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ON THE UNIQUE FACTORIZATION PROBLEM

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B. Jónsson has posed the following problem in [1]. Let $\mathfrak{A} = \langle A, 0, R_\gamma \rangle_{\gamma \in \Gamma}$ be an algebraic structure consisting of a set A, an indexed family of relations R_γ of the finite rank over A, and a distinguished element $0 \in A$ satisfying the condition $\langle 0, \ldots, 0 \rangle \in R_\gamma$ for all $\gamma \in \Gamma$. Let $\mathscr{C}$ be a class of relational structures of the same type. $\mathfrak{A}$ is said to have the *unique factorization property over the class* $\mathscr{C}$ if two following conditions hold true

1. $\mathfrak{A}$ is isomorphic to a direct product of directly indecomposable structures of the class $\mathscr{C}$.
2. Whenever $\mathfrak{A} \cong \prod_{\tau \in T} \mathfrak{A}_\tau \cong \prod_{\sigma \in S} \mathfrak{B}_\sigma$, where $\mathfrak{A}_\tau, \mathfrak{B}_\sigma \in \mathscr{C}$, $\tau \in T$, $\sigma \in S$, and $\mathfrak{A}_\tau, \mathfrak{B}_\sigma$ are directly indecomposable, then card T = card S and there exists a one-to-one map π of T onto S such that $\mathfrak{A}_\tau \cong \mathfrak{B}_{\pi(\tau)}$ for each $\tau \in T$ ($\cong$ denotes an isomorphism).

The problem is to give conditions for algebraic structures to have the unique factorization property.

This problem was solved in [1] for finite structures. Also some results about it are known for finite direct products. The purpose of this paper is to find some sufficient condition in the general case.

Let $\mathfrak{A} = \langle A, 0, R_\gamma \rangle_{\gamma \in \Gamma}$ be an algebraic structure with $\langle 0, \ldots, 0 \rangle \in R_\gamma$ for each $\gamma \in \Gamma$. The element 0 is called the *zero* of $\mathfrak{A}$. We say that $\mathfrak{A}$ is *almost without zero-divisors,* if *card* $A > 1$ and there exist two fixed indices $\gamma_0, \gamma_1 \in \Gamma$ such that R_{γ_0} is a partial binary operation $\oplus$ and R_{γ_1} is an n-ary operation ω $(n > 1)$ with the following properties

(i) $a \oplus 0$, $0 \oplus a$ exist for each $a \in A$ and $a \oplus 0 = 0 \oplus a = a$,

(ii) for arbitrary $a_1, \ldots, a_n \in A$

$$a_1 \ldots a_n \omega = 0 \text{ if and only if } a_j = 0 \text{ for at least one } j \in \{1, \ldots, n\}.$$

The direct product of algebraic structures $\mathfrak{A}_\tau$ of the same type for $\tau \in T$ will be denoted by $\prod_{\tau \in T} \mathfrak{A}_\tau$. We denote by pr_τ the projection of $\prod_{\tau \in T} \mathfrak{A}_\tau$ onto the direct factor $\mathfrak{A}_\tau$. If $\mathfrak{A}_\tau = \langle A_\tau, 0, R_\varphi \rangle_{\gamma \in \Gamma}$ and $\mathfrak{A} = \prod_{\tau \in T} \mathfrak{A}_\tau$, then $\mathfrak{A} = \langle A, 0_A, R_\gamma \rangle_{\gamma \in \Gamma}$ where A is the

Cartesian product of A_τ ($\tau \in T$) and 0_A is an element satisfying $pr_\tau 0_A = 0$ for each $\tau \in T$; further, relations R_γ are performed component by component. Clearly, 0_A is the zero of $\mathfrak{A}$.

Lemma 1. *An algebraic structure almost without zero-divisors is directly indecomposable over the class of structures almost without zero-divisors.*

Proof. Let $\prod_{\tau \in T} \mathfrak{A}_\tau$ be an algebraic structure almost without zero-divisors and so for $\mathfrak{A}_\tau$ ($\tau \in T$). Let $card\ T > 2$. If $\tau', \tau'' \in T$, $\tau' \neq \tau''$, $a, b \in \prod_{\tau \in T} \mathfrak{A}_\tau$ such that $pr\ a \neq 0$, $pr_\tau a = 0$ for $\tau \neq \tau'$ and $pr_{\tau''} b \neq 0$, $pr_\tau b = 0$ for $\tau \neq \tau''$, then $a \neq 0_A \neq b$, however, $ab\ldots b\omega = 0_A$, which is a contradiction.

Notation. Let $\mathfrak{A}_\tau$ ($\tau \in T$) be algebraic structures of the same type, $\mathfrak{A} = \prod_{\tau \in T} \mathfrak{A}_\tau$. Let $T' \subseteq T$. By $\bar{\mathfrak{A}}_\tau$ (or $\overline{\prod_{\tau \in T'} \mathfrak{A}_\tau}$) is denoted a substructure of $\mathfrak{A}$ such that

$$a \in \bar{\mathfrak{A}}_\tau \text{ iff } a \in \mathfrak{A} \text{ and } pr_{\tau'} a = 0 \quad \text{for } \tau' \neq \tau$$

(or $a \in \prod_{\tau \in T'} \mathfrak{A}_\tau$ iff $a \in \mathfrak{A}$ and $pr_\tau a = 0$ for $\tau \in T - T'$, respectively). If $a_\tau \in \mathfrak{A}_\tau$, we denote by $\bar{a}_\tau$ an element of $\bar{\mathfrak{A}}_\tau$ such that $pr_\tau \bar{a}_\tau = a_\tau$.

Lemma 2. *Let $\mathfrak{A}_\tau$ for $\tau \in T$ be algebraic structures almost without zero-divisors and of the same type. Let $\mathfrak{A} = \prod_{\tau \in T} \mathfrak{A}_\tau$.*

(a) *Let $a_1, \ldots, a_n \in \mathfrak{A}$ such that for each $\tau \in T$ there exists at least one $j \in \{1, \ldots, n\}$ with $pr_\tau a_j = 0$. Then $a_1 \ldots a_n \omega = 0_A$.*

(b) *Let $a, b \in \mathfrak{A}$ with either $pr_\tau a = 0$ or $pr_\tau b = 0$ for each $\tau \in T$. Then $a \oplus b$, $b \oplus a$ exist in $\mathfrak{A}$ $a \oplus b = b \oplus a$ and*

$$pr_\tau(a \oplus b) = pr_\tau a \ \text{ or } \ pr_\tau b$$

for each $\tau \in T$.

(c) *$\mathfrak{A}_\tau$ for $\tau \in T$ are also algebraic structures almost without zero-divisors of the same type as $\mathfrak{A}_\tau$.*

Proof. (a) For each $\tau \in T$ we have

$$\begin{aligned} pr_\tau(a_1 \ldots a_n \omega) &= (pr_\tau a_1) \ldots (pr_\tau a_n)\omega = \\ &= (pr_\tau a_1) \ldots (pr_\tau a_{j-1}) 0 (pr_\tau a_{j+1}) \ldots \\ &(pr_\tau a_n)\omega = 0, \text{ thus } a_1 \ldots a_n \omega = 0_A. \end{aligned}$$

(b) For each $\tau \in T$ $pr_\tau a \oplus pr_\tau b$ is defined and equal to $pr_\tau b \oplus pr_\tau a$ (by (i)). From the definition of direct product we get

$$pr_\tau(a \oplus b) = pr_\tau a \oplus pr_\tau b\,,$$

thus $a \oplus b$ is also defined and equal to $b \oplus a$. As $pr_\tau a = 0$ or $pr_\tau b = 0$, it is evident that

$$pr_\tau(a \oplus b) = pr_\tau a \oplus pr_\tau b = pr_\tau a \quad \text{or} \quad pr_\tau b\,.$$

(c) Evident.

Theorem 1. *Let $\mathfrak{A}_\tau$, $\mathfrak{B}_\sigma$ for $\tau \in T$, $\sigma \in S$ be algebraic structures almost without zero-divisors and of the same type. Let $\mathfrak{A} = \prod_{\tau \in T} \mathfrak{A}_\tau$, $\mathfrak{B} = \prod_{\sigma \in S} \mathfrak{B}_\sigma$ and φ be an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. Then there exists an injective mapping $\sigma \to \tau_\sigma$ of S into T such that $\bar{\mathfrak{B}}_\sigma \subseteq \varphi(\bar{\mathfrak{A}}_{\tau_\sigma})$ for each $\sigma \in S$.*

Proof. 1°. Let us choose a fixed $\sigma_0 \in s$ arbitrarily. Let $b \in \bar{\mathfrak{B}}_{\sigma_0}$, $b \neq 0_B$. Then $pr_{\sigma_0} b = b_{\sigma_0} \neq 0$ and $pr_\sigma b = 0$ for $\sigma \neq \sigma_0$. Denote by a an element of $\mathfrak{A}$ such that $\varphi(a) = b$. As φ is an isomorphism, such $a \in \mathfrak{A}$ exists, hence there exists $\tau_0 \in T$ such that $pr_{\tau_0} a \neq 0$. Denote $a_{\tau_0} = pr_{\tau_0} a$. Further, denote by c an element of $\mathfrak{A}$ such that $pr_\tau c = pr_\tau a$ for $\tau \neq \tau_0$ and $pr_{\tau_0} c = 0$. By Lemma 2 (b), $\bar{a}_{\tau_0} \oplus c$ exists and, evidently, $a = \bar{a}_{\tau_0} \oplus c$.

By Lemma 2(a) we obtain $\bar{a}_{\tau_0} c \ldots c\omega = 0_A$. Hence $0_B = \varphi(0_A) = \varphi(a_{\tau_0} c \ldots c\omega) = \varphi(a_{\tau_0})\varphi(c) \ldots \varphi(c)\omega$. According to the definition of direct product we get $0 = pr_\sigma 0_B = (pr_\sigma \varphi(\bar{a}_{\tau_0}))(pe_\sigma \varphi(c)) \ldots (pr_\sigma \varphi(c))\omega$. By (ii) we get

(A) $\quad$ either $pr_\sigma \varphi(\bar{a}_{\tau_0}) = 0$ or $pr_\sigma \varphi(c) = 0$ for each $\sigma \in S$.

2°. If $\varphi(\bar{a}_{\tau_0}) \notin \bar{\mathfrak{B}}_{\sigma_0}$, then there exists $\sigma' \neq \sigma_0$ such that $pr_{\sigma'} \varphi(\bar{a}_{\tau_0}) \neq 0$. By (A), $pr_{\sigma'} \varphi(c) = 0$. As φ is an isomorphism, the existence of $\bar{a}_{\tau_0} \oplus c$ implies the existence of $\varphi(\bar{a}_{\tau_0}) \oplus \varphi(c)$ and, moreover, $\varphi(\bar{a}_{\tau_0}) \oplus \varphi(c) = \varphi(\bar{a}_{\tau_0} \oplus c) = \varphi(a) = b$. Thus $0 = pr_{\sigma'} b = pr_{\sigma'} \varphi(a) = pr_{\sigma'} \varphi(\bar{a}_{\tau_0}) \oplus pr_{\sigma'} \varphi(c) = pr_{\sigma'} \varphi(\bar{a}_{\tau_0}) \oplus 0 = pt_{\sigma'} \varphi(\bar{a}_{\tau_0}) \neq 0$, which is a contradiction. In the same way we obtain a contradiction for $\varphi(c) \notin \bar{\mathfrak{B}}_{\sigma_0}$. In the summary, we have

(B) $\quad \varphi(\bar{a}_{\tau_0}) \in \bar{\mathfrak{B}}_{\sigma_0}, \quad \varphi(c) \in \bar{\mathfrak{B}}_{\sigma_0}$.

3°. By (B) $pr_\sigma \varphi(\bar{a}_{\tau_0}) = 0 = pr_\sigma \varphi(c)$ for each $\sigma \neq \sigma_0$. By (A) either $pr_{\sigma_0} \varphi(\bar{a}_{\tau_0}) = 0$ or $pr_{\sigma_0} \varphi(c) = 0$. Thus either $\varphi(\bar{a}_{\tau_0}) = 0_B$ or $\varphi(c) = 0_B$.

If $\varphi(c) \neq 0_B$, then $\varphi(\bar{a}_{\tau_0}) = 0_B$ and $b = \varphi(a) = \varphi(\bar{a}_{\tau_0}) \oplus \varphi(c) = \varphi(c)$. As φ is a one-to-one mapping, we get $a = c$. However, $0 = pr_{\tau_0} c$, $pr_{\tau_0} a = a_{\tau_0} \neq 0$, and therefore $a \neq c$, which is a contradiction.

It remains $\varphi(c) = 0_B$, i. e. $\varphi(a) = \varphi(\bar{a}_{\tau_0}) \oplus \varphi(c) = \varphi(\bar{a}_{\tau_0})$. As φ is a one-to-one mapping, it implies $a = \bar{a}_{\tau_0}$. Thus, for each $b \in \bar{B}_{\sigma_0}$, $b \neq 0_B$ there exists an index $\tau_0 \in T$ and an element $\bar{a}_{\tau_0} \in \bar{\mathfrak{A}}_{\tau_0}$ such that $\varphi(\bar{a}_{\tau_0}) = b$.

4°. We prove that this index τ_0 is the same for all $b \in \mathfrak{B}_{\sigma_0}$. Let $b_1, b_2 \in \mathfrak{B}_{\sigma_0}$, $b_1 \neq 0_B \neq b_2$. By 3° there exist $\tau_1, \tau_2 \in T$ and elements $\bar{a}_{\tau_1} \in \bar{\mathfrak{A}}_{\tau_1}$, $\bar{a}_{\tau_2} \in \bar{\mathfrak{A}}_{\tau_2}$ such that $\varphi(\bar{a}_{\tau_1}) = b_1$, $\varphi(\bar{a}_{\tau_2}) = b_2$. Clearly $\bar{a}_{\tau_1} \neq 0_A \neq \bar{a}_{\tau_2}$. Suppose $\tau_1 \neq \tau_2$. Then, by Lemma 2(a),

$$0_B = \varphi(0_A) = \varphi(\bar{a}_{\tau_1}\bar{a}_{\tau_2}\ldots\bar{a}_{\tau_2}\omega) = b_1 a_2 \ldots b_2 \omega,$$

however, by Lemma 2(c),

$$0_B \neq b_1 b_2 \ldots b_2 \omega, \quad \text{a contradiction. Thus} \quad \tau_1 = \tau_2.$$

Accordingly, there exists an index $\tau_0 \in T$ such that for each $b \in \mathfrak{B}_{\sigma_0}$, $b \neq 0_B$ there exists $a \in \bar{\mathfrak{A}}_{\tau_0}$ satisfying $\varphi(a) = b$. If $b = 0_B$, we put $a = 0_A$. Clearly $\varphi(0_A) = 0_B$ and $0_A \in \bar{\mathfrak{A}}_{\tau_0}$. Thus $\varphi(\bar{\mathfrak{A}}_{\tau_{\sigma_0}}) \supseteq \mathfrak{B}_{\sigma_0}$, where $\tau_{\sigma_0} = \tau_0$. As $\sigma_0 \in S$ was choosen arbitrarily, the preceding holds true for each $\sigma \in S$.

5° The unicity of τ_σ for each $\sigma \in S$ follows by 4°: If $\mathfrak{B}_\sigma \subseteq \varphi(\bar{\mathfrak{A}}_{\tau_1})$, $\mathfrak{B}_\sigma \subseteq \varphi(\bar{\mathfrak{A}}_{\tau_2})$, $\tau_1 \neq \tau_2$, then for $0_B \neq b \in \mathfrak{B}_\sigma$ we get $b = \varphi(\bar{a}_{\tau_i})$, $\bar{a}_{\tau_i} \in \bar{\mathfrak{A}}_{\tau_i}$, $i = 1, 2$. This yields a contradiction by the reasoning as in 4°. Hence there is a mapping $\sigma \to \tau_\sigma$ of S into T with $\mathfrak{B}_\sigma \subseteq \varphi(\bar{\mathfrak{A}}_{\tau_\sigma})$.

6°. Prove the injectivity of this mapping. Let $\sigma_1, \sigma_2 \in S$, $\tau \in T$ and $\mathfrak{B}_{\sigma_1} \subseteq \varphi(\bar{\mathfrak{A}}_\tau)$, $\mathfrak{B}_{\sigma_2} \subseteq \varphi(\bar{\mathfrak{A}}_\tau)$. Let $\bar{b}_{\sigma_1} \in \mathfrak{B}_{\sigma_1}$, $\bar{b}_{\sigma_2} \in \mathfrak{B}_{\sigma_2}$, $\bar{b}_{\sigma_1} \neq 0_B \neq b_{\sigma_2}$. Then there exist $a_1, a_2 \in \bar{\mathfrak{A}}_\tau$ such that $\varphi(a_1) = \bar{b}_{\sigma_1}$, $\varphi(a_2) = \bar{b}_{\sigma_2}$. Clearly, $a_1 \neq 0_A \neq a_2$. By Lemma 2(c), $\bar{\mathfrak{A}}_\tau$ is almost without zero-divisors, thus $a_1 a_2 \ldots a_2 \omega \neq 0_A$. However, $0_B = \varphi(0_A) \neq$ $\neq \varphi(a_1 a_2 \ldots a_2 \omega) = \bar{b}_{\sigma_1}\bar{b}_{\sigma_2}\omega = 0_B$ by Lemma 2(a), which is a contradiction. Summarizing, $\sigma \to \tau_\sigma$ is an injective mapping.

Theorem 2. *If an algebraic structure $\mathfrak{A} = \langle A, 0, R_\gamma \rangle_{\gamma \in \Gamma}$ is directly decomposable into structures almost without zero-divisors, then $\mathfrak{A}$ has the unique factorization property over the class of structures almost without zero-divisors.*

Proof. Let $\mathfrak{A}_\tau$, $\mathfrak{B}_\sigma$ be algebraic structures almost without zero-divisors of the same type for $\tau \in T$, $\sigma \in S$ and

$$\mathfrak{A} \cong \prod_{\tau \in T} \mathfrak{A}_\tau \cong \prod_{\sigma \in S} \mathfrak{B}_\sigma.$$

By Lemma 1, $\mathfrak{A}_\tau$ and $\mathfrak{B}_\sigma$ are directly indecomposable. Denote by φ the isomorphism of $\prod_{\tau \in T} \mathfrak{A}_\tau$ onto $\prod_{\sigma \in S} \mathfrak{B}_\sigma$. Clearly φ^{-1} is also an isomorphism of $\prod_{\sigma \in S} \mathfrak{B}_\sigma$ onto $\prod_{\tau \in T} \mathfrak{A}_\tau$ and $\varphi^{-1}\varphi = id_A$. By Theorem 1, there exists exactly one $\tau_\sigma \in T$ for each $\sigma \in S$ and just one $\sigma_\tau \in S$ for each $\tau \in T$ such that

$$\varphi(\bar{\mathfrak{A}}_{\tau_\sigma}) \supseteq \bar{\mathfrak{B}}_\sigma, \quad \varphi^{-1}(\mathfrak{B}_{\sigma_\tau}) \supseteq \bar{\mathfrak{A}}_\tau.$$

As $\bar{\mathfrak{A}}_{\tau'} \cap \bar{\mathfrak{A}}_{\tau''} = \{0_A\}$ for $\tau' \neq \tau''$, $\tau', \tau', \tau'' \in T$, we have $\tau_{\sigma_\tau} = \tau$ and $\varphi^{-1}(\mathfrak{B}_{\sigma_\tau}) = \bar{\mathfrak{A}}_\tau$. In the same way $\sigma_{\tau_\sigma} = \sigma$ can be proved and $\varphi(\bar{\mathfrak{A}}_{\tau_\sigma}) = \mathfrak{B}_\sigma$. In other words the mappings $\alpha: \sigma \to \tau_\sigma$ and $\beta: \tau \to \sigma_\tau$ satisfy

$$\alpha\beta = id_S, \quad \beta\alpha = id_T.$$

Hence α and β are subjective and injective and $\bar{\mathfrak{A}}_\tau$ isomorphic to $\mathfrak{B}_{\beta(\tau)}$. It follows that $\mathfrak{A}_\tau$ is isomorphic to $\mathfrak{B}_{\beta(\tau)}$, which completes the proof.

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О ПРОБЛЕМЕ ОДНОЗНАЧНОЙ ФАКТОРИЗАЦИИ

Иван Хайда

Резюме

Предполагаем, что алгебраическая структура $\mathfrak{A}$ выполняет условие однозначной факторизации в классе $\mathscr{C}$, существуют ли прямо неразложимые структуры $\mathfrak{A}_\tau \in \mathscr{C}$ для $\tau \in T$ так, что $\mathfrak{A}$ изоморфна прямому произведению структур $\mathfrak{A}_\tau$ $(\tau \in T)$ и, если $\mathfrak{B}_\sigma \in \mathscr{C}$ для $\sigma \in S$ прямо неразложима и $\mathfrak{A}$ изоморфна прямому произведению структур $\mathfrak{B}_\sigma$ $(\sigma \in S)$, потом существует биекция π множества T на S так, что $\mathfrak{A}_\tau$ изоморфна с $\mathfrak{B}_{\pi(\tau)}$. В этой работе введено понятие алгебраической структуры почти без делителей нуля и доказано, что если алгебраическая структура $\mathfrak{A}$ прямо разложима на структуры $\mathfrak{A}_\tau$ $(\tau \in T)$ почти без делителей нуля (для произвольного множества T), то $\mathfrak{A}$ выполняет условие однозначной факторизации над классом структур почти без делителей нуля.