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ON CHARACTERIZATION OF THE LIPSCHITZIAN COMPOSITION OPERATOR BETWEEN SPACES OF FUNCTIONS OF BOUNDED p-VARIATION

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Introduction

Let $I = [a,b]$ be an interval, $f\colon I \times \mathbb{R} \to \mathbb{R}$ a fixed two-place function, and $\mathcal{F}(I)$ the linear space of all functions $u\colon I \to \mathbb{R}$. The function $F\colon \mathcal{F}(I) \to \mathcal{F}(I)$ given by the formula

$$(Fu)(t) := f\big(t, u(t)\big) \quad t \in I,\ u \in \mathcal{F}(I),$$

is called a composition operator. In [4] it is proved that a composition operator F maps the space $\operatorname{Lip}(I)$ of all Lipschitzian function into itself and is globally Lipschitzian if and only if $f(t,x) = g(t)x + h(t)$, where g, $h \in \operatorname{Lip}(I)$.

This result has been further extended to some other function Banach spaces (see [1–7]). Recently N. Merentes (see [7]) proved an analogous theorem in the space $RV_p[a,b]$ of functions of bounded p-variation in the sense of Riesz $(1 < p < \infty)$. In the present paper we generalize these results in the case that the composition operator F is globally Lipschitzian between spaces $RV_p[a,b]$ and $RV_q[a,b]$ where $1 \leqslant q \leqslant p$. On the other hand, if $1 \leqslant p < q$, the composition operator F is constant.

1. Preliminary results

Given $1 \leqslant p < \infty$ and $u\colon [a,b] \to \mathbb{R}$, we write

$$V_p(u;\pi) := \sup_{\pi} \sum_{i=1}^{n} \frac{|u(t_i) - u(t_{i-1})|^p}{|t_i - t_{i-1}|^{p-1}}$$

for the p-variation of the function u in the sense of Riesz, where the supremum is taken over all partitions $\pi\colon a = t_0 < \ldots < t_n = b$ of the interval $[a,b]$. By

$RV_p = RV_p[a,b]$ we denote the Banach space of all functions u on $[a,b]$ for which the norm

$$\|u\|_p := |u(a)| + \big(V_p(u;[a,b])\big)^{\frac{1}{p}}$$

is finite. Usually, one takes $BV_\infty[a,b]$ as the space $\mathrm{Lip}[a,b]$ of all Lipschitzian functions on $[a,b]$ with the norm

$$\|u\|_{\mathrm{Lip}[a,b]} := |u(a)| + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|}.$$

Moreover, the space $RV_1[a,b]$ is simply denoted by $BV[a,b]$ and it is the classical space of functions of bounded variation on $[a,b]$.

It is easy to see that if $p > 1$, then every function $u \in RV_p[a,b]$ is continuous. More precisely, the inclusions

$$\mathrm{Lip}[a,b] \subset RV_p[a,b] \subset AC[a,b] \subset BV[a,b] \quad (p > 1)$$

hold, where $AC[a,b]$ is the space of all absolutely continuous functions.

Lemma 1 ([8], Riesz). *Let $1 < p < \infty$ be a fixed number. A function u fulfills $u \in RV_p[a,b]$ if and only if $u \in AC[a,b]$ and $u' \in L_p[a,b]$. In that case we also have the equality*

$$V_p(u;[a,b]) = \int_a^b |u'(t)|^p \,\mathrm{d}t.$$

F. Szigeti (see [9], p. 13) proved that the space $RV_p[a,b]$ $(1 < p < \infty)$ is also a Banach algebra.

In [7] it is proved that the composition operator F generated by $f\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ maps the space $RV_p[a,b]$ $(1 < p < \infty)$ into itself and is globally Lipschitzian if and only if $f(t,x) = g(t)x + h(t)$ $(t \in [a,b];\ x \in \mathbb{R})$ for some g, $h \in RV_p[a,b]$. In the case $p = 1$, J. Matkowski and J. Miś (see [6]) proved that the composition operator F, generated by f, maps the space $BV[a,b]$ into itself and satisfies the global Lipschitzian condition if and only if

$$\overline{f}(x,y) = g(x)y + h(x)$$

for two functions g, $h \in NBV[a,b]$, where

$$\overline{f}(x,y) = \lim_{\delta \to 0} f(x - \delta, y) \quad (y \in \mathbb{R})$$

is the left-continuous regularization of f and $NBV[a,b]$ is the subspace of all functions $u \in BV[a,b]$ such that u is continuous on $[a,b]$ from the left.

In this section we will present a characterization of functions $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ for which the composition operator F generated by f maps the space $RV_p[a,b]$ into the space $RV_q[a,b]$ $(1\leqslant q\leqslant p)$ and is globally Lipschitzian. In the case $1\leqslant p<q$, the composition operator is constant.

Theorem 1. *Let p, q be real numbers such that $1<q\leqslant p$. The composition operator F generated by $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ maps the space $RV_q[a,b]$ into the space $RV_p[a,b]$ and is globally Lipschitzian if and only if the function f satisfies the following conditions:*

a) *For all $t\in[a,b]$ there exists $M(t)>0$ such that*

$$|f(t,x)-f(t,y)|\leqslant M(t)|x-y|\quad (x,y\in\mathbb{R}), \tag{1}$$

b)

$$f(t,x)=g(t)x+h(t)\quad (t\in[a,b],\ x\in\mathbb{R}), \tag{2}$$

where g, $h\in RV_q[a,b]$.

Proof. Suppose that there exist g, $h\in RV_q[a,b]$ such that $f(t,x)=g(t)x+h(t)$ $(t\in[a,b],\ x\in\mathbb{R})$. Then the composition F generated by f is given by

$$(Fu)(t)=g(t)u(t)+h(t)\quad (t\in[a,b],\ u\in RV_q[a,b]).$$

Since $F(RV_p[a,b])\subset RV_q[a,b]$ $(1<q\leqslant p)$ and $RV_q[a,b]$ is a Banach algebra, then $Fu\in RV_q[a,b]$ for all $u\in RV_p[a,b]$.

Moreover,

$$\|Fu_1-Fu_2\|_q\leqslant \|g\|_q\|u_1-u_2\|_p\quad (u_1,u_2\in RV_p[a,b]).$$

Thus, the composition operator F maps the space $RV_p[a,b]$ into the space $RV_q[a,b]$ and is globally Lipschitzian.

Suppose now that $F\colon RV_p[a,b]\to RV_q[a,b]$ $(1<q\leqslant p)$ is globally Lipschitzian, then there exists a constant $M>0$ such that

$$\|Fu_1-Fu_2\|_q\leqslant M\|u_1-u_2\|_p\quad (u_1,u_2\in RV_p[a,b]).$$

Let $t\in(a,b]$. Using the definition of the operator F and of the norm $\|.\|_q$ we have

$$\bigl|f\bigl(t,u_1(t)\bigr)-f\bigl(t,u_2(t)\bigr)-f\bigl(a,u_1(a)\bigr)+f\bigl(a,u_2(a)\bigr)\bigr|\leqslant M|t-a|^{1-\frac{1}{q}}\|u_1-u_2\|_p \tag{3}$$

for all u_1, $u_2 \in RV_p[a,b]$.

Define a function $\alpha\colon [a,b] \to \mathbb{R}$ by

$$\alpha(\tau) := \begin{cases} \dfrac{\tau - a}{t - a}, & a \leqslant \tau \leqslant t, \\ 1, & t \leqslant \tau \leqslant b. \end{cases}$$

We have $\alpha \in RV_p[a,b]$ and

$$V_p(\alpha; [a,b]) = \frac{1}{|t-a|^{p-1}}.$$

Let us fix x, $y \in \mathbb{R}$ and define functions $u_i\colon [a,b] \to \mathbb{R}$ $(i = 1, 2)$ by

(4) $$u_1(\tau) := x, \quad \tau \in [a,b], \quad u_2(\tau) := \alpha(\tau)(y - x) + x, \quad \tau \in [a,b].$$

The functions u_i fulfill $u_i \in RV_p([a,b])$ $(i = 1, 2)$ and

$$\|u_1 - u_2\|_p = \big(V_p(\alpha; [a,b])\big)^{\frac{1}{p}} |x - y| = \frac{|x-y|}{|t-a|^{1-\frac{1}{p}}}.$$

Hence, substituting into the inequality (3) the particular functions u_i $(i = 1, 2)$ defined by (4), we obtain

(5) $$|f(t,x) - f(t,y)| \leqslant M \frac{|t-a|^{1-\frac{1}{q}}}{|t-a|^{1-\frac{1}{p}}} |x - y|$$

for all $t \in (a,b]$, x, $y \in \mathbb{R}$.

Now, let $t = a$. Define a function $\beta\colon [a,b] \to \mathbb{R}$ by

$$\beta(\tau) := \frac{\tau - a}{b - a} \quad (\tau \in [a,b]).$$

The function β fulfills $\beta \in RV_p[a,b]$ and

$$V_p(\beta; [a,b]) = \frac{1}{|b-a|^{p-1}}.$$

Let us fix x, $y \in \mathbb{R}$ and define functions $u_i\colon [a,b] \to \mathbb{R}$ $(i = 1, 2)$ by

(6) $$u_1(\tau) := x, \quad \tau \in [a,b], \quad u_2(\tau) := \beta(\tau)(x - y) + y, \quad \tau \in [a,b].$$

The functions u_i fulfill $u_i \in RV_p[a,b]$ $(i = 1, 2)$ and

$$\|u_1 - u_2\|_p = \big(1 + \big(V_p(\beta; [a,b])\big)^{\frac{1}{p}}\big)|x - y| = \Big(1 + \frac{1}{|b-a|^{1-\frac{1}{p}}}\Big)|x - y|.$$

Hence, substituting into the inequality (3) the particular functions u_i ($i = 1, 2$) defined by (6), we obtain

$$|f(a,x) - f(a,y)| \leqslant M|b-a|^{1-\frac{1}{q}}\Big(1 + \frac{1}{|b-a|^{1-\frac{1}{p}}}\Big)|x-y|$$

for all x, $y \in \mathbb{R}$.

Define a function $M\colon [a,b] \to \mathbb{R}$ by

$$M(t) := \begin{cases} M\dfrac{|t-a|^{1-\frac{1}{q}}}{|t-a|^{1-\frac{1}{p}}}, & a < t \leqslant b, \\ M|b-a|^{1-\frac{1}{q}}\Big(1 + \dfrac{1}{|b-a|^{1-\frac{1}{p}}}\Big), & t = a. \end{cases}$$

Hence we have for all $t \in [a,b]$ that there exists $M(t) > 0$ such that the inequality (1) holds. Thus for all $t \in [a,b]$ the function $f(t,.)\colon \mathbb{R} \to \mathbb{R}$ is continuous.

Next we shall prove that f satisfies the equality (2).

Let us fix t, $t_0 \in [a,b]$ such that $t_0 < t$. Since the composition operator F generated by $f\colon [a,b]\times\mathbb{R} \to \mathbb{R}$ is globally Lipschitzian between $RV_p[a,b]$ and $RV_q[a,b]$ ($1 < q \leqslant p$), there exists a constant $M > 0$ such that

$$\text{(7)} \quad \big|f\big(t,u_1(t)\big) - f\big(t,u_2(t)\big) - f\big(t_0,u_1(t_0)\big) + f\big(t_0,u_2(t_0)\big)\big| \leqslant M\|u_1-u_2\|_p|t-t_0|^{1-\frac{1}{q}}$$

for all u_1, $u_2 \in RV_p[a,b]$.

Define a function $\gamma\colon [a,b] \to \mathbb{R}$ by

$$\gamma(\tau) := \begin{cases} \dfrac{\tau-a}{t_0-a}, & a \leqslant \tau \leqslant t_0, \\ -\dfrac{\tau-t}{t-t_0}, & t_0 \leqslant \tau \leqslant t, \\ 0, & t \leqslant \tau \leqslant b. \end{cases}$$

The function γ fulfills $\gamma \in RV_p[a,b]$. Let us fix x, $y \in \mathbb{R}$ and define functions $u_i\colon [a,b] \to \mathbb{R}$ by

$$\text{(8)} \quad \begin{aligned} u_1(\tau) &:= \frac{\gamma(t)}{2}x + \Big(1 + \frac{\gamma(\tau)}{2}\Big)y \quad (\tau \in [a,b]), \\ u_2(\tau) &:= \frac{1+\gamma(\tau)}{2}x + \frac{1-\gamma(\tau)}{2}y \quad (\tau \in [a,b]). \end{aligned}$$

The functions u_i fulfill $u_i \in RV_p[a,b]$ ($i = 1, 2$) and

$$\|u_1 - u_2\|_p = \frac{|x-y|}{2}.$$

Hence, substituting into the inequality (7) the particular functions u_i $(i = 1, 2)$ defined by (8), we obtain

$$(9) \quad \left|f(t,y) - f\Big(t, \frac{x+y}{2}\Big) - f\Big(t_0, \frac{x+y}{2}\Big) + f(t_0, x)\right| \leqslant \frac{M}{2}|t-t_0|^{1-\frac{1}{q}}|x-y|.$$

Since F maps $RV_p[a,b]$ into $RV_q[a,b]$ $(1 < q \leqslant p)$, then for all $x \in \mathbb{R}$ the function $f(.,x)$ is continuous on $[a,b]$. Consequently, letting $t_0 \uparrow t$ in the inequality (9), we get

$$\left|f(t,y) - f\Big(t, \frac{x+y}{2}\Big) - f\Big(t, \frac{x+y}{2}\Big) + f(t,x)\right| = 0$$

for all $t \in [a,b]$ and x, $y \in \mathbb{R}$.

Thus for all $t \in [a,b]$, x, $y \in \mathbb{R}$, we have

$$\frac{f(t,x) + f(t,y)}{2} = f\Big(t, \frac{x+y}{2}\Big).$$

Consequently, for all $t \in [a,b]$ the function $f(t,.)\colon \mathbb{R} \to \mathbb{R}$ satisfies the Jensen equation and since the function $f(t,.)$ is continuous on $\mathbb{R}$, we have that there exist two functions g, $h\colon [a,b] \to \mathbb{R}$ such that

$$f(t,x) = g(t)x + h(t), \quad (t \in [a,b],\ x \in \mathbb{R}).$$

Since $h(t) = f(t,0) = F(0)$, $g(t) = f(t,1) - f(1,0) = F(1) - F(0)$ and F maps $RV_p[a,b]$ into $RV_q[a,b]$, we conclude g, $h \in RV_q[a,b]$. □

Remark 1. It is easy to observe that the above theorem remains true if there exist Banach spaces $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ such that $RV_p[a,b] \hookrightarrow X \hookrightarrow Y \hookrightarrow RV_q[a,b]$ $(1 < q \leqslant p)$ and the composition operator F maps the space X into the space Y and is globally Lipschitzian.

Theorem 2. *Let p, q be real numbers such that $1 < p < q$. If the composition operator F generated by $f\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ maps the space $RV_p[a,b]$ into the space $RV_q[a,b]$ and is globally Lipschitzian, then the function f satisfies the condition*

$$f(t,x) = f(t,0) \quad (t \in [a,b],\ x \in \mathbb{R}).$$

As an immediate consequence of Theorem 2 we obtain that the composition operator F is constant.

Proof. Since the composition operator F generated by $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$, maps the space $RV_p[a,b]$ into the space $RV_q[a,b]$ $(1<p<q)$ and is globally Lipschitzian, there exists a constant $M>0$ such that

$$\|Fu_1-Fu_2\|_q\leqslant M\|u_1-u_2\|_p \quad (u_1,u_2\in RV_p[a,b]).$$

Let us fix t, $t_0\in[a,b]$ such that $t_0<t$. Using the definitions of the operator F and of the norm $\|.\|_q$, we have

$$\begin{aligned}(10)\qquad &\big|f\big(t,u_1(t)\big)-f\big(t,u_2(t)\big)-f\big(t_0,u_1(t_0)\big)+f\big(t_0,u_2(t_0)\big)\big|\\ &\qquad\leqslant M|t-t_0|^{1-\frac{1}{q}}\|u_1-u_2\|_p \quad (u_1,u_2\in RV[a,b]).\end{aligned}$$

Define a function $\alpha\colon[a,b]\to\mathbb{R}$ by

$$\alpha(\tau):=\begin{cases}1, & a\leqslant t\leqslant t_0,\\ -\dfrac{\tau-t}{t-t_0}, & t_0\leqslant\tau\leqslant t,\\ 0, & t\leqslant\tau\leqslant b.\end{cases}$$

The function α fulfills $\alpha\in RV_p[a,b]$ and

$$V_p(\alpha;[a,b])=\frac{1}{|t-t_0|^{p-1}}.$$

Let us fix $x\in\mathbb{R}$ and define functions $u_i\colon[a,b]\to\mathbb{R}$ $(i=1,2)$ by

$$(11)\qquad u_1(\tau):=x\quad \tau\in[a,b],\quad u_2(\tau):=\alpha(\tau)x\quad \tau\in[a,b].$$

The functions u_i fulfill $u_i\in RV_p[a,b]$ $(i=1,2)$ and

$$\|u_1-u_2\|_p=\frac{|x|}{|t-t_0|^{1-\frac{1}{p}}}.$$

Hence, substituting into the inequality (10) the particular functions u_i $(i=1,2)$ defined by (11), we obtain

$$(12)\qquad |f(t,x)-f(t,0)|\leqslant M\frac{|t-t_0|^{1-\frac{1}{q}}}{|t-t_0|^{1-\frac{1}{p}}}|x|.$$

Since $q>p$, letting $t_0\uparrow t$ in the inequality (12) we obtain

$$f(t,x)=f(t,0)\quad (t\in[a,b],\ x\in\mathbb{R}).$$

Next we shall consider the case when the composition operator F generated by $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ maps the space $RV_p[a,b]$ into the space $BV[a,b]$. In this case a similar result holds for the left regularization $f^*\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ of the function f defined by

$$f^*(t,x) := \begin{cases} \lim\limits_{s\uparrow t} f(s,x), & t\in(a,b],\ x\in\mathbb{R},\\ \lim\limits_{s\downarrow a}\lim\limits_{v\uparrow s} f(v,x), & t=a, x\in\mathbb{R}. \end{cases}$$

□

Theorem 3. *Let p be a real number such that $1<p<\infty$. The composition operator F generated by $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ maps the space $RV_p[a,b]$ into the space $BV[a,b]$ and if it is globally Lipschitzian, then the function f satisfies the following conditions:*

a) *For each $t\in[a,b]$ there exists $M(t)>0$ such that*

$$|f^*(t,x)-f^*(t,y)|\leqslant M(t)|x-y|\quad (x,y\in\mathbb{R}), \tag{13}$$

b)

$$f^*(t,x)=g(t)x+h(t)\quad (t\in[a,b],\ x\in\mathbb{R}), \tag{14}$$

where g, $h\in NBV[a,b]$.

P r o o f. Let $t\in[a,b)$ and define a function $\alpha\colon[a,b]\to\mathbb{R}$ by

$$\alpha(t) := \begin{cases} 1, & a\leqslant\tau\leqslant t,\\ \dfrac{\tau-b}{t-b}, & t\leqslant\tau\leqslant b. \end{cases}$$

The function α fulfills $\alpha\in RV_p[a,b]$ and

$$V_p(\alpha,[a,b]) = \frac{1}{|b-t|^{p-1}}.$$

Let us fix x, $y\in K$ and define funcitons $u_i\colon[a,b]\to\mathbb{R}$ ($i=1$, 2) by

$$u_1(\tau):=x\quad \tau\in[a,b],\quad u_2(\tau):=\alpha(\tau)(y-x)+x,\quad \tau\in[a,b]. \tag{15}$$

The functions u_i fulfil $u_i\in RV_p[a,b]$ ($i=1$, 2) and

$$\|u_1-u_2\|_p = \big(V_p(\alpha;[a,b])\big)^{\frac{1}{p}}|x-y| = \Big(1\frac{1}{|b-t|^{1-\frac{1}{p}}}\Big)|x-y|.$$

Since the composition operator F is globally Lipschitzian between $RV_p[a,b]$ and $BV[a,b]$, there exists a constant $M > 0$ such that

$$\left|f\big(b,u_1(b)\big) - f\big(b,u_2(b)\big) - f\big(t,u_1(t)\big) + f\big(t,u_2(t)\big)\right| \leqslant M\|u_1 - u_2\|_p \tag{16}$$

for all u_1, $u_2 \in RV_p[a,b]$.

Hence, substituting into the inequality (16) the particular functions u_i ($i = 1, 2$) defined by (15) we obtain

$$|f(t,y) - f(t,x)| \leqslant M\left[1 + \frac{1}{|b-t|^{1-\frac{1}{p}}}\right]$$

for all $t \in [a,b)$.

In the case $t = b$, by a similar argument as above, we obtain that there exists a constant $M(b) > 0$ such that

$$|f(b,x) - f(b,y)| \leqslant M(b)|x-y| \quad (x,y \in \mathbb{R}).$$

Thus, defining a function $M\colon [a,b] \to \mathbb{R}$ by

$$M(t) := \begin{cases} 1 + \dfrac{1}{|b-t|^{1-\frac{1}{p}}}, & t \in [a,b), \\ M(b), & t = b, \end{cases}$$

we obtain that for each $t \in [a,b)$ there exists $M(t) > 0$ such that

$$|f(t,x) - f(t,y)| \leqslant M(t)|x-y| \quad \big(t \in [a,b),\ x,y \in \mathbb{R}\big). \tag{17}$$

Hence, passing to the limit in the inequality (17), by the definition of f^* we have for all $t \in [a,b]$ that there exists $M(t) > 0$ such that

$$|f^*(t,x) - f^*(t,y)| \leqslant M(t)|x-y| \quad (x,y \in \mathbb{R}).$$

Next we shall prove that f^* satisfies the equality (14).

Let us fix t, $t_0 \in [a,b]$, $n \in N$ such that $t_0 < t$. Define a partition π_n of the interval $[t_0,t]$ by $\pi_n\colon a < t_0 < t_1 < \ldots < t_{2n-1} < t_{2n} = t$, where

$$t_1 - t_{i-1} = \frac{t-t_0}{2n}, \quad i = 1,2,\ldots,2n.$$

Since the composition operator F is globally Lipschitzian between $RV_p[a,b]$ and $BV[a,b]$, there exists a constant $M > 0$ such that

$$\sum_{i=1}^{n}\left|f\big(t_{2i},u_1(t_{2i})\big) - f\big(t_{2i}u_2(t_{2i})\big) - f\big(t_{2i-1},u_1(t_{2i-1})\big) + f\big(t_{2i-1}u_2(t_{2i-1})\big)\right| \leqslant M\|u_1-u_2\|_p \quad (u_1,u_2 \in RV_p[a,b]). \tag{18}$$

Define a function $\alpha\colon [a,b] \to \mathbb{R}$ in the following way:

$$\alpha(\tau) := \begin{cases} 0, & a \leqslant \tau \leqslant t_0, \\ \dfrac{\tau - t_{i-1}}{t_i - t_{i-1}}, & t_{i-1} \leqslant \tau \leqslant t_i,\ i = 1, 3, \dots, 2n-1, \\ -\dfrac{\tau - t_i}{t_i - t_{i-1}}, & t_{i-1} \leqslant \tau \leqslant t_i,\ i = 2, 4, \dots, 2n, \\ 0, & t \leqslant \tau \leqslant b. \end{cases}$$

The function α fulfils $\alpha \in RV_p[a,b]$ and

$$V_p(\alpha; [a,b]) = \frac{2^p n^p}{|t - t|^{p-1}}.$$

Let us fix x, $y \in \mathbb{R}$ and define functions $u_i\colon [a,b] \to \mathbb{R}$ by

$$\begin{aligned} u_1(\tau) &:= \frac{\alpha(\tau)}{2}x + \Bigl(1 - \frac{\alpha(\tau)}{2}\Bigr)y \quad (\tau \in [a,b]), \\ u_2(\tau) &:= \frac{1+\alpha(\tau)}{2}x + \frac{1-\alpha(\tau)}{2}y \quad (\tau \in [a,b]). \end{aligned} \tag{19}$$

The functions u_i fulfil $u_i \in RV_p[a,b]$ $(i = 1,\, 2)$ and

$$\|u_1 - u_2\|_p = \frac{|x-y|}{2}.$$

Hence, substituting into the inequality (18) the particular functions u_i $(i = 1,\, 2)$ defined in (19), we obtain

$$\sum_{i=1}^{n} \Bigl| f(t_{2i}, y) - f\Bigl(t_{2i}, \frac{x+y}{2}\Bigr) - f\Bigl(t_{2i-1}, \frac{x+y}{2}\Bigr) + f(t_{2i-1}, x) \Bigr| \leqslant M\frac{|x-y|}{2} \tag{20}$$

for all x, $y \in \mathbb{R}$.

Since the composition operator F maps the space $RV_p[a,b]$ into the space $BV[a,b]$, then $f(.,x) \in BV[a,b]$ for all $x \in \mathbb{R}$, thus letting $t_0 \uparrow t$ in the inequality (20) we get

$$\Bigl| f^*(t,y) - f^*\Bigl(t, \frac{x+y}{2}\Bigr) - f^*\Bigl(t, \frac{x+y}{2}\Bigr) + f^*)t, x) \Bigr| \leqslant M\frac{|x-y|}{2n} \tag{21}$$

for all x, $y \in \mathbb{R}$, $n \in N$.

Passing to the limit for $n \to \infty$ in the inequality (21), we get

$$\frac{f^*(t,y) + f^*(t,x)}{2} = f^*\Bigl(t, \frac{x+y}{2}\Bigr)$$

for all $t \in [a,b]$, x, $y \in \mathbb{R}$.

Thus for all $t \in [a,b]$, the function $f^*(t,.) \colon \mathbb{R} \to \mathbb{R}$ satisfies the Jensen equation and by property (a) of this theorem we get that there exist two functions g, $h \in NBV[a,b]$ such that

$$f^*(t,x) = g(t)x + h(t) \quad (t \in [a,b],\ x \in \mathbb{R}).$$

$\square$

Remark 2. It is easy to observe that the above theorem remains true if there exists a Banach space $(X, \|.\|_X)$ such that $RV_p[a,b] \hookrightarrow X \hookrightarrow BV[a,b]$ $(1 < p < \infty)$ and the composition operator F maps the space X into the space $BV[a,b]$ and is globally Lipschitzian.

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