Czechoslovak Mathematical Journal

R. Kantrowitz; Michael M. Neumann

Automatic continuity of homomorphisms and derivations on algebras of continuous vector-valued functions

Czechoslovak Mathematical Journal, Vol. 45 (1995), No. 4, 747–756

Persistent URL: `http://dml.cz/dmlcz/128554`

Czechoslovak Mathematical Journal, 45 (120) 1995, Praha

AUTOMATIC CONTINUITY OF HOMOMORPHISMS AND DERIVATIONS ON ALGEBRAS OF CONTINUOUS VECTOR-VALUED FUNCTIONS

R. KANTROWITZ, Clinton, and M. M. NEUMANN, Mississippi State

(Received April 14, 1994)

INTRODUCTION

The question of the extent to which homomorphisms and derivations on complex Banach or Fréchet algebras are automatically continuous has received a considerable amount of attention, particularly in the context of semi-simple algebras. The seminal result in this area is, of course, the automatic continuity of all multiplicative linear functionals on Banach algebras, which leads easily to the continuity of all homomorphisms from a Banach algebra into a semi-simple commutative Banach algebra. This implies, in particular, the uniqueness of the complete norm topology on semi-simple commutative Banach algebras. A famous theorem due to Johnson [6] extends this result to arbitrary semi-simple Banach algebras. Moreover, Johnson and Sinclair have shown that derivations on semi-simple Banach algebras are necessarily continuous; see [7] and [8]. Similar results have been obtained in the case of certain semi-simple commutative Fréchet algebras by Carpenter in [1] and [2], but it remains an intriguing open question, commonly known as Michael's problem, whether all multiplicative linear functionals on commutative Fréchet algebras are continuous. For an excellent account of the classical theory of automatic continuity, we refer to [3], [4], and [13].

In the present paper, we shall discuss the continuity of homomorphisms and derivations on certain algebras of vector-valued functions, which are typically neither commutative nor semi-simple, and seem not to be covered by the classical theory. Given a compact Hausdorff space Ω and an arbitrary unital complex Banach algebra A, let $C(\Omega, A)$ denote the Banach algebra of all A-valued continuous functions on Ω, endowed with pointwise operations and the usual supremum norm $\|\cdot\|_\infty$. Clearly, $C(\Omega, A)$ is commutative if and only if A is, and in this case, semi-simplicity of $C(\Omega, A)$ is equivalent to that of A; see for instance [5]. Further, let B denote an arbitrary complex Fréchet algebra, by which we mean a complete, metrizable topological algebra over $\mathbb{C}$, not assumed to be locally convex.

In this setting, we shall obtain the continuity of all epimorphisms $\nu\colon B \to C(\Omega, A)$ and of all derivations $\delta\colon C(\Omega, A) \to C(\Omega, A)$, provided that Ω contains no isolated points. Simple examples show that this restriction on Ω is essential here. For arbitrary compact Hausdorff spaces Ω, we shall prove that $C(\Omega, A)$ enjoys a unique Banach algebra topology whenever A does and that every derivation on $C(\Omega, A)$ is continuous whenever every derivation on A is continuous.

The basic theory will be developed in a more general context, namely for certain subalgebras of $C(\Omega, A)$ which strongly separate the points of Ω in a sense to be discussed below. Our main results are Theorems 4 and 8, which establish the automatic continuity of a homomorphism with strongly point separating range and of a derivation with strongly point separating domain in $C(\Omega, A)$. This approach is general enough to ensure the continuity of all epimorphisms and derivations on various classical algebras of vector-valued differentiable, analytic and Lipschitz functions. In particular, in turns out that these algebras carry a unique Banach or Fréchet algebra topology.

1. Strong point separation and auxiliary results

Throughout this section, let Ω be a compact Hausdorff space, and consider a unital complex Banach algebra A and a complex Fréchet algebra B, where neither algebra is assumed to be commutative or semi-simple. For an arbitrary linear transformation $\nu\colon B \to C(\Omega, A)$ and any $\omega \in \Omega$, let $\nu_\omega\colon B \to A$ denote the linear mapping given by $\nu_\omega(b) := \nu(b)(\omega)$ for all $b \in B$. We shall investigate the continuity properties of ν in terms of the set Δ_ν of all $\omega \in \Omega$ for which the mapping $\nu_\omega\colon B \to A$ is discontinuous.

Lemma 1. *For every linear transformation $\nu\colon B \to C(\Omega, A)$, the set Δ_ν is open in Ω. Moreover, ν is continuous if and only if Δ_ν is empty.*

Proof. To see that Δ_ν is open, let $(\omega_\lambda)_{\lambda\in\Lambda}$ be a net in $\Omega \setminus \Delta_\nu$ which converges to some $\omega \in \Omega$. Then the operators $\nu_{\omega_\lambda}\colon B \to A$ are continuous for all $\lambda \in \Lambda$ and pointwise bounded on B, since $\sup\{\|\nu_{\omega_\lambda}(b)\| : \lambda \in \Lambda\} \leqslant \|\nu(b)\|_\infty < \infty$ for all $b \in B$. By the principle of uniform boundedness, there exists a neighborhood U of zero in B such that $\|\nu_{\omega_\lambda}(b)\| \leqslant 1$ for all $b \in U$ and $\lambda \in \Lambda$. Since $\nu_{\omega_\lambda}(b) \to \nu_\omega(b)$ by the continuity of the function $\nu(b)$ on Ω, we conclude that $\|\nu_\omega(b)\| \leqslant 1$ for all $b \in U$, which establishes the continuity of the operator $\nu_\omega\colon B \to A$. Thus $\omega \in \Omega \setminus \Delta_\nu$ which shows that Δ_ν is indeed open. The continuity of ν obviously implies that Δ_ν is empty, and the converse follows from another application of the principle of uniform boundedness. □

Since each of the mappings ν_ω is a homomorphism whenever ν is, it follows from Lemma 1 that all homomorphisms from B into $C(\Omega, A)$ are continuous precisely when all homomorphisms from B into A are continuous. This will be the case, for instance, when B is an arbitrary Banach algebra and A is a semi-simple commutative Banach algebra. Conversely, if $\mu\colon B \to A$ is a discontinuous homomorphism and if F is a non-empty subset of Ω which is both open and closed, let $\nu(b)(\omega) := \chi_F(\omega)\,\mu(\omega)$, for all $b \in B$ and $\omega \in \Omega$, where χ_F denotes the characteristic function of F. Then $\nu\colon B \to C(\Omega, A)$ is a discontinuous homomorphism with $\Delta_\nu = F$, but this construction does not preclude the possibility that all epimorphisms from B onto $C(\Omega, A)$ are continuous. One might even suspect that every homomorphism $\nu\colon B \to C(\Omega, A)$, for which the range $\nu(B)$ is large enough to separate the points of Ω, is automatically continuous. However, the following example shows that the ordinary notion of point separation is not strong enough to ensure continuity.

Example. Assume that the Banach algebra A contains a non-zero nilpotent element, and choose a non-zero $u \in A$ with the property that $u^2 = 0$. Also assume that B is a Banach algebra that admits a discontinuous point derivation, which means that there exist a non-zero multiplicative linear functional $\varphi\colon B \to \mathbb{C}$ and a discontinuous linear functional $d\colon B \to \mathbb{C}$ such that $d(xy) = d(x)\,\varphi(y) + \varphi(x)\,d(y)$ holds for all $x, y \in B$. For instance, such discontinuous point derivations exist in abundance on the Banach algebra $C^1([0,1])$ of all continuously differentiable complex-valued functions on the unit interval $[0,1]$; see Section 6 of [3] for further information. Finally, assume that Ω is a compact subset of the complex plane with at least two points, and choose any injective continuous function $f\colon \Omega \to \mathbb{C}$, for instance the identity function on Ω. In this setting, let

$$\nu(b)(\omega) := \varphi(b)\,e + f(\omega)\,d(b)\,u \qquad \text{for all } b \in B \text{ and } \omega \in \Omega,$$

where e denotes the identity element of A. Then it is easily seen that $\nu\colon B \to C(\Omega, A)$ is a homomorphism for which $\nu_\omega\colon B \to A$ is discontinuous if and only if $f(\omega) = 0$. Since f is injective, it follows that ν is discontinuous, although the range $\nu(B)$ separates the points of Ω in the sense that, for all distinct $\omega_1, \omega_2 \in \Omega$, there exists some $b \in B$ for which $\nu(b)(\omega_1) \neq v(b)(\omega_2)$.

We shall see that a strengthened version of point separation will guarantee continuity in this context. A subset G of $C(\Omega, A)$ is said to *strongly separate the points of* Ω if, for any $\omega_1, \omega_2 \in \Omega$ with $\omega_1 \neq \omega_2$, there exists a function $g \in G$ such that $g(\omega_1) = 0$ and $g(\omega_2) \in \operatorname{Inv} A$, where $\operatorname{Inv} A$ denotes the set of invertible elements of A.

Clearly, strong point separation implies ordinary point separation for any subset G of $C(\Omega, A)$. Moreover, if $C(\Omega) := C(\Omega, \mathbb{C})$ is viewed canonically as a subalgebra of $C(\Omega, A)$ and if G is a subset of $C(\Omega, A)$ for which $G \cap C(\Omega)$ is dense in $C(\Omega)$ and closed under addition of complex constants, then it is easily seen that G strongly separates the points of Ω. Similarly, if G is a dense linear subspace of $C(\Omega, A)$ which contains all A-valued constant functions, then G is strongly point separating. On the other hand, in the preceding example, the set $G := \nu(B)$ does not strongly separate the points of Ω, although for $\omega_1, \omega_2 \in \Omega$ with $\omega_1 \neq \omega_2$, it is always possible to find some $b \in B$ for which $\nu(b)(\omega_1)$ and $\nu(b)(\omega_2)$ are distinct and both invertible, and also some $c \in B$ for which $\nu(c)(\omega_1)$ and $\nu(c)(\omega_2)$ are distinct and both non-invertible.

The proofs of our main automatic continuity results will require the following lemma, which states that strong point separation implies an apparently much stronger separation property.

Lemma 2. *Assume that G is a subalgebra of $C(\Omega, A)$ which strongly separates the points of Ω, and let E be any infinite subset of Ω. Then there exist $\omega_n \in E$ and $g_n \in G$ for all $n \in \mathbb{N}$ such that $g_k(\omega_n) = 0$ for $k \geqslant n$ and $g_k(\omega_n) \in \operatorname{Inv} A$ for $k < n$.*

Proof. Choose distinct $t_n \in E$ for all $n \in \mathbb{N}$. By the compactness of Ω, there exists some $t_\infty \in \Omega$ such that $U \cap \{t_n : n \in \mathbb{N}\}$ is infinite for each neighborhood U of t_∞. By discarding one of the t_n's if necessary, we may assume that $t_n \neq t_\infty$ for all $n \in \mathbb{N}$. Hence, by the strong point separation, there exists some $h_n \in G$ such that $h_n(t_n) = 0$ and $h_n(t_\infty) \in \operatorname{Inv} A$ for all $n \in \mathbb{N}$. Let $f_n := h_1 h_2 \ldots h_n \in G$ for all $n \in \mathbb{N}$. Then clearly $f_n(t_1) = f_n(t_2) = \ldots = f_n(t_n) = 0$ and $f_n(t_\infty) \in \operatorname{Inv} A$ for all $n \in \mathbb{N}$. By induction, we may now choose $m(n) \in \mathbb{N}$ for all $n \in \mathbb{N}$ such that $m(n) < m(k)$ and $f_{m(n)}(t_{m(k)}) \in \operatorname{Inv} A$ for $n < k$. Indeed, let $m(1) := 1$, and if the integers $m(1) < \ldots < m(n)$ have already been chosen, we observe that the set $U_n := f_{m(1)}^{-1}(\operatorname{Inv} A) \cap \ldots \cap f_{m(n)}^{-1}(\operatorname{Inv} A)$ is an open neighborhood of t_∞ and hence contains t_k for infinitely many $k \in \mathbb{N}$. Therefore, there exists an integer $m(n+1) > m(n)$ for which $t_{m(n+1)} \in U_n$ and thus $f_{m(k)}(t_{m(n+1)}) \in \operatorname{Inv} A$ for $k = 1, \ldots, n$, which completes the inductive choice. Finally, let $\omega_n := t_{m(n)} \in E$ and $g_n := f_{m(n)} \in G$ for all $n \in \mathbb{N}$. Then $g_k(\omega_n) = 0$ for $k \geqslant n$ and $g_k(\omega_n) \in \operatorname{Inv} A$ for $k < n$, as desired. □

Another ingredient for the proofs of our principal theorems will be the following central result from the theory of automatic continuity, which formalizes the so-called gliding hump technique. A proof of this result is given in Section 2 of [11]; see also Theorem 4.2 of [12] for a non-linear generalization.

Lemma 3. *Consider a sequence of Fréchet spaces X_n for $n = 0, 1, 2, \ldots$ and continuous linear operators $T_n : X_n \to X_{n-1}$ for all $n \in \mathbb{N}$. Further, for $n = 0, 1, 2, \ldots$, let Y_n be a normed linear space and consider continuous linear operators $\pi_n : Y_0 \to Y_n$ for all $n \in \mathbb{N}$. Finally, let $\theta : X_0 \to Y_0$ denote a linear mapping such that $\pi_n \theta T_1 \ldots T_n$:*

$X_n \to Y_n$ *is continuous for all* $n \in \mathbb{N}$. *Then there exists some* $n \in \mathbb{N}$ *such that* $\pi_k \theta T_1 \ldots T_n \colon X_n \to Y_n$ *is continuous for all* $k \in \mathbb{N}$.

2. Continuity of homomorphisms

Let Ω denote a compact Hausdorff space, and consider an arbitrary unital complex Banach algebra A. Whereas ordinary point separation by the range is insufficient to ensure the continuity of a homomorphism into $C(\Omega, A)$, we now show that strong point separation is, in fact, sufficient. Our first main result is the following.

Theorem 4. *Consider an arbitrary Fréchet algebra* B, *a Banach algebra* A *with identity, and a compact Hausdorff space* Ω *without isolated points. If* $\nu \colon B \to C(\Omega, A)$ *is a homomorphism for which* $\nu(B)$ *strongly separates the points of* Ω, *then* ν *is automatically continuous. In particular, every epimorphisms* $\nu \colon B \to C(\Omega, A)$ *is continuous, and the Banach algebra* $C(\Omega, A)$ *has a unique Fréchet algebra topology.*

Proof. We first claim that Δ_ν is a finite set. Suppose to the contrary that Δ_ν is infinite. Then, by Lemma 2, there exist $\omega_n \in \Delta_\nu$ and $b_n \in B$ for all $n \in \mathbb{N}$ such that $\nu(b_k)(\omega_n) = 0$ for $k \geqslant n$ and $\nu(b_k)(\omega_n) \in \operatorname{Inv} A$ for $k < n$. We now apply Lemma 3 to the Fréchet spaces $X_n := B$ for all $n = 0, 1, 2, \ldots$, the Banach spaces $Y_0 := C(\Omega, A)$ and $Y_n := A$, and the operators $T_n \colon B \to B$ and $\pi_n \colon C(\Omega, A) \to A$ given by $T_n(b) := b_n b$ and $\pi_n(f) := f(\omega_n)$ for all $b \in B$, $f \in C(\Omega, A)$, and $n \in \mathbb{N}$. Since $\nu(b_n)(\omega_n) = 0$, the mapping $\theta := \nu$ satisfies

$$\pi_n \theta T_1 \ldots T_n(b) = \nu(b_1 \ldots b_n b)(\omega_n) = \nu(b_1)(\omega_n) \ldots \nu(b_n)(\omega_n)\, \nu(b)(\omega_n) = 0$$

for all $b \in B$ and $n \in \mathbb{N}$. Consequently, by Lemma 3, there exists some $n \in \mathbb{N}$ such that $\pi_{n+1} \theta T_1 \ldots T_n \colon B \to A$ is continuous. Since

$$\pi_{n+1} \theta T_1 \ldots T_n(b) = \nu(b_1)(\omega_{n+1}) \ldots \nu(b_n)(\omega_{n+1})\, \nu(b)(\omega_{n+1})$$

for all $b \in B$ and since the first n factors on the right-hand side are invertible, we conclude that $\nu_{\omega_{n+1}}$ is continuous on B. But this is impossible since ω_{n+1} belongs to Δ_ν. This contradiction shows that Δ_ν is indeed finite. Since Δ_ν is also open by Lemma 1, we conclude that Δ_ν consists only of isolated points. By the assumption on Ω, it follows that Δ_ν is empty and hence that ν is continuous, again by Lemma 1. The assertion on epimorphisms is now immediate, since $C(\Omega, A)$ strongly separates the points of Ω, and the uniqueness of the Fréchet algebra topology on the Banach algebra $C(\Omega, A)$ follows from this and the open mapping theorem for Fréchet spaces. □

Corollary 5. *Assume that Ω has no isolated points and that B is a strongly point separating subalgebra of $C(\Omega, A)$, which is endowed with a Fréchet algebra topology. Then the followwing assertions hold:*

(a) *B is continuously embedded in $C(\Omega, A)$.*

(b) *All point evaluations from B into A are continuous.*

(c) *Every epimorphism from any Fréchet algebra onto B is continuous.*

(d) *B carries a unique Fréchet algebra topology.*

Proof. Assertion (a) follows from Theorem 4 applied to the inclusion mapping $i\colon B \to C(\Omega, A)$, and (b) follows immediately from (a). Consider now an epimorphism $\mu\colon C \to B$ from a Fréchet algebra C onto B. Again by Theorem 4, the composition $i \circ \mu\colon C \to C(\Omega, A)$ is continuous. From the continuity of $i\colon B \to C(\Omega, A)$, it then follows that $\mu\colon C \to B$ has a closed graph and hence is continuous by the closed graph theorem for Fréchet spaces. The final assertion (d) is a consequence of (c). □

The preceding result applies to a number of classical algebras of vector-valued functions: the Banach algebras $C^n([0,1], A)$ of all n-times continuously differentiable A-valued functions on the unit interval for all $n \in \mathbb{N}$, the Fréchet algebra $C^\infty([0,1], A)$ of all infinitely differentiable A-valued functions on $[0,1]$, the vector-valued Lipschitz algebras $\mathrm{Lip}_\alpha([0,1], A)$ and $\mathrm{lip}_\alpha([0,1], A)$ for all $0 < \alpha < 1$, and the Banach algebra of all continuous A-valued functions on the closed unit disc which are analytic on the interior. If B denotes any of these algebras, then Corollary 5 confirms that B has a unique Fréchet algebra topology and that every epimorphism from any Fréchet algebra onto B is automatically continuous. This complements a number of classical results in this context. For instance, it is known that all C^*-algebras and all Arens-Hoffman extensions of semi-simple commutative Banach algebras have a unique complete norm topology; see Lindberg [9].

Moreover, any of the preceding Banach algebras of vector-valued functions may be used to see that an arbitrary unital Banach algebra A may be embedded into a Banach algebra with unique complete norm topology. This result has also been obtained by Loy [10], who used Banach algebras of power series for this purpose.

In Theorem 4 it is crucial to assume that Ω has no isolated points, since, for a finite set Ω, discontinuous epimorphisms onto $C(\Omega, A)$ can be easily constructed from discontinuous epimorphisms onto A. Moreover, in view of Michael's problem, it seems difficult to relax the assumption on Ω even if one chooses $A = \mathbb{C}$. On the other hand, for compact Hausdorff spaces with isolated points, the following positive result holds.

Theorem 6. *Let Ω be an arbitrary compact Hausdorff space. If the Banach algebra A has a unique Fréchet algebra topology, so does $C(\Omega, A)$. Similarly, if A has a unique Banach algebra topology, so does $C(\Omega, A)$.*

Proof. Let τ be any Fréchet algebra topology on $C(\Omega, A)$, and consider the identity mapping ν from $(C(\Omega, A), \tau)$ onto $(C(\Omega, A), \|\cdot\|_\infty)$. By the open mapping theorem, it suffices to show that ν is continuous. The argument used in the proof of Theorem 4 gives that Δ_ν is a finite set of isolated points of Ω. But the following will show that every isolated point $\omega \in \Omega$ belongs to $\Omega \setminus \Delta_\nu$. Indeed, let $\chi_\omega \in C(\Omega) \subseteq C(\Omega, A)$ denote the characteristic function of the singleton $\{\omega\}$, and let $P_\omega(f) := \chi_\omega f$ for all $f \in C(\Omega, A)$. Then P_ω is a linear projection on $C(\Omega, A)$, which is τ-continuous and hence has τ-closed range, denoted by A_ω. Thus A_ω is a Fréchet algebra with respect to τ, which is algebraically isomorphic to A via evaluation at ω. Since the Banach algebra A is supposed to have a unique Fréchet algebra topology, we conclude that the mapping $\nu_\omega \,|\, A_\omega \colon A_\omega \to A$ is continuous with respect to τ. Since $\nu_\omega = (\nu_\omega \,|\, A_\omega) \circ P_\omega$, it follows that ν_ω is τ-continuous and therefore $\omega \notin \Delta_\omega$. Thus $\Delta_\nu = \emptyset$, so the continuity of ν is now a consequence of Lemma 1. The second assertion of Theorem 6 follows by the same argument. □

We close this section with a brief discussion of the locally compact case. If Ω denotes an arbitrary locally compact Hausdorff space, the algebra $C(\Omega, A)$ of all continuous A-valued functions on Ω becomes a topological algebra with respect to the family of semi-norms p_K given by $p_K(f) := \{\|f(u)\| : u \in K\}$ for all $f \in C(\Omega, A)$ and all compact subsets K of Ω. Note that this topology is not metrizable unless Ω is countable at infinity. Recall that a point $\omega \in \Omega$ is said to be *compactly isolated* in Ω if ω is isolated in every compact subset of Ω which contains it. We shall use the same notion of strong point separation as in the compact case.

Theorem 7. *If Ω is a locally compact Hausdorff space with no compactly isolated points and if B denotes an arbitrary Fréchet algebra, then every homomorphism $\nu\colon B \to C(\Omega, A)$ with strongly point separating range is automatically continuous. In particular, all epimorphisms from B onto $C(\Omega, A)$ are continuous.*

Proof. We have to show that, for each compact subset K of Ω, the homomorphism $\nu_K \colon B \to C(K, A)$ given by $\nu_K(b) := \nu(b) \,|\, K$ for all $b \in B$ is continuous. Fix an arbitrary $\omega \in K$ and choose a compact set $L \subseteq \Omega$ in which ω is not isolated. Since the range of ν_L strongly separates the points of L, the proof of Theorem 4 shows that Δ_{ν_L} is a finite set of isolated points in L. Since $\omega \in L$ is not isolated in L, we conclude that $\nu_\omega \colon B \to A$ is continuous. Consequently, Δ_{ν_K} is empty which proves that ν_K is continuous by Lemma 1. □

The preceding theorem applies, for instance, to the Fréchet algebra $H(U, A)$ of all analytic A-valued functions on an arbitrary open subset U of $\mathbb{C}$. It follows that every epimorphisms from a Fréchet algebra onto $H(U, A)$ is automatically continuous and that $H(U, A)$ has a unique Fréchet algebra topology.

3. Continuity of derivations

Again, let Ω be a compact Hausdorff space, and consider an arbitrary complex Banach algebra A with identity. Given a subalgebra B of $C(\Omega, A)$, recall that a linear mapping $\delta\colon B \to C(\Omega, A)$ is said to be a *derivation* if $\delta(fg) = f\,\delta(g) + \delta(f)\,g$ holds for all $f, g \in B$.

Theorem 8. *Assume that Ω has no islated points and that B is a strongly point separating subalgebra of $C(\Omega, A)$, which is endowed with a Fréchet algebra topology. Then every derivation $\delta\colon B \to C(\Omega, A)$ is automativcally continuous. In particular, every derivation $\delta\colon C(\Omega, A) \to C(\Omega, A)$ is continuous.*

Proof. As in the proof of Theorem 4, it suffices to show that Δ_δ is a finite subset of Ω. Suppose that Δ_δ is infinite, and apply Lemma 3 with $G := B$ and $E := \Delta_\delta$ to obtain $\omega_n \in \Delta_\delta$ and $g_n \in B$ for all $n \in \mathbb{N}$ such that $g_k(\omega_n) = 0$ for $k \geqslant n$ and $g_k(\omega_n) \in \operatorname{Inv} A$ for $k < n$. As before, let $X_0 := B$, $Y_0 := C(\Omega, A)$, $X_n := B$, $Y_n := A$, and define $T_n(g) := g_n\,g$ and $\pi_n(f) := f(\omega_n)$ for all $g \in B$, $f \in C(\Omega, A)$, and $n \in \mathbb{N}$. Since the mapping $\theta := \delta$ is a derivation and $g_n(\omega_n) = 0$, we obtain that

$$\begin{aligned}\pi_n\theta\,T_1\ldots T_n(g) &= \delta(g_1\ldots g_n\,g)(\omega_n)\\ &= g_1(\omega_n)\ldots g_n(\omega_n)\,\delta(g)(\omega_n) + \delta(g_1\ldots g_n)(\omega_n)\,g(\omega_n)\\ &= \delta(g_1\ldots g_n)(\omega_n)\,g(\omega_n)\end{aligned}$$

for all $g \in B$ and $n \in \mathbb{N}$. By Corollary 5, the evaluation mapping which takes the function $g \in B$ to $g(\omega_n) \in A$ is continuous, which establishes the continuity of the composition $\pi_n\theta\,T_1\ldots T_n\colon B \to A$ for each $n \in \mathbb{N}$. Consequently, by Lemma 3, there exists some $n \in \mathbb{N}$ such that $\pi_{n+1}\theta\,T_1\ldots T_n\colon B \to A$ is continuous. For each $g \in B$, we have that

$$\pi_{n+1}\theta\,T_1\ldots T_n(g) = g_1(\omega_{n+1})\ldots g_n(\omega_{n+1})\,\delta(g)(\omega_{n+1}) + \delta(g_1\ldots g_n)(\omega_{n+1})\,g(\omega_{n+1}),$$

which implies that the first summand on the right-hand side is a continuous function in $g \in B$. Since $g_1(\omega_{n+1})\ldots g_n(\omega_{n+1})$ is an invertible element of A, we conclude that $\delta_{\omega_{n+1}}$ is continuous on B and therefore $\omega_{n+1} \notin \Delta_\delta$, the desired contradiction. $\square$

Corollary 9. *Assume that Ω has no isolated points and that B and C are strongly point separating subalgebras of $C(\Omega, A)$, both endowed with a Fréchet algebra topology. Then every derivation $\delta\colon B \to C$ is automatically continuous.*

P r o o f. By Corollary 5, we know that the inclusion $i\colon C \to C(\Omega, A)$ is continuous. Moreover, Theorem 8 implies the continuity of $i \circ \delta\colon B \to C(\Omega, A)$. hence the continuity of $\delta\colon B \to C$ follows from the closed graph theorem. □

Note that Corollary 9 establishes the automatic continuity of all derivations on $C^n([0,1], A)$, $C^\infty([0,1], A)$, $\mathrm{Lip}_\alpha([0,1], A)$, $\mathrm{lip}_\alpha([0,1], A)$, and other classical algebras of vector-valued functions. In Theorem 8 and Corollary 9, it is important that Ω contains no isolated points. For arbitrary compact Hausdorff spaces, we finally obtain the following result.

Theorem 10. *If every derivation on the Banach algebra A is continuous, so is every derivation on $C(\Omega, A)$.*

P r o o f. Let $\delta\colon C(\Omega, A) \to C(\Omega, A)$ be a derivation. By the proof of Theorem 8, we know that Δ_δ consists entirely of isolated points of Ω. We now show that every isolated point $\omega \in \Omega$ necessarily belongs to $\Omega \setminus \Delta_\delta$. This will prove that Δ_δ is empty, from which the continuity of δ will follow by Lemma 1. Since ω is isolated, the corresponding characteristic function χ_ω belongs to $C(\Omega)$ and hence to $C(\Omega, A)$. The calculation $\delta(\chi_\omega) = \delta(\chi_\omega\, \chi_\omega) = 2\,\chi_\omega\, \delta(\chi_\omega)$ implies that $\delta(\chi_\omega) = 0$ and consequently that

$$\delta(\chi_\omega\, f)(\omega) = \delta(f)(\omega) \,+\, \delta(\chi_\omega)(\omega)\, f(\omega) = \delta(f)(\omega) = \delta_\omega(f) \quad \text{for all } f \in C(\Omega, A).$$

It is routine to check that the definition $d(a) := \delta(\chi_\omega\, a)(\omega)$ for all $a \in A$ yields a derivation $d\colon A \to A$, which is continuous by the hypothesis on A. Finally, consider any sequence of functions $f_n \in C(\Omega, A)$ which converges uniformly to zero. Then we have that $f_n(\omega) \to 0$ in A and therefore

$$\delta_\omega(f_n) = \delta(\chi_\omega\, f_n)(\omega) = \delta(\chi_\omega\, f_n(\omega))(\omega) = d(f_n(\omega)) \to 0 \quad \text{as } n \to \infty.$$

This shows that $\omega \notin \Delta_\delta$ and hence completes the proof. □

ACKNOWLEDGEMENT

Parts of this paper are based on sections of the first-named author's Doctoral Dissertation, which was prepared under the guidance of Professor John A. Lindberg, Jr. at Syracuse University in Syracuse, New York, USA.

References

[1] *R. L. Carpenter*: Uniqueness of topology for commutative semi-simple F-algebras. Proc. Amer. Math. Soc. *29* (1971), 113–117.
[2] *R. L. Carpenter*: Continuity of derivations in F-algebras. Amer. J. Math. *93* (1971), 500–502.
[3] *H. G. Dales*: Automatic continuity: a survey. Bull. London Math. Soc. *10* (1978), 129–183.
[4] *H. G. Dales*: Banach algebras and automatic continuity. forthcoming monograph.
[5] *A. Hausner*: Ideals in a certain Banach algebra. Proc. Amer. Math. Soc. *8* (1957), 246–249.
[6] *B. E. Johnson*: The uniqueness of the (complete) norm topology. Bull. Amer. Math. Soc. *73* (1967), 537–539.
[7] *B. E. Johnson*: Continuity of derivations on commutative algebras. Amer. J. Math. *91* (1969), 1-10.
[8] *B. E. Johnson and A. M. Sinclair*: Continuity of derivations and a problem of Kaplansky. Amer. J. Math. *90* (1968), 1067–1073.
[9] *J. A. Lindberg, Jr.*: Homomorphisms of Banach algebras into Arens-Hoffman extensions of a semi-simple algebra. J. London Math. Soc. *13* (1976), 111–121.
[10] *R. J. Loy*: Uniqueness of the Fréchet space topology on certain topological algebras. Bull. Austral. Math. Soc. *4* (1971), 1–7.
[11] *M. M. Neumann*: Automatic continuity of linear operators. Functional Analysis: Surveys and Recent Results II. North-Holland Math. Studies 38, 1980, pp. 269–296.
[12] *M. M. Neumann and V. Pták*: Automatic continuity, local type and causality. Studia Math. *82* (1985), 61–90.
[13] *A. M. Sinclair*: Automatic continuity of linear operators. London Math. Soc. Lecture Note Series 21. Cambridge University Press, Cambridge, 1976.

Authors' addresses: *R. Kantrowitz*, Department of Mathematics, Hamilton College, Clinton, NY 13323, USA, e-mail: `rkantrow@itsmail.hamilton.edu`; *M. M. Neumann*, Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762, USA, e-mail: `neumann@math.msstate.edu`.