Imrich Abrhan; Ladislav Satko

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r–SYSTEMS OF UNARY ALGEBRAS II

(Maximal and minimal subalgebras of the direct products of unary algebras)

IMRICH ABRHAN * — LADISLAV SATKO **

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ABSTRACT. Maximal and minimal subalgebras of the direct product of an r-system of unary algebras are studied. They are characterized by J-simple subalgebras and J-subalgebras. These algebras are also studied in the direct product of an r-system of unary algebras.

1. Introduction

We continue in our investigation from [1]. All concepts and definitions are from this work. We only repeat notions which are most frequently used. Let $\mathbf{A} = \langle A; F\rangle$ be a unary algebra. Then $\mathcal{P}(\mathbf{A})$ is the set of all subsets $N \subseteqq A$ such that $N \neq \emptyset$ and $\langle N; F\rangle$ is a subalgebra of the algebra $\mathbf{A}$. If $\mathbf{A} = \langle A; F\rangle$ is a unary algebra and $x \in A$, then $F^+(x)$ is the set of all $y \in A$ which have the following property: There exist $f_1, \dots, f_k \in F$ such that $y = f_k(\dots f_1(x)\dots)$.

In [1], we have described maximal and greatest J-classes in the direct product of an r-system of unary algebras. Now we shall describe maximal, greatest and minimal subalgebras in this direct product. We make use of the notion of J-subalgebra of an algebra $\mathbf{A}$. This is the generalization of the notion of the maximal subalgebra. (See [4].) [1; Examples 2 and 3] show how to derive well-known results concerned minimal and maximal left ideals of the direct product of semigroups from our results.

In this introduction, we prove one theorem dealing with subalgebras generated by one element in the direct product of an r-system of unary algebras. It is convenient to start with the following lemma.

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LEMMA 1.1. *Let* $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ *be an arbitrary system of unary algebras,* $\alpha \in A = \prod(A_i \mid i \in I)$. *Then* $[\alpha] \subseteqq \prod([\alpha(i)] \mid i \in I)$.

Proof. Let $\beta \in [\alpha]$. Then either $\beta = \alpha$ or $\beta \in F^+(\alpha)$. Hence either $\beta(i) = \alpha(i)$ or $\beta(i) \in F^+(\alpha(i))$ for any $i \in I$. (See [1; Lemma 2.1].) Thus, $\beta(i) \in \{\alpha(i)\} \cup F^+(\alpha(i)) = [\alpha(i)]$ for any $i \in I$. □

THEOREM 1.1. *Let* $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ *be an* r*-system of unary algebras.* $\alpha \in A = \prod(A_i \mid i \in I)$. *Then* $[\alpha] = \prod([\alpha(i)] \mid i \in I)$ *if and only if either* $\alpha \in F^+(\alpha)$, *or there exists* $j \in I$ *such that* $\alpha(j) \notin F^+(\alpha(j))$, *and* $|[\alpha(i)]| = 1$ *for any* $i \in I$, $i \neq j$.

Proof.

a) Let us suppose $[\alpha] = \prod([\alpha(i)] \mid i \in I)$. Obviously, either $\alpha(i) \in F^+(\alpha(i))$ for any $i \in I$, or there exists such $j \in I$ that $\alpha(j) \notin F^+(\alpha(j))$. Thus, for the r-system of unary algebras we have only to prove there does not exist $j, k \in I$, $j \neq k$, such that $\alpha(j) \notin F^+(\alpha(j))$ and $|[\alpha(k)]| \geq 2$. Suppose, to the contrary, the existence of such elements. Let β be an arbitrary element of $\prod([\alpha(i)] \mid i \in I)$ such that $\beta(j) = \alpha(j)$ and $\beta(k) \neq \alpha(k)$. Evidently, such element exists, and $\beta \neq \alpha$. An assumption $[\alpha] = \prod([\alpha(i)] \mid i \in I)$ implies $\beta \in F^+(\alpha)$. However, in an r-system, $\beta \in F^+(\alpha)$ if and only if $\beta(i) \in F^+(\alpha(i))$ for any $i \in I$. This contradicts $\beta(j) = \alpha(j) \notin F^+(\alpha(j))$. Thus, if there exists $j \in I$ such that $\alpha(j) \notin F^+(\alpha(j))$, then $|[\alpha(i)]| = 1$ for any $i \in I$, $i \neq j$.

b) Let $\alpha \in F^+(\alpha)$. From this, $\alpha(i) \in F^+(\alpha(i))$ for any $i \in I$. Let β be an arbitrary element of $\prod([\alpha(i)] \mid i \in I)$. Then $\beta(i) \in [\alpha(i)] = \{\alpha(i)\} \cup F^+(\alpha(i)) = F^+(\alpha(i))$ for any $i \in I$. It means, for an r-system, $\beta \in F^+(\alpha) = [\alpha]$, and thus $\prod([\alpha(i)] \mid i \in I) \subseteqq [\alpha]$.

As the second possibility, we suppose the existence of $j \in I$ such that $\alpha(j) \notin F^+(\alpha(j))$ and $|[\alpha(i)]| = 1$ for any $i \in I$, $i \neq j$. Let β be an arbitrary element of $\prod([\alpha(i)] \mid i \in I)$. Of course, if $\beta = \alpha$, then $\beta \in [\alpha]$. The case $\beta \neq \alpha$ is possible only if $\beta(j) \neq \alpha(j)$ and $\{\beta(i)\} = \{\alpha(i)\} = F^+(\alpha(i)) = [\alpha(i)]$ for any $i \in I$, $i \neq j$. Obviously, $\beta(i) \in F^+(\alpha(i))$ for any $i \in I$, $i \neq j$. For $i = j$ we have: $\beta(j) \in [\alpha(j)]$ and $\beta(j) \neq \alpha(j)$. Thus, $\beta(j) \in F^+(\alpha(j))$. It is evident that $\beta(i) \in F^+(\alpha(i))$ for any $i \in I$. Therefore, for an r-system this implies $\beta \in F^+(\alpha)$ and consequently $\beta \in [\alpha]$. It means that, also in this case, we have $\prod([\alpha(i)] \mid i \in I) \subseteqq [\alpha]$. This completes the proof of our theorem. □

2. Minimal (J-simple) subalgebras

Let $\mathbf{A} = \langle A; F \rangle$ be a (unary) algebra without proper subalgebras. It means that $N \in \mathcal{P}(\mathbf{A})$ implies $N = A$ and $\mathcal{P}(\mathbf{A}) = \{A\}$. In such a case, for any $x \in A$, $[x]J = A$. Thus, A/J is a one-element set. Such an algebra is called a J-simple algebra.

Let $\mathbf{N} = \langle N; F \rangle$ be a subalgebra of a unary algebra $\mathbf{A} = \langle A; F \rangle$ such that for any $N' \in \mathcal{P}(\mathbf{A})$ the condition $N' \subseteqq N$ implies $N' = N$. Then the algebra $\mathbf{N}$ is called a minimal subalgebra of $\mathbf{A}$. It means that $\{N' \in \mathcal{P}(\mathbf{A}) \mid N' \subseteqq N\}$ $\{N\}$.

Let $\mathbf{A} = \langle A; F \rangle$ be a unary algebra, and $\mathbf{N} = \langle N; F \rangle$ be a subalgebra of the algebra $\mathbf{A}$. Let $N' \subseteqq N$. Then $\mathbf{N}' = \langle N'; F \rangle$ is a subalgebra of $\mathbf{A}$ if and only if it is a subalgebra of $\mathbf{N}$. Hence, for any $N' \subseteqq N$, $N' \in \mathcal{P}(\mathbf{A})$ if and only if $N' \in \mathcal{P}(\mathbf{N})$. Thus, $\mathbf{N} = \langle N; F \rangle$ is a minimal subalgebra of $\mathbf{A}$ if and only if $\mathcal{P}(\mathbf{N}) = \{N\}$. From this it follows that $\mathbf{N} = \langle N; F \rangle$ is a minimal subalgebra of a unary algebra $\mathbf{A} = \langle A; F \rangle$ if and only if $\mathbf{N}$ is a J-simple subalgebra of $\mathbf{A}$.

Let $\mathbf{A} = \langle A; F \rangle$ be a unary algebra, and $K = \bigcap\{N \mid N \in \mathcal{P}(\mathbf{A})\}$. If $K \neq \emptyset$, then $\langle K; F \rangle$ is a J-simple subalgebra of $\mathbf{A}$, and the algebra $\langle K; F \rangle$ is called the kernel of $\mathbf{A}$.

Now we focus on the connection between minimal J-classes and minimal algebras in unary algebras. The Lemmas 2.1 and 2.3 are valid also for an arbitrary universal algebra. As the definition of the set $F^+(x)$ for universal algebras is rather complicated, we state these lemmas only for unary algebras.

LEMMA 2.1. *Let $\mathbf{A} = \langle A; F \rangle$ be a (unary) algebra and $a \in A$. Then the class $[a]J$ is minimal in A/J if and only if $[a]J = [a]$.*

P r o o f.

a) The class $[a]J$ is minimal if and only if, for any $b \in [a]$, $[b] = [a]$. Let $[b] = [a]$ for any $b \in [a]$. Since $[a]J \subseteqq [a]$, for any $b \in [a]$, $b \in [b]J = [a]J \subseteqq [a]$. Therefore, $[a] \subseteqq [a]J \subseteqq [a]$.

b) Let $[a]J = [a]$. Then, for any $b \in [a] = [a]J$, $[b] = [a]$. This completes the proof. □

LEMMA 2.2. *Let $\mathbf{A} = \langle A; F \rangle$ be a (unary) algebra. Then for any nonempty subset N of A the following conditions are equivalent:*

a) *$\langle N; F \rangle$ is a J-simple subalgebra of $\mathbf{A}$.*
b) *$N = [x] = F^+(x)$ for any $x \in N$.*
c) *There exists $x \in A$ such that $N = [x]J \in \mathcal{P}(\mathbf{A})$.*
d) *There exists $x \in A$ such that $N = [x]J$, and $[x]J$ is a minimal class in the set A/J.*

Proof.

1) Obviously, a) implies b).

2) Let, for any $x \in N$, $N = [x]$. Then we have $[x]J \subseteqq [x] = N \subseteqq [x]J$. Therefore, $N = [x]J \in \mathcal{P}(\mathbf{A})$, and b) implies c).

3) Assume an existence of $x \in A$ such that $N = [x]J \in \mathcal{P}(\mathbf{A})$, and let $y \in A$ be such that $[y]J \leq [x]J = N \in \mathcal{P}(\mathbf{A})$. Thus, $[y] \subseteqq N$ and $y \in N = [x]J$. Therefore, $[y]J = [x]J$, which implies that $N = [x]J$ is a minimal element in the partially ordered set A/J. We have proved c) implies d).

4) Let $x \in A$ be such that $N = [x]J$, and $[x]J$ be a minimal element of A/J. By Lemma 2.1, $N = [x]J = [x]$. Clearly, this is true for any $x \in N$. Let $N' \in \mathcal{P}(\mathbf{A})$ have the property $\emptyset \neq N' \subseteqq N$. Then, for any $x \in N'$, $[x] \subseteqq N' \subseteqq N = [x]$. Thus, $N' = N$, and $\langle N; F\rangle$ is a J-simple subalgebra of $\mathbf{A}$. Therefore d) implies a), and the proof is complete. □

LEMMA 2.3. *Let $\mathbf{A} = \langle A; F\rangle$ be a (unary) algebra and $\emptyset \neq K \subseteqq A$. Then $\langle K; F\rangle$ is the kernel of $\mathbf{A}$ if and only if there exists $a \in A$ such that $[a]J$ is the smallest element of A/J and $K = [a]J$. (See also* [3].*)*

Proof.

1) Let $\langle K; F\rangle$ be the kernel of the algebra $\mathbf{A}$. By Lemma 2.2 c), there exists $a \in A$ such that $K = [a]J$. Let $b \in A$ be an arbitrary element. Then $a \in [a]J = K \subseteqq [b]$. Thus, $[a] \subseteqq [b]$ and $K = [a]J \leq [b]J$ for any $b \in A$. From this it easily follows that K is the smallest element of the partially ordered set A/J.

2) Let $[a]J = K$ be the smallest element in A/J. By Lemma 2.2 d), a), $\langle K; F\rangle$ is a J-simple subalgebra of $\mathbf{A}$. Then, by Lemma 2.2 b), $K = [a]$. Let N be an arbitrary element of $\mathcal{P}(\mathbf{A})$ and $b \in N$. Since $K = [a]J$ is the smallest element of A/J, we have $K = [a]J \leq [b]J$. Therefore $K = [a] \subseteqq [b] \subseteqq N$. It means that $\langle K; F\rangle$ is the kernel of the algebra $\mathbf{A}$. □

Now we focus on J-simple subalgebras in a direct product of unary algebras.

THEOREM 2.1. *Let $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ be an r-system of unary algebras, $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$. Let $N \in \mathcal{P}(\mathbf{A})$. Then $\mathbf{N} = \langle N; F\rangle$ is a J-simple subalgebra of the algebra $\mathbf{A}$ if and only if for any $i \in I$ there exists a J-simple subalgebra $\mathbf{N}_i = \langle N_i; F\rangle$ of the algebra $\mathbf{A}_i$ such that $\mathbf{N} = \prod(\mathbf{N}_i \mid i \in I)$.*

Proof.

1) Let $\mathbf{N} = \langle N; F\rangle$ be a J-simple subalgebra of $\mathbf{A}$. By Lemma 2.2 b), $N = [\alpha] = F^+(\alpha)$ for any $\alpha \in N$. From this it follows $\alpha \in F^+(\alpha)$. By Theorem 1.1, $N = [\alpha] = \prod([\alpha(i)] \mid i \in I)$. Now we show that $\langle[\alpha(i)]; F\rangle$ is a J-simple subalgebra of $\mathbf{A}_i$ for any $i \in I$. Suppose to the contrary that there exists $j \in I$ such that $\langle[\alpha(j)]; F\rangle$ is not a J-simple subalgebra of $\mathbf{A}_j$. Then there exists

$M_j \in \mathcal{P}(\mathbf{A}_j)$ such that $M_j \subset [\alpha(j)]$. If we denote $M = \prod(X_i \mid X_i = [\alpha(i)]$ for any $i \in I$, $i \neq j$, and $X_j = M_j)$, then $M \subset \prod([\alpha(i)] \mid i \in I) = [\alpha]$. Since $M \in \mathcal{P}(\mathbf{A})$, this contradicts the assumption that $[\alpha]$ is a J-simple subalgebra of $\mathbf{A}$. Thus, for any $i \in I$, $\langle[\alpha(i)]; F\rangle$ is a J-simple subalgebra of $\mathbf{A}_i$, and $\mathbf{N} = \prod(\langle[\alpha(i)]; F\rangle \mid i \in I)$.

2) Let $\mathbf{N} = \prod(\mathbf{N}_i \mid i \in I)$, where $\mathbf{N}_i = \langle N_i; F\rangle$ is a J-simple subalgebra of $\mathbf{A}_i$ for any $i \in I$. By Lemma 2.2 b), $N_i = [x_i] = F^+(x_i)$ for any $x_i \in N_i$. Let $\alpha \in A = \prod(A_i \mid i \in I)$ be an element such that $\alpha(i) \in N_i$ for any $i \in I$. Then $N_i = [\alpha(i)] = F^+(\alpha(i))$ for any $i \in I$. From this it easily follows that $\alpha(i) \in F^+(\alpha(i))$ for any $i \in I$. As $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ is an r-system of unary algebras, $\alpha \in F^+(\alpha)$. By Theorem 1.1, $N = \prod(N_i \mid i \in I) = \prod([\alpha(i)] \mid i \in I) = [\alpha] = F^+(\alpha)$. By Lemma 2.2 b) and a), $\mathbf{N} = \langle N; F\rangle$ is a J-simple subalgebra of an algebra $\mathbf{A}$. □

THEOREM 2.2. *Let $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ be an r-system of unary algebras, $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$. Let $\emptyset \neq K \subseteqq A = \prod(A_i \mid i \in I)$. Then $\mathbf{K} = \langle K; F\rangle$ is the kernel of $\mathbf{A}$ if and only if for any $i \in I$ there exists the kernel $\mathbf{K}_i = \langle K_i; F\rangle$ of $\mathbf{A}_i$, and $\mathbf{K} = \prod(\mathbf{K}_i \mid i \in I)$.*

Proof.

1) Let us suppose that $\mathbf{K} = \langle K; F\rangle$ be the kernel of $\mathbf{A}$. Then $\langle K; F\rangle$ is a J-simple subalgebra of $\mathbf{A}$. By Theorem 2.1, $\mathbf{K} = \prod(\mathbf{K}_i \mid i \in I)$ and $\mathbf{K}_i = \langle K_i; F\rangle$ is a J-simple subalgebra of $\mathbf{A}_i$ for any $i \in I$. Let α be an element of A such that $\alpha(i) \in K_i$ for any $i \in I$. By Lemma 2.2 c), $K_i = [\alpha(i)]J \in \mathcal{P}(\mathbf{A}_i)$ for any $i \in I$. We prove that $[\alpha(i)]J$ is the smallest element of the set A_i/J for any $i \in I$.

Let $j \in I$ be fixed, and choose an arbitrary $y_j \in A_j$. Let $\beta \in A$ be such that $\beta(j) = y_j$. Since $K = \prod(K_i \mid i \in I) = \prod([\alpha(i)] \mid i \in I)$, we have $\alpha \in K$. Therefore, $K = [\alpha]J$. However, $[\alpha]J$ is the smallest element of A/J. Hence $[\alpha]J \leq [\beta]J$. According to [1; Lemma 2.3], $[\alpha(i)]J \leq [\beta(i)]J$ for any $i \in I$. Obviously, also $[\alpha(j)]J \leq [\beta(j)]J = [y_j]J$. It means that $[\alpha(j)]J$ is the smallest element of A_j/J. Since $j \in I$ was an arbitrary index from I, we get $[\alpha(j)]J$ is the smallest element of A_j/J for any $j \in I$. Thus, $\langle K_j; F\rangle$ is the kernel of $\mathbf{A}_j$ for any $j \in I$.

2) Let $\langle K_i; F\rangle$ be the kernel of $\mathbf{A}_i$ for any $i \in I$. Let $\alpha \in A$ be such that $\alpha(i) \in K_i$ for any $i \in I$. Then $K_i = [\alpha(i)]J$, and $[\alpha(i)]J$ is the smallest element of A_i/J for every $i \in I$. Let $K = \prod(K_i \mid i \in I)$. According to Theorem 2.1, $\langle K; F\rangle$ is a J-simple subalgebra of $\mathbf{A}$. Since $\alpha \in K$, by Lemma 2.2, $K = [\alpha]J = [\alpha] = F^+(\alpha)$. Hence $\alpha \in F^+(\alpha)$. Let β be an arbitrary element of

A. As $[\alpha(i)]J$ is the smallest element of A_i/J, we have $[\alpha(i)]J \le [\beta(i)]J$ for any $i \in I$. In this case, by [1; Lemma 2.4], we have $[\alpha]J \le [\beta]J$, and $[\alpha]J = K$ is the smallest element of A/J. Hence $\langle K; F\rangle$ is the kernel of $\mathbf{A}$. □

3. J-Subalgebras

Let $\mathbf{A} = \langle A; F\rangle$ be a (unary) algebra, $b \in A$. Set

$$N(b) = \{x \in A \mid [b]J \not\leq [x]J\}.$$

The set $N(b)$ is of great importance for this section. (See also [4].) Clearly, $x \in N(b)$ if and only if either $[x]J < [b]J$, or the classes $[x]J$ and $[b]J$ are incomparable.

LEMMA 3.1. *Let $\mathbf{A} = \langle A; F\rangle$ be a unary algebra. Then for any $a \in A$, $a \neq b$, $a \in N(b)$ if and only if $b \notin F^+(a)$.*

P r o o f.

a) Let $a \neq b$ and $a \in N(b)$. Then $[b]J \not\leq [a]J$. It follows immediately that either $[a]J < [b]J$, or classes $[a]J$ and $[b]J$ are incomparable. In the first case, $[a] \subset [b]$ holds. It means that $a \in F^+(b)$ and $b \notin F^+(a)$. For the other case, $[a] \nsubseteq [b]$ and $[b] \nsubseteq [a]$. Hence, in both cases, $b \notin F^+(a)$.

b) Let $a \neq b$ and $a \notin N(b)$. Then $[b]J \le [a]J$, and thus $[b] \subseteqq [a]$. From the last inclusion we conclude $b \in F^+(a) \cup \{a\}$, and, by the condition $a \neq b$, $b \in F^+(a)$. This completes our proof. □

R e m a r k . Let $\mathbf{A} = \langle A; F\rangle$ be a unary algebra. Then, for any $b \in A$, $b \notin N(b)$, and we get the next proposition:

Let $\mathbf{A} = \langle A; F\rangle$ be a unary algebra, $b \in A$, and $b \in F^+(b)$. Then, for any $a \in A$, $a \in N(b)$ if and only if $b \notin F^+(a)$.

DEFINITION 3.1. (*See also* [3].) *Let $\mathbf{N} = \langle N; F\rangle$ be a subalgebra of a unary algebra $\mathbf{A} = \langle A; F\rangle$. Let there exist $b \in A$ such that $N = N(b)$. Then $\mathbf{N}$ is called a J-subalgebra of $\mathbf{A}$.*

Now we formulate three lemmas in which some useful properties of the set $N(b)$ are stated.

LEMMA 3.2. *Let $\mathbf{A} = \langle A; F\rangle$ be a unary algebra.*

a) *Let $b \in A$ and $N(b) \neq \emptyset$. Then $\langle N(b); F\rangle$ is a J-subalgebra of $\mathbf{A}$.*
b) *If $N \in \mathcal{P}(\mathbf{A})$ and $N \neq A$, then $\langle N(b); F\rangle$ is a J-subalgebra of $\mathbf{A}$ for any $b \in A \setminus N$.*

P r o o f . (See also [4].)

a) Let $\mathbf{A} = \langle A; F \rangle$ be a unary algebra, $b \in A$, and $N(b) \neq \emptyset$. For any $x \in A$ and any $f \in F$, $f(x) \in [x]$, and thus $[f(x)] \subseteqq [x]$. It means for J-classes that $[f(x)]J \leq [x]J$.

Now we prove $\langle N(b); F \rangle$ is a subalgebra of $\mathbf{A}$. Suppose that there exist $x \in N(b)$ and $f \in F$ such that $f(x) \notin N(b)$. From this it follows that $[b]J \leq [f(x)]J$. However, $[b]J \leq [f(x)]J \leq [x]J$ contradicts our assumption $x \in N(b)$. From this we conclude that $\langle N(b); F \rangle$ is a subalgebra of $\mathbf{A}$. Clearly, it is a J-subalgebra.

b) Let $N \in \mathcal{P}(\mathbf{A})$, $N \neq A$, and $b \in A \setminus N$. Then, for any $x \in N$, $[x] \subseteqq N$, and thus $[b] \nsubseteqq [x]$. Now there are two possibilities. Either $[x] \subset [b]$ or $[x] \nsubseteqq [b]$. These conditions imply $x \in N(b)$, and therefore $N(b) \neq \emptyset$. The rest of the proof follows from a). □

LEMMA 3.3. *Let $\mathbf{A} = \langle A; F \rangle$ be a unary algebra, and $b \in A$. Then the following conditions are equivalent:*

a) $N(b) = \emptyset$.
b) $[b]J$ *is the smallest element of* A/J.
c) $\langle [b]J; F \rangle$ *is the kernel of* $\mathbf{A}$.

P r o o f . We have $N(b) = \emptyset$ if and only if, for any $a \in A$, $[b]J \leq [a]J$. This is true if and only if $[b]J$ is the smallest element of A/J. This proves the equivalence of a) and b).

The equivalence of b) and c) is proved in Lemma 2.3. □

LEMMA 3.4. *Let $\mathbf{A} = \langle A; F \rangle$ be a unary algebra and $b \in A$. If $N(b) = \emptyset$, then $b \in F^+(b)$.*

P r o o f . Suppose to the contrary that $N(b) = \emptyset$ and $b \notin F^+(b)$. Then, for any $f \in F$, $f(b) \in F^+(b) \subset [b]$. Therefore $[f(b)] \subset [b]$ and thus $[f(b)]J < [b]J$. Hence $f(b) \in N(b)$ and $N(b) \neq \emptyset$, which is a contradiction to our assumption $N(b) = \emptyset$. □

Now we concentrate on the set $N(\alpha)$ in a direct product of an r-system of unary algebras.

THEOREM 3.1. *Let $\{\mathbf{A}_i = \langle A_i; F \rangle \mid i \in I\}$ be an r-system of unary algebras, $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$ and $\alpha \in A = \prod(A_i \mid i \in I)$. Then $N(\alpha) = \emptyset$ if and only if $N(\alpha(i)) = \emptyset$ for any $i \in I$.*

P r o o f .

a) Let $N(\alpha) = \emptyset$. By Lemma 3.3, $\langle [\alpha]J = K; F\rangle$ is the kernel of $\mathbf{A}$. According to Theorem 2.2, for any $i \in I$ there exists the kernel $\mathbf{K}_i = \langle K_i; F\rangle$ of the algebra $\mathbf{A}_i$, and $\mathbf{K} = \langle K; F\rangle = \prod(\mathbf{K}_i \mid i \in I)$. Thus, $K = [\alpha]J = \prod(K_i \mid i \in I)$. Hence, for any $i \in I$, $\alpha(i) \in K_i$. Since $\langle K_i; F\rangle$ is the kernel of the algebra $\mathbf{A}_i$, by Lemma 2.3, $[\alpha(i)]J = K_i$ for any $i \in I$, and thus, by Lemma 3.3, we have $N(\alpha(i)) = \emptyset$ for any $i \in I$.

b) If $N(\alpha(i)) = \emptyset$ for any $i \in I$, then, by Lemma 3.3, $\langle [\alpha(i)]J; F\rangle$ is the kernel of the algebra $\mathbf{A}_i$ for any $i \in I$. Let $K = \prod([\alpha(i)]J \mid i \in I)$. Then $\alpha \in K$, and, by Theorem 2.2, $\langle K; F\rangle$ is the kernel of $\mathbf{A}$. Therefore, by Lemma 2.3, $K = [\alpha]J$, and, by Lemma 3.3, $N(\alpha) = \emptyset$. □

THEOREM 3.2. *Let $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ be an r-system of unary algebras, and $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$. Let $\alpha \in A = \prod(A_i \mid i \in I)$ be such that $\alpha \in F^+(\alpha)$. Then*

$$N(\alpha) = \bigcup\Big\{\prod\big(X_i \mid\ X_i = A_i \text{ for any } i \in I,\ i \neq j,\ \text{and } X_j = N(\alpha(j))\big) \mid\ j \in I\Big\}.$$

P r o o f . If $N(\alpha) = \emptyset$, the proof follows from Theorem 3.1.

1) Let $N(\alpha) \neq \emptyset$. According to Theorem 3.1, there exists $k \in I$ such that $N(\alpha(k)) \neq \emptyset$. Let us denote $\bigcup\{\prod(X_i \mid\ X_i = A_i$ for any $i \in I$, $i \neq j$, and $X_j = N(\alpha(j))) \mid j \in I\}$ by Q. Clearly, $Q \neq \emptyset$ and $Q \in \mathcal{P}(\mathbf{A})$.

2) In both cases $\alpha \in F^+(\alpha)$ or $\alpha \notin F^+(\alpha)$, we show that $Q \subseteqq N(\alpha)$. Let $\beta \in Q$. Then there exists $k \in I$ such that $\beta(k) \in N(\alpha(k))$. According to the definition of the set $N(\alpha(k))$ and Lemma 3.1, $\beta(k) \neq \alpha(k)$ and $\alpha(k) \notin F^+(\beta(k))$. Hence $\alpha \neq \beta$, and, as $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ is an r-system of unary algebras, $\alpha \notin F^+(\beta)$. By Lemma 3.1, $\beta \in N(\alpha)$, and thus $Q \subseteqq N(\alpha)$.

3) Finally we show that $\alpha \in F^+(\alpha)$ implies $Q = N(\alpha)$. Let $\beta \in N(\alpha)$. Obviously, $\beta \neq \alpha$ and $\alpha \notin F^+(\beta)$. Hence, there exists $k \in I$ such that $\alpha(k) \notin F^+(\beta(k))$. According to $\alpha \in F^+(\alpha)$, we have $\alpha(k) \neq \beta(k)$. Therefore $\beta(k) \in N(\alpha(k))$. It follows that $\beta \in Q$ and thus $N(\alpha) \subseteqq Q$. Hence $N(\alpha) = Q$. □

In the following theorem, we describe $N(\alpha)$ in the case $\alpha \notin F^+(\alpha)$.

THEOREM 3.3. *Let* $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ *be an* r*-system of unary algebras,* $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$, $A = \prod(A_i \mid i \in I)$, $\alpha \in A$, *and* $\alpha \notin F^+(\alpha)$. *Let* $I_1 = \{i \in I \mid \alpha(i) \notin F^+(\alpha(i))\}$. *Then*

$$\begin{aligned} N(\alpha) = \Big(\bigcup\{\textstyle\prod(X_i \mid X_i = A_i \ &\textit{for any } i \in I,\ i \neq j, \ \textit{and} \\ &X_j = N(\alpha(j)) \ \textit{for } j \in I \setminus I_1, \ \textit{whereas} \\ &X_j = N(\alpha(j)) \cup \{\alpha(j)\} \textit{for } j \in I_1) \mid j \in I\}\Big) \setminus \{\alpha\}. \end{aligned}$$

Proof.

1) Let $\alpha \notin F^+(\alpha)$. By Lemma 3.4, $N(\alpha) \neq \emptyset$. Since we consider an r-system of unary algebras, there exists at least one $k \in I$ such that $\alpha(k) \notin F^+(\alpha(k))$. Thus, the set I_1 is nonempty, and, by Lemma 3.4, $N(\alpha(i)) \neq \emptyset$ for any $i \in I_1$.

Let $M_j = N(\alpha(j)) \cup \{\alpha(j)\}$ for any $j \in I_1$, and $M_j = N(\alpha(j))$ for any $j \in I \setminus I_1$. Let $Q = \bigcup\{\prod(X_i \mid X_i = A_i$ for any $i \in I$, $i \neq j$, and $X_j = N(\alpha(j))) \mid j \in I\}$, and let $R = \big(\bigcup\{\prod(X_i \mid X_i = A_i$ for any $i \in I$, $i \neq j$, and $X_j = \{\alpha(j)\}) \mid j \in I_1\}\big) \setminus \{\alpha\}$. Evidently, $Q \cup R = \big(\bigcup\{\prod(X_i \mid X_i = A_i$ for any $i \in I$, $i \neq j$, and $X_j = M_j) \mid j \in I\}\big) \setminus \{\alpha\}$.

2) We prove $Q \cup R \subseteqq N(\alpha)$. In the part 2) of the proof of Theorem 3.2, we have proved $Q \subseteqq N(\alpha)$. Thus, it is sufficient to prove $R \subseteqq N(\alpha)$.

Let $\beta \in R$. Clearly, $\beta \neq \alpha$. Suppose $\beta \notin N(\alpha)$. By Lemma 3.1, $\alpha \in F^+(\beta)$. Hence $\alpha(i) \in F^+(\beta(i))$ for any $i \in I$. However, the condition $\beta \in R$ implies the existence of $k \in I_1$ such that $\beta(k) = \alpha(k) \notin F^+(\alpha(k)) = F^+(\beta(k))$. This contradicts the assumption $\alpha \in F^+(\beta)$. Thus $\beta \in N(\alpha)$ and $R \subseteqq N(\alpha)$.

3) Finally we prove that $N(\alpha) \subseteqq Q \cup R$. Let $\beta \in N(\alpha)$. Obviously, $\beta \neq \alpha$, and, by Lemma 3.1, $\alpha \notin F^+(\beta)$. Let $I_2 = \{i \in I \mid \alpha(i) \notin F^+(\beta(i))\}$. Evidently, $I_2 \neq \emptyset$. Let $j \in I_2$. If $\alpha(j) \neq \beta(j)$, then, by Lemma 3.1, $\beta(j) \in N(\alpha(j))$ and thus $\beta \in Q$.

If $\alpha(j) = \beta(j) \notin F^+(\beta(j)) = F^+(\alpha(j))$, then $j \in I_1$, and thus $\beta \in R$. So $N(\alpha) \subseteqq Q \cup R$, and the theorem is proved. □

Now we use the obtained results to describe maximal and greatest subalgebras of the direct product of an r-system of unary algebras. Let $\mathbf{A} = \langle A; F\rangle$ be a (unary) algebra. Let $\langle N; F\rangle$ be a subalgebra of $\mathbf{A}$ such that $N \subset A$, and there does not exist $N' \in \mathcal{P}(\mathbf{A})$ such that $N \subset N' \subset A$. In this case, $\langle N; F\rangle$ is called a maximal subalgebra of $\mathbf{A}$. We denote the set of all $N \in \mathcal{P}(\mathbf{A})$ such that $\langle N; F\rangle$ is a maximal subalgebra of $\mathbf{A}$ by $\mathcal{P}_{\max}(\mathbf{A})$.

We also denote by N^* an element from $\mathcal{P}(\mathbf{A})$ such that $N^* \neq A$, and $N \subseteq N^*$ for any $N \in \mathcal{P}(\mathbf{A}) \setminus A$. The described set need not exist. If it does, then the algebra $\langle N^*; F\rangle$ is called the greatest subalgebra of the algebra $\mathbf{A}$.

The following lemma is useful for the next part of this section. (For the proof of this lemma, see [2].)

Lemma 3.5. *Let $\mathbf{A} = \langle A; F\rangle$ be a unary algebra, and $\mathbf{A}$ is not J-simple. Let $\emptyset \neq N \subset A$. Then*

a) *$\langle N; F\rangle$ is a maximal subalgebra of $\mathbf{A}$ if and only if there exists a maximal element $[x]J$ of the partially ordered set A/J such that $N = A \setminus [x]J$;*

b) *$N = N^*$ if and only if there exists a greatest element $[x]J$ of the set A/J and $N = A \setminus [x]J$.*

Lemma 3.6. *Let $\mathbf{A} = \langle A; F\rangle$ be a unary algebra and $\alpha \in A$. Then J-class $[\alpha]J$ is a maximal element in the set A/J if and only if $[\alpha]J = A \setminus N(\alpha)$.*

Proof.

a) Let $[\alpha]J$ be a maximal element in the set A/J. According to the definition of the set $N(\alpha)$, $A \setminus N(\alpha) = \{x \in A \mid [\alpha]J \leq [x]J\}$. Obviously, in this case, we have $A \setminus N(\alpha) = [\alpha]J$.

b) Let $A \setminus N(\alpha) = [\alpha]J$. Then we have $[\alpha]J = \{x \in A \mid [\alpha]J \leq [x]J\}$. Hence, for any $x \in A$ such that $[\alpha]J \leq [x]J$, $x \in [\alpha]J$. Thus $[\alpha]J = [x]J$, and $[\alpha]J$ is a maximal element in A/J. ❑

According to Lemmas 3.5 and 3.6, if $N \in \mathcal{P}_{\max}(\mathbf{A})$, then there exists a maximal J-class $[\alpha]J$ of A/J such that $N = A \setminus [\alpha]J = N(\alpha)$. Therefore any maximal subalgebra of a unary algebra is a J-subalgebra of $\mathbf{A}$. Moreover, $N(\alpha) \in \mathcal{P}_{\max}(\mathbf{A})$ if and only if $N(\alpha) \neq \emptyset$ and $N(\alpha) = A \setminus [\alpha]J$.

Now we can state the results concerning maximal and greatest subalgebras of a direct product of an r-system of unary algebras.

Theorem 3.4. *Let $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ be an r-system of unary algebras, $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$, $A = \prod(A_i \mid i \in I)$, $\alpha \in A$, and $\alpha \in F^+(\alpha)$. Let $I_1 = \{i \in I \mid N(\alpha(i)) \neq \emptyset\}$. Then $N(\alpha) \in \mathcal{P}_{\max}(\mathbf{A})$ if and only if $I_1 \neq \emptyset$, $N(\alpha(i)) \in \mathcal{P}_{\max}(\mathbf{A}_i)$ for any $i \in I_1$, and $\mathbf{A}_i$ is a J-simple algebra for any $i \in I \setminus I_1$.*

P r o o f .

a) Let $N(\alpha) \in \mathcal{P}_{\max}(\mathbf{A})$. Therefore $N(\alpha) \neq \emptyset$. By Theorem 3.1, the set $I_1 = \{i \in I \mid N(\alpha(i)) \neq \emptyset\}$ is nonempty. According to Lemma 3.6, $A \setminus N(\alpha)$ is a maximal element of A/J. Since $\alpha \in A \setminus N(\alpha)$, we get $A \setminus N(\alpha) = [\alpha]J$. By [1; Theorem 3.1], $[\alpha(i)]J$ is a maximal element of a set A_i/J for any $i \in I$. By Lemma 3.6, $[\alpha(i)]J = A_i \setminus N(\alpha(i))$ for any $i \in I$, and thus, if $N(\alpha(i)) \neq \emptyset$, then $N(\alpha(i)) = A_i \setminus [\alpha(i)]J \in \mathcal{P}_{\max}(\mathbf{A}_i)$ for each $i \in I_1$.

Further, $N(\alpha(i)) = \emptyset$, and $[\alpha(i)]J$ is a maximal element of A_i/J for any $i \in I \setminus I_1$. Therefore, for any $i \in I \setminus I_1$, $[\alpha(i)]J = A_i \setminus \emptyset = A_i$, and, by Lemma 3.3, the algebra $\langle [\alpha(i)]J; F \rangle = \langle A_i; F \rangle$ is the kernel of $\mathbf{A}_i$. Thus, any algebra $\mathbf{A}_i$ is a J-simple unary algebra for any $i \in I \setminus I_1$. Now Theorem 3.2 implies the rest of this part of the proof.

b) Let $N(\alpha) = \bigcup\{\prod(X_i \mid X_i = A_i \text{ for any } i \in I,\ i \neq j, \text{ and } X_j = N(\alpha(j))) \mid j \in I\}$, $N(\alpha(i)) \in \mathcal{P}_{\max}(\mathbf{A}_i)$ for any $i \in I_1 \neq \emptyset$, and $\mathbf{A}_i$ be a J-simple unary algebra for any $i \in I \setminus I_1$. Then, by Lemma 2.2, for any $i \in I \setminus I_1$, $[\alpha(i)]J = A_i$. By Lemmas 3.5 and 3.6, $[\alpha(i)]J$ is a maximal element in the set A_i/J. From $I_1 \neq \emptyset$ we have $N(\alpha) \neq \emptyset$. By Lemma 3.6, $[\alpha]J = A \setminus N(\alpha)$, and thus $N(\alpha) = A \setminus [\alpha]J \in \mathcal{P}_{\max}(\mathbf{A})$. □

THEOREM 3.5. *Let $\{\mathbf{A}_i = \langle A_i; F \rangle \mid i \in I\}$ be an r-system of unary algebras, $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$, $A = \prod(A_i \mid i \in I)$, $\alpha \in A$, and $\alpha \in F^+(\alpha)$. Let $I_1 = \{i \in I \mid N(\alpha(i)) \neq \emptyset\}$. Then $N(\alpha) = N^*$ (i.e. $\langle N(\alpha); F \rangle$ is the greatest subalgebra of $\mathbf{A}$) if and only if $I_1 \neq \emptyset$, $N(\alpha(i)) = N_i^*$ is the greatest subalgebra of $\mathbf{A}_i$ for any $i \in I_1$, and $\mathbf{A}_i$ is a J-simple unary algebra for any $i \in I \setminus I_1$.*

P r o o f . By Lemmas 3.5 and 3.6, it is obvious that $N(\alpha) = N^*$ if and only if $N(\alpha) \neq \emptyset$ and $N(\alpha) = A \setminus [\alpha]J$, where $[\alpha]J$ is the greatest element of A/J. Now we can continue similarly as in the proof of Theorem 3.4, only instead of [1; Theorem 3.1] we must use [1; Theorem 3.3]. □

Now we state two theorems which, in the case $\alpha \notin F^+(\alpha)$, describe conditions for $N(\alpha) \in \mathcal{P}_{\max}(\mathbf{A})$ and $N(\alpha) = N^*$. These theorems represent direct consequences of [1; Theorems 3.2 and 3.4].

THEOREM 3.6. *Let $\{\mathbf{A}_i = \langle A_i; F \rangle \mid i \in I\}$ be an r-system of unary algebras, $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$, $A = \prod(A_i \mid i \in I)$, $\alpha \in A$, and $\alpha \notin F^+(\alpha)$. Then $N(\alpha) \in \mathcal{P}_{\max}(\mathbf{A})$ if and only if there exists $i \in I$ such that $N(\alpha(i)) \in \mathcal{P}_{\max}(\mathbf{A}_i)$ and $\alpha(i) \notin F^+(\alpha(i))$.*

P r o o f.

a) Let us suppose $N(\alpha) \in \mathcal{P}_{\max}(\mathbf{A})$ and $\alpha \notin F^+(\alpha)$. Then $N(\alpha) \neq \emptyset$, and $[\alpha]J$ is a maximal element of A/J. According to [1; Lemma 3.2], there exists $i \in I$ such that $[\alpha(i)]J$ is a maximal element of A_i/J and $\alpha(i) \notin F^+(\alpha(i))$. Thus $N(\alpha(i)) \neq \emptyset$ and $[\alpha(i)]J = A_i \setminus N(\alpha(i))$. So, $N(\alpha(i)) = A_i \setminus [\alpha(i)]J \in \mathcal{P}_{\max}(\mathbf{A}_i)$ and, of course, $\alpha(i) \notin F^+(\alpha(i))$.

b) Suppose, there is $i \in I$ such that $N(\alpha(i)) \in \mathcal{P}_{\max}(\mathbf{A}_i)$ and $\alpha(i) \notin F^+(\alpha(i))$. Then $N(\alpha(i)) \neq \emptyset$, and $[\alpha(i)]J$ is a maximal element of A_i/J. As we consider an r-system of unary algebras, $\alpha \notin F^+(\alpha)$, and, by [1; Theorem 3.2], $[\alpha]J$ is a maximal element of A/J. Obviously, $N(\alpha) \neq \emptyset$. Therefore $N(\alpha) = A \setminus [\alpha]J \in \mathcal{P}_{\max}(\mathbf{A})$. □

R e m a r k 2. By [1; Lemma 1.3], from $\alpha \notin F^+(\alpha)$ it follows that $[\alpha]J = \{\alpha\}$. Clearly, in this case, $N(\alpha) \in \mathcal{P}_{\max}(\mathbf{A})$ if and only if $\{\alpha\}$ is a maximal class of A/J and $N(\alpha) = A \setminus \{\alpha\}$.

We can derive this result also from Theorem 3.3 in the following way. If $j \in I$ is such that $\alpha(j) \notin F^+(\alpha(j))$, and $[\alpha(j)]J = \{\alpha(j)\}$ is a maximal element of A_j/J, then $M_j = N(\alpha(j)) \cup \{\alpha(j)\} = (A_j \setminus \{\alpha(j)\}) \cup \{\alpha(j)\} = A_j$, and thus $\prod(X_i \mid X_i = A_i$ for any $i \in I$, $i \neq j$, and $X_j = M_j) = A$. Therefore

$$N(\alpha) = \Big(\bigcup\{\prod(X_i \mid X_i = A_i \text{ for any } i \in I,\ i \neq j, \text{ and } X_j = M_j) \mid j \in I\}\Big) \setminus \{\alpha\} = A \setminus \{\alpha\}.$$

The following theorem is the direct consequence of [1; Theorem 3.4].

THEOREM 3.7. *Let $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ be an r-system of unary algebras, $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$, $A = \prod(A_i \mid i \in I)$, $\alpha \in A$, and $\alpha \notin F^+(\alpha)$. Then $N(\alpha) = N^*$ is the greatest subalgebra of the algebra $\mathbf{A}$ if and only if there exists $i \in I$ such that $\alpha(i) \notin F^+(\alpha(i))$, $N(\alpha(i)) = N_i^*$ is the greatest subalgebra of $\mathbf{A}_i$, and $|A_j| = 1$ for any $j \in I$, $i \neq j$.*

In the following, we show how these results can be applied to M-automata and direct products of M-automata by describing M-automata as unary algebras.

Let M be a monoid with a unit element 1, S be a nonempty set, and $\delta\colon S \times M \to S$ be a mapping such that:

a) $\delta(s,1) = s$ for any $s \in S$,
b) $\delta(\delta(s,m),m') = \delta(s,mm')$ for any $s \in S$, and $m, m' \in M$.

Then the couple $\mathbf{A} = (S;\delta)$ is called an M-automaton. (See for example [5].)

Let $\mathbf{A} = (S;\delta)$ be an M-automaton. Let $\emptyset \neq N \subseteqq S$, $\delta(s,m) \in N$ for any $s \in N$, and $m \in M$. Then the couple $\mathbf{N} = (N;\delta)$ we call a subautomaton of $\mathbf{A}$. We denote by $\mathcal{P}(\mathbf{A})$ the set of all subsets N of S such that $\mathbf{N} = (N;\delta)$ is a subautomaton of $\mathbf{A}$.

Let $a \in S$ and $[a]_A = \{\delta(a,m) \in S \mid m \in M\}$. Obviously, $a \in [a]_A$. Further, for any $\delta(a,m) \in [a]_A$ and any $m' \in M$, $\delta(\delta(a,m),m') = \delta(a,mm') \in [a]_A$. Therefore, $([a]_A;\delta)$ is a subautomaton of $\mathbf{A}$. On the other hand, if $(N;\delta)$ is an arbitrary subautomaton of $\mathbf{A}$ and $a \in N$, then, for any $m \in M$, $\delta(a,m) \in N$. Hence $[a]_A \subseteqq N$. Thus, we have proved that $[a]_A = \bigcap\{N \in \mathcal{P}(\mathbf{A}) \mid a \in N\}$. Subautomaton $([a]_A;\delta)$ is a subautomaton of $\mathbf{A}$ which is generated by the element $a \in S$.

For the set S we define a relation J_A in the following way: $a\, J_A\, b$ if and only if $[a]_A = [b]_A$. Evidently, J_A is an equivalence on the set S. For the set S/J_A we define a binary operation $\leq$ by: $[a]J_A \leq [b]J_A$ if and only if $[a]_A \subseteqq [b]_A$. It is obvious that the relation $\leq$ is a partial order on the set S/J_A.

Let $b \in S$. By $N(b)$ we denote the set $\{x \in S \mid [b]J_A \not\leq [x]J_A\}$. In the case $N(b) \neq \emptyset$, it is not too difficult to prove that $N(b) \in \mathcal{P}(\mathbf{A})$.

We call a subautomaton $\mathbf{N} = (N;\delta)$ of an M-automaton $\mathbf{A} = (S;\delta)$ a minimal (J_A-simple) subautomaton of $\mathbf{A}$ if there does not exist an element $N' \in \mathcal{P}(\mathbf{A})$ such that $N' \subset N$.

Let $\mathbf{N} = (N;\delta)$ be a subautomaton of the M-automaton $\mathbf{A}$. If $N \neq S$, and there does not exist $N' \in \mathcal{P}(\mathbf{A})$ such that $N \subset N' \subset S$, the subautomaton $\mathbf{N}$ will be called a maximal subautomaton of $\mathbf{A}$.

Remark 3. Let $\mathbf{A} = (S;\delta)$ be an M-automaton. For any $m \in M$ we define a function $f_m \colon S \to S$ such that $f_m(s) = \delta(s,m)$ for any $s \in S$. Let $F_M = \{f_m \mid m \in M\}$. Then we can consider a unary algebra $\mathbf{S} = \langle S; F_M\rangle$ which is assigned to the M-automaton $\mathbf{A} = (S;\delta)$.

Let I be a set and $|I| \geq 2$. Let $\mathbf{A}_i = (S_i;\delta_i)$ be an M_i-automaton for any $i \in I$. Let $S = \prod(S_i \mid i \in I)$ and $M = \prod(M_i \mid i \in I)$ be the Cartesian product. We denote also by M the direct product of the monoids M_i. Define the mapping $\delta \colon S \times M \to S$ by $\delta(\sigma,\mu) = \nu$ if and only if $\delta_i(\sigma(i),\mu(i)) = \nu(i)$ for any $i \in I$. Then the couple $\mathbf{A} = (S;\delta)$ is an M-automaton. We call this M-automaton a direct product of M_i-automata $\mathbf{A}_i = (S_i;\delta_i)$. This automaton will be denoted $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$.

If the direct product of M_i-automata is defined for any $i \in I$ and common M, then to any M_i-automaton $\mathbf{A}_i = (S_i;\delta_i)$ we can assign an M-automaton $\tilde{\mathbf{A}}_i = (S_i;\tilde{\delta}_i)$ such that $\tilde{\delta}_i(s_i,\mu) = s_i'$ if and only if $\delta_i(s_i,\mu(i)) = s_i'$ for any $s_i \in S_i$ and $\mu \in M$. By Remark 3, to any M-automaton $\tilde{\mathbf{A}}_i = (S_i,\tilde{\delta}_i)$ we assign the unary algebra $\mathbf{S}_i = \langle S_i, F_M \rangle$. All of these unary algebras are of the same type. In the same way as in [1; Example 1], we can prove that the system $\{\mathbf{S}_i \mid i \in I\}$ is an r-system of unary algebras. Moreover, the direct product $\mathbf{S} = \prod(\mathbf{S}_i \mid i \in I)$ of this r-system is the unary algebra which is assigned, by Remark 3, to the direct product of M_i-automata $\mathbf{A}_i$.

In this way, to any system $\{\mathbf{A}_i \mid i \in I\}$ of M_i-automata the r-systen $\{\mathbf{S}_i \mid i \in I\}$ of unary algebras is assigned. For this assignment we can formulat the following propositions.

Let $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$ be the direct product of M_i-automata $\mathbf{A}_i = (S_i; \delta_i)$, and $|I| \geq 2$. Let $\{\mathbf{S}_i \mid i \in I\}$ be the r-system of unary algebras which is assigned to the system $\{\mathbf{A}_i \mid i \in I\}$ of M_i-automata. Let $\mathbf{S} = \prod(\mathbf{S}_i \mid i \in I)$ be the direct product of unary algebras $\mathbf{S}_i = \langle S_i; F_M \rangle$. Then:

a) for any nonempty subset N of S, $N \in \mathcal{P}(\mathbf{A})$ if and only if $N \in \mathcal{P}(\mathbf{S})$;
b) for any $i \in I$ and any nonempty subset N_i of the set S_i, $N_i \in \mathcal{P}(\mathbf{A}_i)$ if and only if $N_i \in \mathcal{P}(\mathbf{S}_i)$;
c) for any nonempty subset N of S, $N = [\alpha]_A$ if and only if $N = [\alpha]$;
d) for any $i \in I$ and any nonempty subset N_i of the set S_i, $N_i = [x_i]_A$ if and only if $N_i = [x_i]$.

In this way, M-automata can be described as unary algebras and we can use our results concerning unary algebras to find e.g. all minimal or maximal subautomata, and, if it exists, the greatest subautomaton of a given M-automaton. Also, related questions concerning the direct product of M-automata can be solved.

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* *Department of Mathematics*
Faculty of Machinery Engineering
Slovak Technical University
nám. Slobody 17
SK-812 31 Bratislava
Slovakia

** *Department of Mathematics*
Faculty of Electrical Engineering
Slovak Technical University
Ilkovičova 3
SK-812 19 Bratislava
Slovakia