Klein bottle, i.e. $\chi(S) \geqq 0$. Then, clearly, every triangulation in $\mathcal{T}_k^S$ is infinite (since $k \geqq 7$).

Given a triangulation $K \in \mathcal{T}_k^S$ we shall be concerned with finding

$$\operatorname{iso}(K) = \min_L \operatorname{iso}(L)\,,$$

where the minimum is taken over all finite subcomplexes L of K, and $\operatorname{iso}(L)$ is the quotient of the number of the edges of L lying in less than 2 triangles of L and the number of triangles of L. Usually we shall use the quantity $I(L) := 1/\operatorname{iso}(L)$, seeking to maximize $I(L)$.

For a given finite subcomplex L of $K \in \mathcal{T}_k^S$ we shall use the following notation:

∂L	...	the graph consisting of all edges of L belonging to less than two triangles of L
p, q, r	...	the number of vertices, edges, and triangles of L, respectively
p_1, q_1	...	the number of inner vertices (link is a cycle) and inner edges (lying in two triangles of L)
p_0, q_0	...	the number of vertices and edges of ∂L
n	...	the number of components of ∂L
$c_1, \dots, c_n$	...	numbers of edges in components of ∂L
$d_1, \dots d_{p_0}$	...	degrees (in L) of vertices lying in ∂L
d	...	$\sum d_i$
B_i	...	number of components of ∂L having exactly i edges ($i = 0, 1, 2, 3, \dots$)

Sometimes we will have to emphasize that certain of the above variables refer to a complex L. In such a case we shall write $p(L)$, $q(L)$, $r(L)$, $p_0(L)$, $q_0(L)$, $n(L)$, etc. with the obvious meaning. For example, $I(L) = r(L)/q_0(L)$.

It is clear that the above introduced quantities are not independent. The following equalities hold:

$$p_0 + p_1 = p \tag{2.2}$$

$$q_0 + q_1 = q \tag{2.3}$$

$$\sum_{i=1}^{n} c_i = q_0 \tag{2.4}$$

$$\sum_{i=1}^{p_0} d_i = d \tag{2.5}$$

$$\sum_{i} iB_i = q_0 \tag{2.6}$$

$$\sum_{i} B_i = n \tag{2.7}$$

We finish this section by an obvious result characterizing finite subcomplexes of triangulations from $\mathcal{T}_k^S$.

PROPOSITION 2.1. *Let L be a finite pure 2-dimensional complex which is homeomorphic to a subset of the surface S. Then L is a subcomplex of a triangulation from $\mathcal{T}_k^S$ if and only if each vertex of L with the link in L isomorphic to a cycle C_ℓ is of degree $\ell \geqq k$.* □

3. Restrictions

Our aim is to determine the maximum of $I(L) = r(L)/q_0(L)$ taken over all finite complexes L which are subcomplexes of triangulations from $\mathcal{T}_k^S$. We shall confine ourselves to a subset of such complexes L by showing that, given L a subcomplex of K, there is a complex L' related to L (which is a subcomplex of some $K' \in \mathcal{T}_k^S$, and K' is "close" to K) such that

(a) L' satisfies several restrictions stated in this section,
(b) $q_0(L') \leqq \max\{q_0(L), k\}$, and
(c) $I(L') \geqq I(L)$.

To show that L' is a subcomplex of some K' we shall use Proposition 2.1, usually without mentioning it.

Restriction 1. *L is pure.*

Proof. Let L' be L minus all edges and vertices which are not contained in any triangle of L. □

Restriction 2. *L is strongly connected, i.e. any two triangles A, B are connected by a sequence of triangles starting with A and ending with B such that any two consecutive triangles share a common edge.*

Proof. For L' take the strongly connected component of L with largest $I(L')$. □

Restriction 3. *The boundary ∂L of L consist of disjoint cycles.*

Proof. If L has vertices with the link consisting of more than one component, construct L' as follows. For each such vertex v take the strongly connected components of star(v, L) in the same cyclic order as they follow each other embedded in the surface. Between any two consecutive strong components with the exception of one (chosen arbitrarily) add two triangles as shown in Figure 1. It is easy verified that the obtained complex L' has the required properties. □

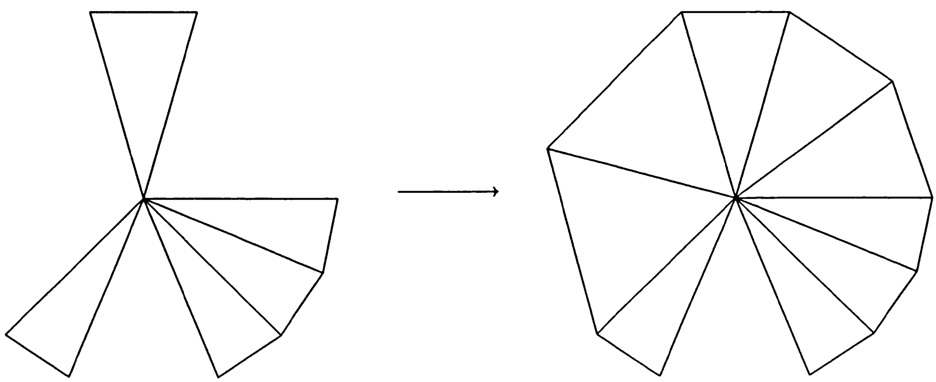


Figure 1.

RESTRICTION 4. $I(L) \geqq 1$.

Proof. If $I(L) < 1$, take L' to consist of a vertex of degree k and k triangles containing this vertex. Then we get $I(L') = 1$. □

RESTRICTION 5. *L has no vertices of degree two.*

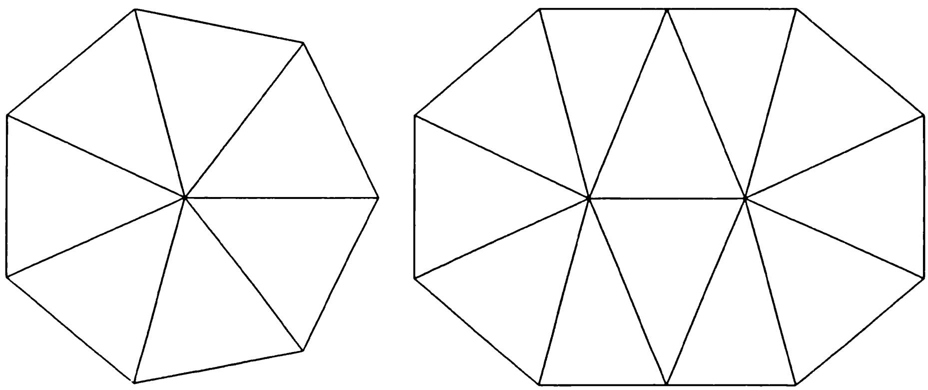
Proof. If v is a vertex of degree 2, it lies in ∂L. Let L_1 be L minus the triangle containing v. Since $I(L) \geqq 1$ by Restriction 4,

$$I(L_1) = \big(r(L) - 1\big)/\big(q_0(L) - 1\big) \gneqq I(L).$$

By repeating this for vertices of degree two in L_1, and so on, we must stop sooner or later with a nonempty complex L' without vertices of degree two. □

RESTRICTION 6. *If L is not equal to any of the two complexes of Figure 2, then for every cycle C on ∂L of length c, the sum of degrees (in L) of vertices of C is at least $3c + 3$.*

Proof. Triangulate the cycles of $\partial L \setminus C$ to get a triangulation T of a surface with C as the boundary. (Note that T is not necessarily a subcomplex of some K in $\mathcal{T}_k^S$.) It is clear that $I(T) \geqq I(L)$. Degrees on C are the same in L and in T. Hence it suffices to prove that their sum (in T) is at least $3c + 3$. Since $I(T) \geqq 1$, it can be shown that T must contain an inner vertex. By using this and Restriction 5, it can be shown that the only two possibilities for T with the sum of degrees $\leqq 3c + 2$ are the complexes represented in Figure 2. The details are left to the reader. □



(a) (b)

Figure 2.

Restriction 7. *Each cycle C in ∂L bounds on S.*

Proof. By "induction" on $\chi(S)$. If S is the 2-sphere, i.e., $\chi(S) = 2$, then the restriction is clear. Suppose now that $0 \leqq \chi(S) < 2$ and that we have proved all the results of this paper for surfaces S' with $\chi(S') > \chi(S)$. Suppose, moreover, that ∂L contains a cycle C which is not bounding. Then, since ∂L is bounding, L is homeomorphic to a subset of a surface S' with $\chi(S') > \chi(S)$, and by Proposition 2.1 there is $N' \in \mathcal{T}_k^{S'}$ containing L as a subcomplex. By "induction hypothesis", L satisfies the main bounds of Theorems 4.1 and 4.2 for the surface S'. Consequently, it fits also the larger upper bounds for S. □

Restriction 8. *The number n of components of ∂L is less or equal to the number of ends of K.*

Proof. Let C be a cycle of ∂L. If C bounds a finite subcomplex of K, then add the triangles of this subcomplex to L. By doing this for all cycles of ∂L we obtain L' in which there is at least one end bounded by every boundary cycle. □

To conclude, we state the main result of this section more explicitly.

Theorem 3.1. *Let $K \in \mathcal{T}_k^S$ and let L be a finite subcomplex of L. Then there is a triangulation $K' \in \mathcal{T}_k^S$ and a finite subcomplex L' of K' such that:*

(a) *L' has properties determined by Restrictions 1, 2, 3, 4, 5, 7 and 8.*
(b) *If L is none of the examples of Figure 2, then L' also fits the Restriction 6.*
(c) $q_0(L') \leqq \max\{q_0(L), k\}$.
(d) $I(L') \geqq I(L)$.
(e) *The number of ends of K' is (less or) equal to the number of ends of K.*

□

4. The main bound

Throughout this section we assume S, K and L are given and they satisfy the restrictions stated for L' in Theorem 3.1. The restrictions 1 – 6 enable us to derive some additional equalities and inequalities. By Restriction 1, every edge of ∂L is contained in one triangle of L, and every inner edge is contained in two triangles, thus

$$3r = 2q_1 + q_0 . \tag{4.1}$$

By Restriction 3 we have

$$p_0 = q_0 \tag{4.2}$$

and by Restrictions 2 and 7, the Euler formula is

$$p - q + r = \chi(S) - n. \tag{4.3}$$

By using (2.3) and (4.1) we get from (4.3) equality $2p - q_0 - r = 2\chi(S) - 2n$, and then by (2.6) and (2.7):

$$2p - r - \sum_i (i-2)B_i = 2\chi(S). \tag{4.4}$$

Let us count the sum of degrees of vertices of L. Since the inner vertices have the degree at least k, it is clear that:

$$\sum_{v \in V(G)} \deg(v) \geqq kp_1 + \sum_i d_i = kp_1 + d.$$

But by counting the degrees, each edge is counted twice, thus $2q \geqq kp_1 + d$. After eliminating q and p_1 by using (2.2), (2.3), (4.2), and (4.1), we have:

$$kp - 3r - (k+1)q_0 + d \leqq 0. \tag{4.5}$$

From this inequality and (4.4) we eliminate p, obtaining:

$$(k-6)r - 2(k+1)q_0 + 2d + k\sum_i (i-2)B_i \leqq -2k\chi(S). \tag{4.6}$$

By Restriction 6, with the only exception of complexes of Figure 2, we may assume that the sum of degrees on a boundary i-gon of L is at least $3i+3$. So $d = \sum d_j \geqq \sum (3i+3)B_i$, and we have:

$$-d + 3\sum_i (i+1)B_i \leqq 0.$$

Now (4.6) and the above inequality imply that

$$(k-6)r - 2(k+1)q_0 + \sum_i (ki - 2k + 6i + 6)B_i \leqq -2k\chi(S). \tag{4.7}$$

By adding the equation (2.6) multiplied by $\frac{1}{3}(k+24)$, the last inequality transforms to

$$3(k-6)r - (5k-18)q_0 + (2k-6)\sum_{i \geqq 3} (i-3)B_i \leqq -6k\chi(S). \tag{4.8}$$

By observing that $(2k-6)(i-3)B_i \geqq 0$ for each $i \geqq 3$, we derive:

$$r \leqq \frac{5k-18}{3(k-6)} q_0 - \frac{2k\chi(S)}{k-6}. \tag{4.9}$$

We recall once more that (4.9) holds for all complexes satisfying our restrictions, except possibly for the complexes of Figure 2. But those clearly fit in the equality (4.9). Our main theorem follows:

THEOREM 4.1. *Let $K \in \mathcal{T}_k^S$ ($k \geqq 7$) and let L be a finite subcomplex of K. Then*

$$I(L) \leqq \frac{5k-18}{3(k-6)} - \frac{2k\chi(S)}{(k-6)q_0(L)}\,. \tag{4.10}$$

This bound is sharp.

Proof. Let L' be a complex obtained from L by Theorem 3.1. We have shown by (4.9) that $I(L') \leqq (5k-18)/(3(k-6)) - 2k\chi(S)/[(k-6)q_0(L')]$. Since $I(L) \leqq I(L')$, the same bound holds for L. Since $q_0(L') \geqq 3$, this bound is greater or equal to 1. Hence, the Restriction 4 may be assumed to hold already for L.

Note that in this case the condition (c) of Theorem 3.1 reduces to

$$q_0(L') \leqq q_0(L),$$

and (4.10) follows from the above.

Sharpness of (4.10) will be proved in the next section by exhibiting infinitely many examples for which equality holds. □

Finally we shall consider a related problem. Let $\mathcal{T}_{k,m}^S$ be the class of all triangulations from $\mathcal{T}_k^S$ which have at most m ends. Note that the reduction $K \to K'$ of Theorem 3.1 preserves the classes $\mathcal{T}_{k,m}^S$, i.e. if $K \in \mathcal{T}_{k,m}^S$, then also $K' \in \mathcal{T}_{k,m}^S$. The problem of finding the maximal $I(L)$ in the class $\mathcal{T}_{k,m}^S$ can have a lower maximal value than the unrestricted problem in $\mathcal{T}_k^S$. Note that, by Restriction 8, we have another inequality:

$$n \leqq m\,. \tag{4.11}$$

From inequality (4.8), and using (2.6) and (2.7) we obtain:

$$(k-6)r - (k-4)q_0 \leqq -2k\chi(S) + 2(k-3)n\,. \tag{4.12}$$

If the right-hand side of (4.12) is non-negative, i.e. $(k-3)n \geqq k\chi(S)$, then, we get by using $q_0 \geqq 3n$ and $n \leqq m$:

$$\begin{aligned}(k-6)I(L) &\leqq k-4+\big(-2k\chi(S)+2(k-3)n\big)/q_0\\ &\leqq k-4-\frac{2k\chi(S)}{3n}+2(k-3)/3\\ &\leqq \big[5k-18-2k\chi(S)/m\big]/3\,.\end{aligned} \tag{4.13}$$

Otherwise, if $(k-3)n < k\chi(S)$, we get from (4.12):

$$(k-6)I(L) \leqq k-4-2/q_0\,. \tag{4.14}$$

Note that this bound was obtained under the assumption that L satisfies Restrictions 1–8, and in particular that $I(L) \geqq 1$ and that L does not equal either of the exceptions from Figure 2.

THEOREM 4.2. *Let $K \in \mathcal{T}^S_{k,m}$ and let L be a finite subcomplex of K. Then*

$$I(L) \leqq \frac{\left(5 - \frac{2\chi(S)}{m}\right)k - 18}{3(k-6)}, \qquad \text{if} \quad (k-3)m \geqq k\chi(S),$$

and otherwise

$$I(L) \leqq \frac{k-4}{k-6} - \frac{2}{(k-6)q_0(L)}.$$

Proof. Note that the first bound is always greater than the second and this bound is $> \frac{k-1}{k-2}$. Thus, this result follows from (4.13). To get the second bound, proceed as follows: If $I(L) \leqq 1$ or L is one of the exceptions in Figure 2, then $I(L) \leqq (k-1)/(k-2)$. But this is less than the bound in the theorem. Otherwise, (4.14) applies. □

Note. The first bound of Theorem 4.2 is sharp in many cases (see Section 5), while the second bound is not.

5. Sharpness of our Bounds

The bound (4.10) of Theorem 4.1 is sharp in the sense that there exist examples which attain this bound. For $k = 7, 8, 9, 10, 11, 12, 13$ and 15 we shall exhibit infinitely many planar examples for which equality in (4.10) holds. Thus also the bound $I(L) < (5k-18)/(3k-18)$ is the best possible for subcomplexes L of triangulations from $\mathcal{T}^S_k$.

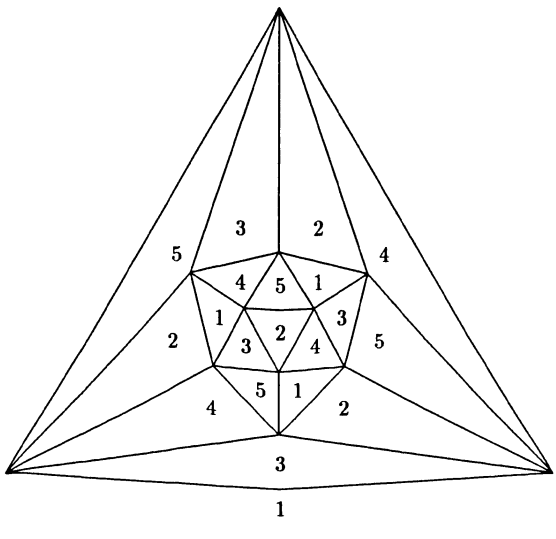


Figure 3.

We start constructing complex L by taking a *basic piece*. For $k = 7$, 9, 11, 13, 15, the basic piece is represented in Figure 3, and for $k = 8, 10, 12$ it is shown in Figure 4.

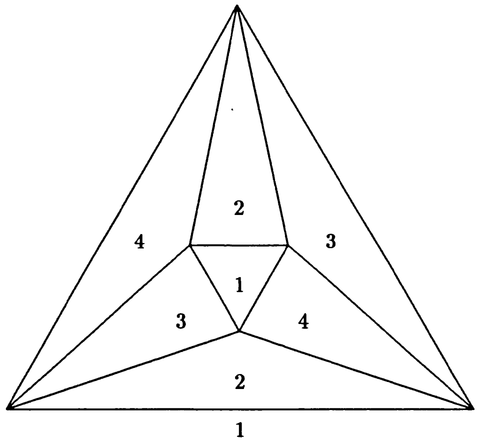


Figure 4

Each basic piece has some marked triangles which we call *distinguished triangles*. For several values of k the distinguished triangles are as follows:

$k = 7$: triangles marked 1 (in Figure 3)
$k = 9$: triangles marked 1 or 2
$k = 11$: triangles marked $1, 2$ or 3
$k = 13$: triangles marked $1, 2, 3$, or 4
$k = 15$: all triangles
$k = 8$: triangles marked 1 or 2 (in Figure 4)
$k = 10$: triangles marked $1, 2$ or 3
$k = 12$: all triangles

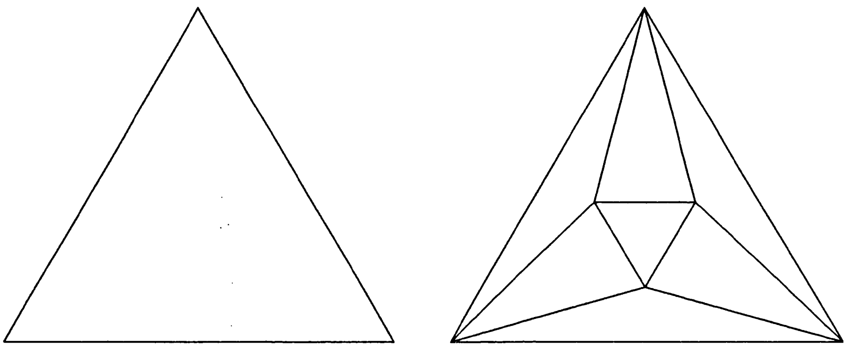


Figure 5.

In any one of the distinguished triangles we may add another basic piece together with six other triangles in between as shown in Figure 5 (always in such a way that the outside triangle of the basic piece which is added in the middle, is distinguished). Thus we obtain a larger picture with several new distinguished triangles instead of the one which was " filled up". We may continue this process in an arbitrary way and as long as we want.

Finally we define L as follows. At the beginning, in the starting basic piece, L consists of those triangles which are not distinguished. By each addition of a basic piece into a distinguished triangle we add to L the six new triangles and the non-distinguished triangles of the added basic piece. After we stop adding basic pieces we fill up each of the remaining distinguished triangles by adding an octahedron in it, i.e. adding a triangle instead of the basic piece, as shown in Figure 5. For each such filling we add to L the six triangles of the octahedron, with the exception of the middle one.

Note that, by Proposition 2.1, L is a subcomplex of a triangulation in $\mathcal{T}_k^{\text{sphere}}$, and that the boundary ∂L of L consists of disjoint triangles, each having the sum of degrees at its boundary equal to 12. Consequently, it is easy to see that for L, the inequalities (4.5) and (4.7) are, in fact, equalities, and so is (4.8). Since $B_i = 0$ for $i \geqq 4$, (4.8) is equivalent to (4.10). By verifying that L satisfies restrictions $1-6$, we see that this is an example which attains the bound (4.10).

The same examples also show that the bound of Theorem 4.2 is sharp, at least for values of k treated above and for those numbers m of ends which can be obtained as the numbers of distinguished triangles (i.e., for $k = 7$ every even $m \geqq 4$ is attainable).

6. Bounds on isoperimetric numbers

Let $\mathcal{C}_k^S$ and $\mathcal{C}_{k,m}^S$ ($k \geqq 7$, $m \geqq 1$) be the classes of cubic graphs which arise as duals of triangulations in $\mathcal{T}_k^S$ and $\mathcal{T}_{k,m}^S$, respectively. It is easy to see that Theorems 4.1 and 4.2 imply the following results:

THEOREM 6.1. *Let $G \in \mathcal{C}_k^S$ ($k \geqq 7$). Then its isoperimetric number $i(G)$ is bounded by:*

$$i(G) \geqq \frac{3(k-6)}{5k-18}$$

and this bound is the best possible. □

THEOREM 6.2. *Let $G \in \mathcal{C}_{k,m}^S$. Then*

$$i(G) \geqq \frac{3(k-6)}{(5-2\chi(S)/m)k-18}, \qquad \text{if} \quad (k-3)m \geqq k\chi(S),$$

and

$$i(G) \geqq \frac{k-6}{k-4}, \qquad \textit{otherwise}.$$

□

It is worth mentioning that to bound the isoperimetric number of a graph, many times inequalities involving the spectrum of the graph are used (e.g. Cheeger-like [3] inequality [1, 7]). At this place we shall undertake the converse way. Using inequalities of Theorems 6.1 and 6.2 we shall derive bounds for the spectral radii of graphs in $\mathcal{C}_k^S$ and $\mathcal{C}_{k,m}^S$.

COROLLARY 6.3. *Let $G \in \mathcal{C}_k^S$ $(k \geqq 7)$. Then its spectral radius $\rho(G)$ is at most*

$$\rho(G) \leqq \frac{6}{5k-18}\sqrt{6(k-4)(k-3)}.$$

Proof. It is shown in [7] that $\rho^2(G) \leqq 9 - i^2(G)$. The inequality of the corollary then follows from Theorem 6.1. □

In the same way we get from Theorem 6.2 the following corollary:

COROLLARY 6.4. *Let $G \in \mathcal{C}_{k,m}^S$. Then*

$$\rho(G) \leqq \frac{6\sqrt{6\left(k-4-\frac{\chi(S)}{3m}k\right)\left(k-3-\frac{\chi(S)}{2m}k\right)}}{\left(5-\frac{2\chi(S)}{m}\right)k-18} \qquad \textit{if} \quad (k-3)m \geqq k\chi(S),$$

and

$$\rho(G) \leqq \frac{2}{k-4}\sqrt{(k-3)(2k-9)} \qquad \textit{otherwise}.$$

□

7. Cyclically t-edge connected graphs

Recall that a graph G is *cyclically t-edge connected* if the omission of any t or fewer edges results in a graph having at most one component that contains cycles.

Let $N \in \mathcal{T}_k^S$, L be a finite subcomplex of N, and $G \in \mathcal{C}_k^S$ the dual graph of N. If G is cyclically t-edge connected, then we have, besides the restrictions of Section 3, also the following one:

Every boundary cycle of L is of length at least t.

Since, if C is a cycle in ∂L of length $t-1$ or less, the removal of the edges dual to C will disconnect G (by Restriction 7). The dual graph of L contains a cycle since $I(L)$ is assumed to be at least 1. By the proof of Restriction 8, C bounds an end of N, hence also the dual of this part contains cycles, a contradiction with the cyclic t-edge connectivity of G.

By the above restriction, $B_i = 0$ for $i < t$. Thus, if we multiply (2.6) by $(kt - 2k + 6t + 6)/t$ and subtract it from inequality (4.7), we get:

$$t(k-6)r - \big[k(t+2) - 4t - 6\big]q_0 + (2k-6)\sum_{i \geqq t}(i-t)B_i \leqq -2kt\chi(S)\,. \tag{7.1}$$

Now, since $(2k-6)(i-t)B_i \geqq 0$ for each i, an improved bound for $i(G)$ follows:

THEOREM 7.1. *If $G \in \mathcal{C}_k^S$ is cyclically t-edge connected, then*

$$i(G) \geqq \frac{t(k-6)}{(t+2)k - 4t - 6}\,. \tag{7.2}$$

□

We remark that the above lower bound is increasing as a function of t, and the highest one is obtained for $t = k$ (no graph from $\mathcal{C}_k^S \setminus \mathcal{C}_{k+1}^S$ is cyclically $(k+1)$-edge connected).

The corresponding improved bounds for cyclically t-edge connected graphs from $\mathcal{C}_{k,m}^S$ can be derived from (7.1) in the same way as Theorem 4.2 is derived from (4.8).

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