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Mathematica Slovaca, Vol. 35 (1985), No. 2, 185--193

Persistent URL: `http://dml.cz/dmlcz/128590`

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TOPOLOGIES CORRESPONDING TO METRICS ON L-GROUPS

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Choe, Conrad, Jakubík, Holland, Wolk and other authors dealt with Birkhoff's problem: "Which directed groups are topological groups and topological lattices in the interval topology?"

They investigated the classes of lattice ordered groups which form topological groups with respect to the interval topology.

In the presented paper there are investigated valuations on a lattice ordered group G with values in an abelian lattice ordered group H; further there is studied a topology on G which is defined by means of positive valuations.

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§1. Metrics on l-groups

Definition. Let v: $G \to H$ be a mapping of an l-group $(G, +, \vee, \wedge)$ into a commutative l-group $(H, +, \vee, \wedge)$ fulfilling the condition:

$$v(x)+v(y)=v(x\vee y)+v(x\wedge y), \text{ for all } x, y\in G. \tag{I}$$

Then v is called *an l-valuation on G.*

If there holds $x\geqslant y \Rightarrow v(x)\geqslant v(y)$ $(x>y\Rightarrow v(x)>v(y)$, respectively, for $x, y\in G$, then an l-valuation v on G is called *isotone* (*positive*, respectively) or shortly, *an li-valuation* (*an lp-valuation*, respectively).

A particular case of this notion (in the case when H is the additive group of all reals with the natural linear order) was investigated by G. Birkhoff [1, p. 230].

Let us remark that the condition (I) is trivial in the case of a linear order on H and the condition (I) has the form $v(x)+v(y)=v(x+y)+v(0)$ in the case of $x, y\in G$, $x\wedge y=0$, which is near the condition defining the group homomorphism.

1.1. Proposition. *For an l-valuation v on an l-group G there holds:* $v(x) = v(x^+)+v(x^-)-v(0)$, *for all* $x\in G$.

Let us now consider the notion of a metric.

1.2. Proposition. *If v: $G \to H$ is an li-valuation on an l-group G and d: $G \times G \to H$ is a mapping such that $d(x, y) = v(v \vee y) - v(x \wedge y)$, for all $x, y \in G$, then there holds:*

a) $d(x, x) = 0$, $d(x, y) = d(y, x)$, $d(x, y) \geqslant 0$,

b) $d(x, y) + d(y, z) \geqslant d(x, z)$,

c) $d(a \vee x, a \vee y) + d(a \wedge x, a \wedge y) = d(x, y)$,

for all $x, y, z, a \in G$.

Proof. The assertions of *a*) are obviously valid.

c) $d(a \vee x, a \vee y) + d(a \wedge x, a \wedge y) = \{v[(a \vee x) \vee (a \vee y)] - v[(a \vee x) \wedge (a \vee y)]\} + \{v[(a \wedge x) \vee (a \wedge y)] - v[(a \wedge x) \wedge (a \wedge y)]\} = v(a \vee (x \vee y)) - v(a \vee (x \wedge y)) + v(a \wedge (x \vee y)) - v(a \wedge (x \wedge y)) = [v(a) + v(x \vee y)] - [v(a) + v(x \wedge y)] = v(x \vee y) - v(x \wedge y) = d(x, y)$.

b) With regard to *a*) and *c*) we have $d(x, y) + d(y, z) = d(x \quad y, y) + d(x \wedge y, y) + d(y \vee z, y) + d(y \wedge z, y) \geqslant d(x \vee y \vee z, y \vee z) + d(y \wedge x, y) + d(y \vee z, y) + d(x \wedge y \wedge z, x \wedge y) = v(x \vee y \vee z) - v(y \vee z) + v(y) - v(y \wedge x) + v(y \vee z) - v(y) + v(x \wedge y) - v(x \wedge y \wedge z) \geqslant v(x \vee y) - v(x \wedge y) = d(x, y)$.

Definition. Let v: $G \to H$ be an li-valuation on an l-group G. Then a mapping d: $G \times G \to H$ is called *an l-pseudometric mapping on* G iff there holds:

$$d(x, y) = v(x \vee y) - v(x \wedge y), \text{ for all } x, y \in G. \tag{II}$$

For an l-pseudometric mapping d on G there holds:

$$d(x \vee y, y) = d(x, x \wedge y), \text{ for } x, y \in G.$$

1.3. Proposition. *Let v be an li-valuation on an l-group G and d be an l-pseudometric mapping corresponding to. v. Then v is a positive l-valuation iff there holds:*

$$d(x, y) = 0 \Rightarrow x = y, \text{ for } x, y \in G.$$

Proof. 1. If v is an lp-valuation and $x, y \in G$, $x \neq y$, then $x \vee y > x \wedge y$ and $d(x, y) = v(x \vee y) - v(x \wedge y) > 0$ holds.

2. If the l-valuation v fails to be positive, then there exist $x, y \in G$ such that $x > y$ and $v(x)$ non$>$ $v(y)$ hold. We have $0 \leqslant d(x, y) = v(x) - v(y)$, i.e., $v(x) \geqslant v(y)$. Further $v(x) = v(y)$ and $d(x, y) = 0$ implies $x = y$, a contradiction.

Definition. Let v: $G \to H$ be an lp-valuation on an l-group G. Then a mapping d: $G \times G \to H$ fulfilling (*II*) is called *an l-metric mapping on G (an l-metric on G)*.

Let us put now a natural condition on an l-metric.

Definition. Let d be an l-metric on an l-group G. Then we say that d *is compatible with group operation* if and only if there holds:

$$d(x+a, y+a) = d(x, y), \text{ for all } x, y \in G. \qquad (III)$$

Let us introduce a mapping $\bar{v}$: $G \to H$ in the following way: $\bar{v}(g) = v(g) - v(0)$, for all $g \in G$. Then there holds:

1.4. Theorem. *Let v be an lp-valuation on an l-group G and d be an l-metric corresponding to v. Then d is compatible with group operation iff the mapping $\bar{v}$ is a group homomorphism.*

Proof. $\Leftarrow$: $d(x+a, y+a) = v[(x+a) \vee (y+a)] - v[(x+a) \wedge (y+a)] = v[(x \vee y)+a] - v[(x \wedge y)+a] = \bar{v}[(x \vee y)+a] + v(0) - \{\bar{v}[(x \wedge y)+a] + v(0)\} = \bar{v}(x \vee y) + \bar{v}(a) - \bar{v}(x \wedge y) - \bar{v}(a) = \bar{v}(x \vee y) - \bar{v}(x \wedge y) = v(x \vee y) - v(0) - [v(x \wedge y) - v(0)] = v(x \vee y) - v(x \wedge y) = d(x, y)$.

$\Rightarrow$: Let $x, y \in G$. Denote $x \wedge y = u$, $u \wedge 0 = u_1$. Then we have $\bar{v}(x+y) = [\bar{v}(x+y) - \bar{v}(x+u)] + [\bar{v}(x+u) - \bar{u}(x+u_1)] + [\bar{v}(x+u_1) - \bar{v}(x)] + \bar{v}(x) = \bar{v}(x) + d(x+y, x+u) + d(x+u, x+u_1) - d(x+u_1, x+0) = \bar{v}(x) + d(y, u) + d(u, u_1) - d(u_1, 0) = \bar{v}(x) - d(0, u_1) + d(u_1, u) + d(u, y) = \bar{v}(x) - (\bar{v}(0) - \bar{v}(u_1)) + (\bar{v}(u) - \bar{v}(u_1)) + (\bar{v}(y) - \bar{v}(u)) = \bar{v}(x) + \bar{v}(y)$.

1.5. Theorem. *Let v be an lp-valuation on an l-group G, d be an l-metric corresponding to v. Then the following assertions are equivalent:*

1. *$\bar{v}$ is a group homomorphism.*
2. *$d(x+a, y+a) = d(x, y)$, for each $a, x, y \in G$.*
3. *$d(a+x, a+y) = d(x, y)$, for each $a, x, y \in G$.*

Proof. 1.$\Leftrightarrow$2. (see 1.4). The relation 1.$\Leftrightarrow$3. can be verified analogously.

1.6. Proposition. *Let d be an l-metric compatible with group operation on an l-group G and v be a corresponding lp-valuation. Then there holds:*

1. *$d(x, y) = v(|x-y|) - v(0)$,*
2. *$d(x, y) = d(-x, -y)$,*
3. *$v(x) - v(0) = d(x^+, 0) - d(x^-, 0)$, for all $x, y \in G$.*

Proof. 1. $d(x, y) = d(x - (x \wedge y), y - (x \wedge y)) = v[(x - (x \wedge y)) \vee (y - (x \wedge y))] - v(0) = v[(x \vee y) - (x \wedge y)] - v(0) = v(|x-y|) - v(0)$.

2. $d(x, y) = d(x-y, 0) = d(-y, -x) = d(-x, -y)$.

3. $v(x) = \bar{v}(x) + v(0) = \bar{v}(x^+ + x^-) + v(0) = \bar{v}(x^+) + \bar{v}(x^-) + v(0) = v(x^+) + v(x^-) - v(0) = [d(x^+, 0) + v(0)] + [v(0) - d(x^-, 0)] - v(0) = d(x^+, 0) - d(x^-, 0) + v(0)$.

§2. Topologies corresponding to l-metrics

Now, let us investigate topologies corresponding to l-metrics on l-groups. We shall use the following notation in this paragraph:

Let G be an l-group, $G \neq \{0\}$, H be a commutative l-group, v: $G \to H$ be an lp-valuation on G, which defines an l-metric d on G compatible with group operation and let $\bar{v}$ be a group homomorphism corresponding to the lp-valuation v. Let us denote $U_h = \{g \in G: \bar{v}(|g|) < |h|\}$, for $0 \neq h \in H$ and $\Omega_g^H = \{U_h + g: 0 \neq h \in H\}$, for each $g \in G$. Then we have $\Omega_g^H = \Omega_0^H + g$, for each $g \in G$ and $U_h = \{g \in G: d(0, g) < |h|\}$ (see 1.6). Then:

2.1. Proposition. $\bigcap \Omega_0^H = \{0\}$ *holds.*

Proof. If $x \in \bigcap \Omega_0^H = \bigcap \{U_h: 0 \neq h \in H\}$ then $\bar{v}(|x|) < |h|$, for each $0 \neq h \in H$. But $\bar{v}(|x|) \neq 0$ implies $\bar{v}(|x|) < \bar{v}(|x|)$, contradiction. Thus $v(|x|) = 0$ and $0 = \bar{v}(|x|) = d(0, x)$ (see 1.6). Finally, $x = 0$ (see 1.3).

Remark. *Ker* $\bar{v}$ is a normal subgroup in G in the case when d is an l-metric compatible with group operation (see 1.4). *Ker* $\bar{v}$ is not an l-ideal in G and thus $\bar{v}$ is no l-homomorphism in general. This fact follows from Proposition 2.2.

2.2. Proposition. *If* $x, y \in Ker\, \bar{v}$, $x \neq y$, *then* $x \parallel y$.

Proof. If $x, y \in Ker\, \bar{v}, y < x$, then $x - y > 0$ and $0 = \bar{v}(0) < \bar{v}(x - y) = \bar{v}(x) - \bar{v}(y)$, a contradiction.

Definition. Let G be an l-group and $\emptyset \neq M \subseteq G$, $0 \in M$. Then we say that *M is dense in G* iff for each $g_1, g_2 \in G$ such that $g_1 > g_2$ there exists an element $m \in M$ with the property $g_2 < m < g_1$.

If a subset in G exists which is dense in G, then we say that *G is dense.*

2.3. Theorem. *Under the above denotations the following assertions are equivalent:*

1. *Ω_g^H is a complete system of neighbourhoods of $g \in G$ of a nondiscrete topology on G.*
2. *H is a linearly ordered group and $Im\, \bar{v}$ is dense in H.*

Proof. 1. $\Rightarrow$ 2.: If H is not linearly ordered, then elements $h, k \in H$ exists such that $h \wedge k = 0$. Then $x \in U_h \cap U_k$ implies $\bar{v}(|x|) \leqslant h \wedge k = 0$ and thus $|x| \in Ker\, \bar{v}$. From this and from 2.2 $x = 0$ follows, i.e., $U_h \cap U_k = \{0\}$, a contradiction. Further, for each $h \in H$, $h > 0$ we have $U_h \neq \{0\}$ and this implies the existence of $g \in G$ such that $0 < \bar{v}(|g|) < h$. It means that $Im\, \bar{v}$ is dense in H.

2. $\Rightarrow$.1.: If H is linearly ordered and $U_h, U_k \in \Omega_0^H$, then we can suppose that $|k| \geqslant |h|$ and thus $U_k \supseteq U_h$. It means that $(\Omega_g^H, \supseteq)$ is a linearly ordered set. From the density of $Im\, \bar{v}$ in H it follows that each $U_h \in \Omega_0^H$ is a non zero set and thus $U_h + g \neq \{g\}$, for each $g \in G$.

The system Ω_g^H is a complete system of neighbourhoods of an element g, for each $g \in G$, in a nondiscrete topology.

Definition. This topology is called *the topology corresponding to the l-metric on* G and denoted $\tau(\Omega_0^H)$.

Remark. 1. $(G, \tau(\Omega_0^H))$ is a Kuratowski space (see 2.1).

2. We can easily see that $\tau(\Omega_0^H)=\tau(\Omega_0^{Im\,\bar{v}})$ and then we shall denote $\tau(\Omega_0)$ only. For the investigation of the topology $\tau(\Omega_0)$ it is sufficient to consider an epimorphism $\bar{v}\colon G\to Im\,\bar{v}$.

Until the end of section 2 we assume that H is a linearly ordered group and $Im\,\bar{v}$ is dense in H.

2.4. Lemma. *Let H be a dense linearly ordered group. Then for each $h\in H$, $0<h$ there exists $x\in H$ such that $0<2x<h$.*

Proof. From the density of H there follows the existence of $x\in H$ such that $0<x<h$. If $2x$ non$<h$, then we choose $0<x_1<h-x$. Thus $2x\geqslant h$, i.e., $x\geqslant h-x>x_1$ and further $h=h-x+x>2x_1$.

Definition. *Let (G,Σ) be a topological group and let $(G,\geqslant)$ be a partially ordered group. A topology $\tau(\Sigma)$, defined by a complete system Σ of neighbourhoods of zero, is called a weak locally convex topology,* iff *for each $U\in\Sigma$ there exists $V\in\Sigma$ with the property:*

$$v_1,\ v_2\in V,\ g\in G,\ v_1\geqslant g\geqslant v_2\Rightarrow g\in U. \qquad (K)$$

2.5. Lemma. *If $(G,\geqslant)$ is an l-group and (G,Σ) is a topological group with a weak locally convex topology, then $(G,\geqslant,\Sigma)$ is a tl-group.*

Proof. If $U\in\Sigma$ and $g\in G$, then there exist $V,W\in\Sigma$ such that $\pm W\subseteq V\subseteq U$ and V has the property (K). It implies that for each element $x\in W$ there holds $0\leqslant|-g^+\vee(x+g^-)| = [-g^+\vee(x+g^-)]\vee[g^+\wedge(-g^--x)]\leqslant(0\vee x)\vee[(g^++|x|)\wedge(-g^-+|x|)]=(0\vee x)\vee[(g^+\wedge-g^-)+|x|]=|x|$. From this we have $-x\leqslant-g^+\vee(x+g^-)\leqslant x$ and $-x,x\in V$ implies $-g^+\vee(x+g^-)\in U$. Finally, $(G,\Sigma,\geqslant)$ is a tl-group (see [4,1.1]).

2.6. Theorem. *Let G be an l-group and d be an l-metric on G. Then G is a topological l-group with the topology $\tau(\Omega_0)$ corresponding to d.*

Remark. Further, the tl-group described in Theorem 2.6 is denoted by (G,Ω_0).

Proof of Theorem 2.6. First, we shall prove that (G,Ω_0) is a topological group:

1. For each $U_h,U_k\in\Omega_0$ there exists $U_l\in\Omega_0$ such that $U_l\subseteq U_h\cap U_k$ (see 2.3).

2. If $U_k\in\Omega_0$, then we shall prove that there exists $U_x\in\Omega_0$ such that $U_x\pm U_x\subseteq U_k$:

For $|k|\in H$ there exist element $x,x_1\in H$ such that $0<2x<x_1$, $0<2x<|k|$ (see Lemma 2.4) and from this we have $0<4x<2x_1<|k|$. Then for each $a,b\in U_x$ there holds $\bar{v}(|a+b|)\leqslant\bar{v}(|a|+|b|+|a|)=\bar{v}(|a|)+\bar{v}(|b|)+\bar{v}(|a|)<3x<4x<k|$, i.e., $a+b\in U_k$. It means that $U_x+U_x\subseteq U_k$; $-U_x=U_x$ is clear.

3. If $U_k\in\Omega_0$ and $u\in U_k$, then we shall prove that there exists $U_x\in\Omega_0$ such that $U_x+u\subseteq U_k$: We have $\bar{v}(|u|)<|k|$ and there exists an element $x\in H$ such that

$0<2x<|k|-\bar{v}(|u|)$ (see 2.4). Then for each $g\in U_x$ there holds $\bar{v}(|g+u|)\leqslant \bar{v}(|g|+|u|+|g|) = \bar{v}(|g|) + \bar{v}(|u|) + \bar{v}(|g|) = 2\bar{v}(|g|) + \bar{v}(|u|)< 2x+\bar{v}(|u|)<|k|$ and thus $U_x+u\subseteq U_k$.

4. If $U_k\in\Omega_0$ and $g\in G$ then we shall prove that $-g+U_k+g\subseteq U_k$: For each $x\in U_k$ there holds $\bar{v}(|-g+x+g|) = \bar{v}(-g+|x|+g) = -\bar{v}(g) + \bar{v}(|x|) + \bar{v}(g) = \bar{v}(|x|)<|k|$, i.e., $-g+U_k+g\subseteq U_k$.

Summing up, (G,Ω_0) is a topological group. The rest of this proof follows from 2.5 when we prove that the topology $\tau(\Omega_0)$ is weak locally convex: Let $U_h\in\Omega_0$ and $U_h=\{g\in G\colon \bar{v}(|g|)<h\}$ hold. Then an element $x\in H$ exists such that $0<2x<h$ (see 2.3 and 2.4). For each $v_1, v_2\in U_x$ and $g\in G$ such that $v_1\geqslant g\geqslant v_2$ there holds: $0\leqslant\bar{v}(g^+)\leqslant\bar{v}(v_1\vee 0)\leqslant\bar{v}(|v_1|)<x$, $0\leqslant\bar{v}(-g\ )\leqslant\bar{v}(-v_2\vee 0)\leqslant\bar{v}(|v_2|)<x$. We have $\bar{v}(|g|) = \bar{v}(g^+-g^-) = \bar{v}(g^+) + \bar{v}(-g^-)<2x<h$, i.e., $g\in U_h$. We proved that for each $U_h\in\Omega_0$ there exists $U_x\in\Omega_0$ with the property (K).

2.7. Proposition. *The mapping $\bar{v}\colon G\to H$ is a continuous mapping of the topological space $(G,\tau(\Omega_0))$ into the topological space (H,ι), where ι is the interval topology on H.*

Proof. $(G,\tau(\Omega_0))$ and (H,ι) are topological groups (H is linearly ordered) and it is sufficient to prove that for each $h\in H$, $h>0$ there exists a neighbourhood $U\in\Omega_0$ such that $\bar{v}(U)\subseteq(-h,h)$. But for a neighbourhood $U_h\in\Omega_0$ there holds: If $x\in U_h$, then $\bar{v}(|x|)<h$ and $|\bar{v}(x)|\leqslant\bar{v}(|x|)<h$ hold, i.e., $-h<\bar{v}(x)<h$.

2.8. Theorem. *If an lp-valuation v on an l-group G is a lattice homomorphism, then $\bar{v}$ is an l-isomorphism and G is a commutative l-group. If, moreover $\tau(\Omega_0)$ is a nondiscrete topology corresponding to an l-metric on G, then G is a dense linearly ordered group and $\tau(\Omega_0)$ is the interval topology on G.*

Proof. It can be easily shown that v is a lattice homomorphism if and only if $\bar{v}$ is a lattice homomorphism. Then $\bar{v}$ is an l-homomorphism according to 1.4 and thus *Ker* $\bar{v}$ is an l-ideal in G. From this and from 2.2 it follows that *Ker* $\bar{v}=\{0\}$ and it implies that $\bar{v}$ is an l-isomorphism and G is a commutative l-group.

Now, if $\tau(\Omega_0)$ is a nondiscrete topology corresponding to an l-metric d on G, then H is a linearly ordered group and *Im* $\bar{v}$ is a dense set in H^+. It implies that G is a linearly ordered group also and we shall prove that G is dense: If $0<g\in G$, then $0<\bar{v}(g)$ and thus an element $x\in G$ exists such that $0<\bar{v}(x)<\bar{v}(g)$. Finally, $0<x<g$ holds. Further, if $0<g\in G$ is an element, then for each $x\in U_{v(g)}$ there holds $\bar{v}(|x|)<\bar{v}(g)$, i.e., $|x|<g$ and $U_{\bar{v}(g)}\subseteq(-g,g)$. If $x\in(-g,g)$, then $|x|<g$ and thus $\bar{v}(|x|)<\bar{v}(g)$. It means that $x\in U_{\bar{v}(g)}$. Finally, we have $(-g,g)=U_{v(g)}$ and thus $\tau(\Omega_0)$ is the interval topology.

2.9. Theorem. *Let (G,Ω_0) be a tl-group with a topology corresponding to an l-metric, K be its (topological) component and $K\neq\{0\}$ hold. Then there holds:*

1. *K is an l-ideal in G.*

2. *$\bar{v}(K)$ is l-isomorphic with the linearly ordered additive group R of real numbers.*

Proof. 1. A component K is a clopen set in G and thus there exists a neighbourhood $U \in \Omega_0$ such that $U \subseteq K$. Further, neighbourhoods $V, W \in \Omega_0$ exist such that $V \subseteq W \subseteq U$, $V \vee -V \subseteq W$ and if $w_1, w_2 \in W$, $g \in G$, $w_1 \geqslant g \geqslant w_2$ hold, then $g \in U$ (see the proof of Theorem 2.6). Now, if $k \in K$, $x \in G$ exist such that $k \geqslant x \geqslant 0$, then $k = \sum_{i=1}^{l} \pm v_i \leqslant \sum_{i=1}^{l} |v_i|$, for suitable elements $v_i \in V$ ($i = 1, 2, \ldots, l$), because K is a subgroup in G generated by V. We have $|v_i| = v_i \vee -v_i \in W$ and $\sum_{i-1}^{l} |v_i| \geqslant x \geqslant 0$. With regard to [2, p. 105, Cor. 2] there exist elements $x_1, x_2, \ldots, x_n \in G$ such that $v_i \geqslant x_i \geqslant 0$ for $i = 1, 2, \ldots, l$ and $x = \sum_{i=1}^{l} x_i$. The fact $v_i \in W$ implies $x_i \in U$ for $i = 1, 2, \ldots, n$ and together $x = \sum_{i=1}^{l} x_i \in \langle U \rangle = K$ holds, where $\langle U \rangle$ is the subgroup in G generated by U. K is a convex normal subgroup and therefore an l-ideal in G.

2. First, we prove that $\bar{v}(K)$ is an archimedean linearly ordered group: If elements $h_1, h_2 \in \bar{v}(K)$ exist such that $0 < h_1$, $0 < h_2$ and $nh_2 < h_1$, for each natural number n, then there exists a neighbourhood $U \in \Omega_0$, $U \subseteq U_{h_2}$ such that $U \subseteq K$. For each $x \in \langle U \rangle$ there holds $x = \sum_{i=1}^{l} \pm u_i$, for suitable elements $u_i \in U$ ($i = 1, 2, \ldots, 1$). From this we have $\bar{v}(|x|) = \bar{v}\left(\left|\sum_{i=1}^{l} \pm u_i\right|\right) \leqslant \sum_{i=1}^{l} \bar{v}(|u_i|) < l \cdot h_2 < h_1$. Further we have $K = \langle U \rangle \subseteq U_{h_1}$. It means that $\bar{v}(|a|) < h_1$, for each $a \in K$. Now, there exists an element $t \in K$ such that $\bar{v}(t) = h_2$, i.e., $\bar{v}(|a|) < \bar{v}(t)$, for each $a \in K$. It implies a contradiction for $a = t$. Finally, $\bar{v}(K)$ is an archimedean linearly ordered group and $\bar{v}(K)$ is l-isomorphic with a subgroup T of R. The mapping $\bar{v}$ is continuous (see 2.7) and thus $\bar{v}(K)$ is a connected subgroup in (H, ι), where ι is the interval topology. $\bar{v}(K)$ is also a closed subset in (H, ι) (see [5,1.4 and 1.3]). If we denote this l-isomorphism by $f\colon \bar{v}(K) \to T$, then f is a continuous mapping of the topological space $(\bar{v}(K), \iota)$ onto the topological space (T, ι). Further, (T, ι) is a connected space. It means that $(T, +, \iota)$ is a connected subgroup in $(R, +, \iota)$, which implies $T = R$.

§ 3. Another topology corresponding to an lp-valuation

Let us investigate another topology corresponding to an lp-valuation, namely: Let G be an l-group, $G \neq \{0\}$, H be a commutative l-group, $v\colon G \to H$ be an lp-valuation on G which defines an l-metric d compatible with a group operation and $\bar{v}$ be a group homomorphsm corresponding to an lp-valuation v (see Theorem 1.3). Let us denote $U^h=\{g\in G\colon \bar{v}(|g|)\ \text{non}\geqslant |h|\}$, for each $h\in H$, $h\neq 0$ and $\Omega^g=\{U^h+g\colon 0\neq h\in H\}$ for each $g\in G$. Then $\Omega^g=\Omega^0+g$, for each $g\in G$ and $U^h=\{g\in G\colon d(0,g)\ \text{non}\geqslant |h|\}$ (see 1.6).

3.1. Proposition. $\bigcap\Omega^0=\{0\}$.

Proof. For $x\in\bigcap\Omega^0=\bigcap\{U^h\colon 0\neq h\in H\}$ there holds $\bar{v}(|x|)<|h|$ or $\bar{v}(|x|)\parallel |h|$, for $0\neq h\in H$. If $\bar{v}(|x|)\neq 0$, then $\bar{v}(|x|)<\bar{v}(|x|)$ or $\bar{v}(|x|)\parallel\bar{v}(|x|)$, a contradiction in both cases. Thus $\bar{v}(|x|)=0$ and $0=\bar{v}(|x|)=d(0,x)$ — see 1.6. Finally $x=0$ (see 1.3) and $\bigcap\Omega^0=\{0\}$.

3.2. Theorem. *The following assertions are equivalent on an l-group G:*

1. *Ω^g is a complete system of neighbourhoods of an element $g\in G$ of a topology $\tau(\Omega^0)$ on G.*
2. *For any two elements $h_1, h_2\in H^+\setminus\{0\}$ with $h_1\wedge h_2=0$ there exists an element $p\in H$, $p>0$ such that there holds:*

$$h_i\leqslant\bar{v}(|g|)\Rightarrow p\vee h_i\leqslant\bar{v}(|g|),\ \text{for each}\ g\in G,\ i\in\{1,2\}.$$

Proof. 1.$\Rightarrow$2.: There exists $p\in H$ such that $U^p\subseteq U^{h_1}\cap U^{h_2}$ and we can suppose that $p>0$ without loss of generality. If $g\in G$ exists such that $h_1\leqslant\bar{v}(|g|)$ and $p\vee h_1\ \text{non}\leqslant\bar{v}(|g|)$, for example, then $p\ \text{non}\leqslant\bar{v}(|g|)$ and thus $g\in U^p\setminus U^{h_1}$, a contradiction.

2.$\Rightarrow$1.: From the fact $\Omega^g=\Omega^0+g$ it follows that it is sufficient to prove, that for $U^{h_1}, U^{h_2}\in\Omega^0$ there exists $U^p\in\Omega^0$ such that $U^p\subseteq U^{h_1}\cap U^{h_2}$: Let $U^{h_1}, U^{h_2}\in\Omega^0$ hold. Now, we can suppose that $h_1>0$ and $h_2>0$. If there exists an element $p\in H$, $0<p\leqslant h_1\wedge h_2$, then $U^p\subseteq U^{h_1}\cap U^{h_2}$ and $U^p\in\Omega^0$. If $h_1\wedge h_2=0$, then let p be as in the condition 2. For each $g\in U^p$ we have $\bar{v}(|g|)\ \text{non}\geqslant p$. If $\bar{v}(|g|)\geqslant h_1$, for example, then $\bar{v}(|g|)\geqslant p\vee h_1$, a contradiction. It means that $g\in U^{h_1}$. Similarly $g\in U^{h_2}$ holds, i.e., $U^p\subseteq U^{h_1}\cap U^{h_2}$.

3.3. Corollary. *If $\tau(\Omega^0)$ is a nondiscrete topology on G and $\bar{v}$ is an l-homomorphism, then $\bar{v}$ is an l-isomorphism, G is a dense linearly ordered commutative group and $\tau(\Omega^0)$ is the interval topology on G.*

Proof. If $\bar{v}$ is an l-homomorphism, then analogously to the proof of 2.8 we can prove that $\bar{v}$ is an l-isomorphism. Now, if $g_1, g_2\in G$, $g_1\neq 0\neq g_2$, $g_1\wedge g_2=0$, then with regard to 3.1 there exist $h_1, h_2\in H$ such that $g_1\ \text{non}\in U^{h_1}$, $g_2\ \text{non}\in U^{h_2}$. Further, there exists $h\in H$ such that $U^h\subseteq U^{h_1}\cap U^{h_2}$, $U^h\neq\{0\}$. This implies $\bar{v}(g_1)\geqslant|h|$, $\bar{v}(g_2)\geqslant|h|$ and $\bar{v}(g_1\wedge g_2)\geqslant|h|$, $h=0$, a contradiction. Finally, G and H are linearly ordered groups, i.e., $\tau(\Omega^0)=\tau(\Omega_0)$. The rest follows from 2.8.

Remark. If (G,Σ) is a tl-group and $\bar{v}$ is an identity on G, then for each $g\in G$, $g\neq 0$ there exists a neighbourhood $U\in\Sigma$ such that $U\subseteq U^g$, (see [4,2.2]).

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Received January 12. 1983.

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ТОПОЛОГИИ, ПРИНАДЛЕЖАЩИЕ МЕТРИКАМ НА Л-ГРУППАХ

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Резюме

В этой статье исследуются оценки на структурно упорядоченной группе, которые имеют значения в абелевой структурно упорядоченной группе. Далее изучается топология, которая определена при помощи положительной оценки.