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Mathematica Slovaca, Vol. 56 (2006), No. 4, 379--385

Persistent URL: `http://dml.cz/dmlcz/128565`

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Mathematica Slovaca

CHROMATIC NUMBERS OF THE STRONG PRODUCT OF ODD CYCLES

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(*Communicated by Martin Škoviera*)

ABSTRACT. The problem of determining the chromatic numbers of the strong product of cycles is considered. A construction is given proving $\chi(G) = 2^p + 1$ for the strong product of p odd cycles of lengths at least $2^p + 1$. Several consequences are discussed. In particular, it is proved that the strong product of p factors has chromatic number at most $2^p + 1$ provided that each factor admits a homomorphism to a sufficiently long odd cycle C_{m_i}, $m_i \geq 2^p + 1$.

1. Introduction

The *strong product* of graphs G and H is the graph $G \boxtimes H$ with vertex set $V(G) \times V(H)$. The vertices (x_1, x_2) and (y_1, y_2) are connected by an edge of $E(G \boxtimes H)$ whenever $\big(x_1 y_1 \in E(G)$ & $x_2 = y_2\big)$ or $\big(x_2 y_2 \in E(H)$ & $x_1 = y_1\big)$ or $\big(x_1 y_1 \in E(G)$ & $x_2 y_2 \in E(H)\big)$. For the product of p factors $G_1, G_2, \dots, G_p$ we use the notation $\mathop{\boxtimes}\limits_{i=1}^{p} G_i = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_p$. The strong product is one of the standard graph products ([10]). The edge set of the strong product of two factors can be seen as the union of the edge sets of the Cartesian and the direct product. The *direct edges* are the edges which connect vertices with both coordinates different and the *Cartesian edges* connect pairs of vertices with one coordinate equal. In a product of more than two factors, one can define *pure direct product edges* (all coordinates different), *pure Cartesian edges* (all but one coordinate equal), and *mixed edges.* Perhaps the most popular application of the strong product of graphs is in the information theory, where the zero error

2000 Mathematics Subject Classification: Primary 05C15.
Keywords: strong product, chromatic number, odd cycle, minimal independent dominating set.

This work was supported in part by the Ministry of Education, Science and Sport of Slovenia under grant L2-4049.

capacity of a noisy channel is defined in terms of independence numbers of strong products of the graph related to the channel ([11], [8]). More on graph products can be found in [4].

A *proper k-coloring* of vertices of a graph G is a function f from $V(G)$ onto a set (of colors) X, $|X| = k$, such that $xy \in E(G)$ implies $f(x) \neq f(y)$. The smallest number k for which a proper k-coloring exists is the *chromatic number* of G, $\chi(G)$. A *clique* of a graph is its maximal complete subgraph. The *clique number* $\omega(G)$ of a graph G is the number of vertices in its largest clique.

A *homomorphism* from a graph G to a graph H is a function f from $V(G)$ to $V(H)$ such that $f(u)$ and $f(v)$ are adjacent vertices of H whenever u and v are adjacent vertices of G. It is well known that a homomorphism $G \to K_k$ is just a proper k-coloring of G.

The chromatic numbers of strong products of even cycles are easy to determine. The problem is incomparably more difficult for cycles of odd length. For product of two odd cycles a formula for the chromatic numbers was derived by Vesztergombi [13] and by Hell and Roberts [2]. In the survey [7] Klavžar posed the problem considered here: Determine the chromatic number of the strong product of several odd cycles, in particular for three factors.

In this paper we give a construction which proves $\chi(G) = 2^p+1$ for a product of p odd cycles of lengths at least $2^p + 1$, which implies:

THEOREM 1. *If $G = \boxtimes_{i=1}^{p} C_{m_i}$ is the strong product of p odd cycles and the shortest cycle is of length at least $2^p + 1$, then $\chi(G) \leq 2^p + 1$.*

For arbitrary graphs G_1 and G_2 it is known that $\chi(G_1 \boxtimes G_2) \leq \chi(G_1)\chi(G_2)$. If $\chi(G_1) = \omega(G_1)$ and $\chi(G_2) = \omega(G_2)$, then clearly we have $\chi(G_1 \boxtimes G_2) = \chi(G_1)\chi(G_2)$. There are examples including products of odd cycles where $\chi(G_1 \boxtimes G_2) < \chi(G_1)\chi(G_2)$.

It is well known that the existence of a homomorphism from G to H implies that the chromatic number of H is an upper bound for the chromatic number of G. Since homomorphisms in factors clearly give a homomorphism in a product, we have:

COROLLARY 1. *If there are homomorphisms from G_i to odd cycles C_{m_i}, $m_i \geq 2^p + 1$, then $\chi\Big(\boxtimes_{i=1}^{p} G_i\Big) \leq 2^p + 1$.*

Remark 1. The factoring of a graph with respect to the strong product can be done in polynomial time ([1]), but the problem of H-coloring is NP-hard unless H is bipartite ([3]). (A graph G is said to be *H-colorable* if there is a homomorphism from G to H.) Therefore the theorem does not give rise to an efficient algorithm. However, the theorem might be useful in some special cases. For example, the graph on Fig. 1 clearly admits a homomorphism to the

5-cycle. Hence the strong product of two such graphs has chromatic number 5 by Theorem 1.

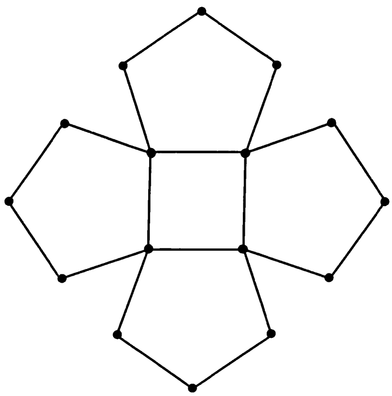

FIGURE 1. The conference graph (proposed to be called the Schwenk graph) of the conference Kalamazoo 2000, where a talk with this result was given.

The rest of the paper is organized as follows. In the next section we give the basic construction and in the following section we prove the main theorem. In Section 4 we overview the known results for the product of three cycles.

2. The construction

We will define a labeling of the vertices of the product of p cycles of length 2^p+1.

Assuming that the vertices of the cycles are assigned numbers $0, 1, \ldots, 2^p$ as usual, each vertex of the product has its given coordinates. By definition, two vertices are connected by a Cartesian edge when they differ by 1 (modulo 2^p+1) in exactly one coordinate and have all other coordinates equal. Furthermore, two vertices are connected by a direct edge if they differ by 1 in all coordinates. Recall that there are also some other mixed edges in the product of more than two factors.

Define the *label* of a vertex $v = (v_1, v_2, \ldots, v_p) \in \boxtimes^p C_{2^p+1}$ as

$$L(v) = \left(\sum_{i=1}^{p} v_i 2^{i-1} \right) \mod (2^p+1)\,.$$

Clearly, the label of origin is 0. Furthermore, the difference between labels of two adjacent vertices in the ith layer is exactly 2^{i-1}. (A *layer* is a subgraph of a product with $p-1$ coordinates fixed.) Labels along a walk from origin around the cycle in the ith layer form the sequence $\dots, 0, 2^{i-1}, 2 \cdot 2^{i-1}, \dots$. Hence each label appears exactly once on each layer because 2^i and 2^p+1 are relatively prime for all i.

This construction is a generalization of the well-known construction of a five-coloring of the strong product of two five cycles, perhaps first given in [13]. Analogous construction was used by Jha to prove theorems on smallest independent dominating sets in the direct product ([5]).

3. Proof of Theorem 1

Clearly, more than 2^p colors are needed for a proper coloring of a strong product of p odd cycles of lengths at least 2^p+1. A direct argument is as follows. Assume there is a proper coloring with 2^p colors. There are complete subgraphs on 2^p vertices in the product. Take any such subgraph. For coloring it, 2^p colors are needed. Now try to extend this coloring along one of the factors, hence try to color a subgraph isomorphic to $C_{2^p+1} \boxtimes \left(\boxtimes^{p-1} K_2\right)$. For each new layer one has to use the same set of colors which were already used two layers backwards. This clearly leads to contradiction, because we cannot properly color the last layer.

We now show that 2^p+1 colors are sufficient for coloring a strong product of p odd cycles provided the cycles are long enough. First we prove:

Lemma 1. *If $G = \boxtimes^p C_{2^p+1}$ is the strong product of p odd cycles of length 2^p+1, then $\chi(G) = 2^p+1$.*

Proof. Assume the factors are cycles of length 2^p+1. By the claim above, $\chi > 2^p$. We prove $\chi \le 2^p+1$ using the construction which has been given in the previous section. Let labels be colors. We claim that this is a proper coloring.

The coordinates of any two neighbours in the strong product can differ in any number of coordinates, but the coordinates can differ only by 1, 0 or -1. Hence the labels of two neighbours can differ by

$$\alpha_0 2^0 + \alpha_1 2^1 + \alpha_2 2^2 + \dots + \alpha_{p-1} 2^{p-1} \mod (2^p+1)$$

with $\alpha_i \in \{-1, 0, 1\}$. The sum is obviously bounded by -2^p and 2^p. Furthermore, it is straightforward that the sum equals 0 if and only if all α_i's are 0. Hence, no pair of neighbours has the same color, the assignment is a proper coloring, which completes the proof. □

Using the last lemma it is easy to prove the theorem. In a product of p odd cycles of lengths at least $2^p + 1$, one can inductively extend the construction by inserting two consecutive layers and repeating the coloring pattern of the neighbouring layers. We omit straightforward details.

4. Chromatic numbers of strong product of three cycles

It may be interesting to note that even in the case of three factors, there are still some cases for which the chromatic number is not known. The products with unknown chromatic numbers are given in the table below.

Recently, an exact formula for the special case when two cycles are of length 5 was given ([6; Theorem 9]): For $k \geq 2$, $\chi(C_5 \boxtimes C_5 \boxtimes C_{2k+1}) = 10 + \lceil \frac{5}{k} \rceil$ and some bounds were obtained for the general case. Some of the bounds follow from independence numbers given in [12]. The bound $\chi(C_7 \boxtimes C_9 \boxtimes C_9) \leq 11$ was obtained by a successful coloring ([15]) with generalized Petford Welsh algorithm ([9], [14]).

We know that the "infinite" families with unknown chromatic numbers are in fact finite, but we do not know how large they are. The author has a proof of this assertion which is long and tedious. Since it does not give a good upper bound on the families, it is omitted here.

Bounds for chromatic numbers.

lower bound	cycle lengths			upper bound
11	5	7	7	12
11	5	7	9	12
11	5	9	9	12
10	7	7	7	12
10	7	7	9	12
10	7	7	11,13, ...	11
10	7	9	9	11
10	7	9	11,13, ...	11

Acknowledgement

The author wishes to thank to the referee for careful reading of the manuscript.

REFERENCES

[1] FEIGENBAUM, J.—SCHÄFFER, A.: *Finding the prime factors of strong direct product graphs in polynomial time*, Discrete Math. **109** (1992), 77–102.

[2] HELL, P.—ROBERTS, F.: *Analogues of the Shannon capacity of graph*, Ann. Discrete Math. **12** (1982), 155–168.

[3] HELL, P.—NEŠETŘIL, J.: *On the complexity of H-coloring*, J. Combin. Theory Ser. B **48** (1990), 92–110.

[4] IMRICH, W.—KLAVŽAR, S.: *Product Graphs: Structure and Recognition*, John Wiley & Sons, New York, 2000.

[5] JHA, P. K.: *Smallest independent dominating sets in Kronecker products of cycles*, Discrete Appl. Math. (To appear).

[6] KLAVŽAR, S.: *Strong products of χ-critical graphs*, Aequationes Math. **45** (1993), 153–162.

[7] KLAVŽAR, S.: *Coloring graph products — A survey*, Discrete Math. **155** (1996), 135–145.

[8] LOVÁSZ, L.: *On the Shannon capacity of a graph*, IEEE Trans. Inform. Theory **25** (1979), 1–7.

[9] PETFORD, A.—WELSH, D.: *A randomised 3-colouring algorithm*, Discrete Math. **74** (1989), 253–261.

[10] SABIDUSSI, G.: *Graph multiplication*, Math. Z. **72** (1960), 446–457.

[11] SHANNON, C. E.: *The zero-error capacity of a noisy channel*, IRE Trans. Inform. Theory **2** (1956), 8–19.

[12] VESEL, A.: *The independence number of the strong product of cycles*, Comput. Math. Appl. **36** (1998), 9–21.

[13] VESZTERGOMBI, F.: *Some remarks on the chromatic number of the strong product of graphs*, Acta Cybernet. **4** (1978/79), 207–212.

[14] ŽEROVNIK, J.: *A randomized algorithm for k-colorability*, Discrete Math. **131** (1994), 379–393.

[15] ŽEROVNIK, J.: *Pomen "temperature" pri nekaterih hevristikah kombinatorične optimizacije.* In: Proceedings DSI, Portorož, april 2000, Slovensko Društvo Informatika Ljubljana 2000, pp. 604 609. (Slovene)

Received September 20, 2004

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