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CONTINUABILITY, BOUNDEDNESS, AND CONVERGENCE TO ZERO OF SOLUTIONS OF A PERTURBED NONLINEAR ORDINARY DIFFERENTIAL EQUATION

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Dedicated to Professor Jaromír Vosmanský on the occasion of his sixtieth birthday

1. Introduction

In this paper we obtain results on the asymptotic behavior of the solutions of the second order nonlinear differential equation

$$(a(t)x')' + h(t,x,x') + q(t)f(x)g(x') = e(t,x,x'). \tag{$*$}$$

We give conditions which ensure that the solutions of $(*)$ are continuable, conditions which imply that all solutions are bounded, and conditions ensuring that all or certain classes of solutions tend to zero as $t \to \infty$. We also include some examples to illustrate our results. Sets of conditions which guarantee these same conculusions for special cases of $(*)$ can be found in numerous places in the literature (see [1–9, 11–30]). In addition to being for a more general equation, the results here are new even when $(*)$ is specialized to the forms previously studied.

2. Boundedness and continuability

Consider the equation

$$(a(t)x')' + h(t,x,x') + q(t)f(x)g(x') = e(t,x,x') \tag{1}$$

where $a, q\colon [t_0, \infty) \to \mathbb{R}$, $f, g\colon \mathbb{R} \to \mathbb{R}$, $h, e\colon [t_0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ are continous, $a(t) > 0$, $q(t) > 0$, and $g(x') > 0$. We will write (1) as the system

$$\tag{2} \begin{aligned} x' &= y, \\ y' &= (-a'(t)y - h(t,x,y) - q(t)f(x)g(y) + e(t,x,y))/a(t). \end{aligned}$$

For any function Q we let $Q(t)_+ = \max\{Q(t), 0\}$ and $Q(t)_- = \max\{-Q(t), 0\}$ so that $Q(t) = Q(t)_+ - Q(t)_-$. We define $G(y) = \int_0^y [s/g(s)]\,\mathrm{d}s$, $F(x) = \int_0^x f(s)\,\mathrm{d}s$ and assume that

$$\tag{3} xf(x) > 0 \quad \text{for } x \neq 0.$$

In addition, we assume that there are nonnegative continuous functions $k, b, r, w\colon [t_0, \infty) \to \mathbb{R}$ and constants $0 \leqslant m \leqslant 1$, $K_1 \geqslant 0$, $K_2 \geqslant 0$, and $C_1 > 0$ such that

$$\tag{4} |e(t,x,y)| \leqslant k(t)F^{\frac{1}{2}}(x) + b(t)G^{\frac{m}{2}}(y) + r(t),$$

$$\tag{5} -w(t)y^2 \leqslant yh(t,x,y),$$

$$\tag{6} g(y) \geqslant C_1,$$

and

$$\tag{7} y^2/g(y) \leqslant K_1 G(y) + K_2.$$

We first give a continuability result for (2).

Theorem 1. *If* (3)–(7) *hold and* $G(y) \to \infty$ *as* $|y| \to \infty$, *then all solutions of* (2) *can be defined for all* $t \geqslant t_0$.

Proof. Suppose there is a solution $(x(t), y(t))$ of (2) and $T > t_0$ such that

$$\lim_{t \to T^-} [|x(t)| + |y(t)|] = \infty.$$

Define

$$V(t) = a(t)G(y(t))/q(t) + F(x(t));$$

then differentiating and applying (3)–(5) and (7) we have

$$\begin{aligned} V'(t) &= a(t)y(t)y'(t)/q(t)g\,(y(t)) + G\,(y(t))\,(a(t)/q(t))' + y(t)f\,(x(t)) \\ &= -a'(t)y^2(t)/q(t)g\,(y(t)) + G\,(y(t))\,(a(t)/q(t))' \\ &\quad + y(t)\,[e(t, x(t), y(t)) - h(t, x(t), y(t))]\,/q(t)g\,(y(t)) \\ &\leqslant K_1 a'(t)_- G(y(t))/q(t) + G(y(t))(a(t)/q(t))' + w(t)y^2(t)/q(t)g(y(t)) \\ &\quad + k(t)|y(t)|F^{\frac{1}{2}}(x(t))/q(t)g(y(t)) + b(t)|y(t)|G^{\frac{m}{2}}(y(t))/q(t)g(y(t)) \\ &\quad + |y(t)|r(t)/q(t)g(y(t)) + K_2 a'(t)_-/q(t). \end{aligned}$$

From (7),

$$w(t)y^2(t)/q(t)g(y(t)) \leqslant w(t)[K_1G(y(t)) + K_2]/q(t)$$
$$\leqslant K_1\,(w(t)/a(t))\,V(t) + K_2w(t)/q(t).$$

Clearly,

$$(8) \qquad |y(t)|\,[a(t)F(x(t))/q(t)g(y(t))]^{\frac{1}{2}} \leqslant (K_1+1)V(t) + K_2a(t)/q(t).$$

Next, we have

(9)

$$\begin{aligned}
&b(t)|y(t)|G^{\frac{m}{2}}(y(t))/q(t)g(y(t))\\
&\quad= [b(t)/(a(t)q(t)g(y(t)))^{\frac{1}{2}}][a^{\frac{1}{2}}(t)|y(t)|/(q(t)g(y(t)))^{\frac{1}{2}}]\\
&\qquad\times [a^{\frac{1}{2}}(t)G^{\frac{1}{2}}(y(t))/q^{\frac{1}{2}}(t)]^m(q(t)/a(t))^{\frac{m}{2}}\\
&\quad\leqslant [b(t)/(a(t)q(t)g(y(t)))^{\frac{1}{2}}](q(t)/a(t))^{\frac{m}{2}}\{a(t)y^2(t)/q(t)g(y(t)) + [a(t)G(y(t))/q(t)]^m\}\\
&\quad\leqslant [b(t)/(a(t)q(t)g(y(t)))^{\frac{1}{2}}](q(t)/a(t))^{\frac{m}{2}}\{(a(t)/q(t))[K_1G(y(t)) + K_2]\\
&\qquad+ 1 + a(t)G(y(t))/q(t)\}\\
&\quad\leqslant [b(t)/(a(t)q(t))^{\frac{1}{2}}](q(t)/a(t))^{\frac{m}{2}}(K_1+1)V(t)/C_1^{\frac{1}{2}}\\
&\qquad+ [b(t)/(a(t)q(t))^{\frac{1}{2}}](q(t)/a(t))^{\frac{m}{2}}(K_2a(t)/q(t)+1)/C_1^{\frac{1}{2}}.
\end{aligned}$$

Also,

$$\begin{aligned}
(10) \qquad |y(t)|r(t)/q(t)g(y(t)) &\leqslant r(t)\left[a(t)y^2(t)/q(t)+1\right]/g(y(t))(a(t)q(t))^{\frac{1}{2}}\\
&\leqslant r(t)\,[a(t)(K_1G(y(t)) + K_2)/q(t)]\,/(a(t)q(t))^{\frac{1}{2}}\\
&\quad+ r(t)/g(y(t))(a(t)q(t))^{\frac{1}{2}}\\
&\leqslant K_1r(t)V(t)/(a(t)q(t))^{\frac{1}{2}}\\
&\quad+ r(t)\,[K_2a(t)/q(t) + 1/g(y(t))]\,/(a(t)q(t))^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$V'(t) \leqslant P_1(t)V(t) + P_2(t)$$

where

$$\begin{aligned}
P_1(t) = {}& K_1(a'(t)_- + w(t))/a(t) + (a(t)/q(t))'_+/\,(a(t)/q(t))\\
&+ (K_1+1)k(t)/\,[a(t)q(t)C_1]^{\frac{1}{2}} + [b(t)/(a(t)q(t))^{\frac{1}{2}}](q(t)/a(t))^{\frac{m}{2}}(K_1+1)/C_1^{\frac{1}{2}}\\
&+ K_1r(t)/(a(t)q(t))^{\frac{1}{2}}
\end{aligned}$$

and

$$P_2(t) = K_2(a'(t)_- + w(t))/q(t) + K_2k(t)[a(t)/q^3(t)C_1]^{\frac{1}{2}} \\ + [b(t)/(a(t)q(t))^{\frac{1}{2}}](q(t)/a(t))^{\frac{m}{2}}(K_2a(t)/q(t) + 1)/C_1^{\frac{1}{2}} \\ + r(t)[K_2a(t)/q(t) + 1/C_1]/(a(t)q(t))^{\frac{1}{2}}.$$

Now $\int_{t_0}^{T} P_2(s)\,\mathrm{d}s = K_3$ for some positive constant K_3, so by Gronwall's inequality we have

$$V(t) \leqslant [V(t_0) + K_3]\exp\int_{t_0}^{t} P_1(s)\,\mathrm{d}s \leqslant [V(t_0) + K_3]\exp\int_{t_0}^{T} P_1(s)\,\mathrm{d}s.$$

Hence, we see that $a(t)G(y(t))/q(t)$ is bounded on $[t_0, T)$ and since $q(t)/a(t)$ is bounded on $[t_0, T]$ we have $G(y(t))$ is bounded on $[t_0, T)$. But this implies that $y(t)$ is bounded on $[t_0, T)$. An integration shows that $x(t)$ is also bounded on $[t_0, T)$ contradicting the assumption that $(x(t),\ y(t))$ is a solution of (2) with finite escape time. □

To obtain a boundedness result for solutions of (1) we will ask that

$$\int_{t_0}^{\infty} [a'(s)_-/a(s)]\ \mathrm{d}s < \infty, \tag{11}$$

$$\int_{t_0}^{\infty} [w(s)/a(s)]\ \mathrm{d}s < \infty, \tag{12}$$

$$\int_{t_0}^{\infty} \left[k(s)/(a(s)q(s))^{\frac{1}{2}}\right]\ \mathrm{d}s < \infty, \tag{13}$$

$$\int_{t_0}^{\infty} \left[b(s)/(a(s)q(s))^{\frac{1}{2}}\right](q(s)/a(s))^{\frac{m}{2}}\ \mathrm{d}s < \infty, \tag{14}$$

$$\int_{t_0}^{\infty} \left[r(s)/(a(s)q(s))^{\frac{1}{2}}\right]\ \mathrm{d}s < \infty, \tag{15}$$

$$\int_{t_0}^{\infty} [(a(s)/q(s))'_+/(a(s)/q(s))]\ \mathrm{d}s < \infty, \tag{16}$$

and

$$F(x) \to \infty \text{ as } |x| \to \infty. \tag{17}$$

Theorem 2. *If (3)–(7) and (11)–(17) hold, then every continuable solution of (1) is bounded.*

P r o o f. Let $x(t)$ be a continuable solution of (1) and define V as in the proof of Theorem 1. Then, from the proof of Theorem 1 we have

$$V'(t) \leqslant P_1(t)V(t) + P_2(t). \tag{18}$$

Notice that (11) implies that $a(t)$ is bounded from below and (16) implies that $a(t)/q(t)$ is bounded from above. Hence, $a(t) \geqslant a_1$ and $q(t) \geqslant K_4 a(t) \geqslant K_4 a_1$ for some positive constants a_1 and K_4. This implies that

$$\begin{aligned} P_2(t) \leqslant\ & K_5(a'(t)_- + w(t))/a(t) + K_6 k(t)/(a(t)q(t))^{\frac{1}{2}} \\ & + K_7[b(t)/(a(t)q(t))^{\frac{1}{2}}](q(t)/a(t))^{\frac{m}{2}} + K_8 r(t)/(a(t)q(t))^{\frac{1}{2}} \end{aligned}$$

for some positive constants K_j, $j = 5, 6, 7, 8$. By (11)–(16) we have $\int_{t_0}^{\infty} P_i(s)\,\mathrm{d}s < \infty$ for $i = 1, 2$. Hence, integrating (18) and applying Gronwall's inequality we obtain that $V(t)$ is bounded. Thus, $F(x(t))$ is bounded and so by (17) $x(t)$ is bounded. □

Corollary 3. *If, in addition to* (3)–(7) *and* (11)–(17), $G(y) \to \infty$ *as* $|y| \to \infty$, *then all solutions of* (1) *are bounded.*

P r o o f. By Theorem 1 all solutions of (1) are continuable, so by Theorem 2, all solutions of (1) and bounded. □

Remark. When comparing boundedness results such as Theorem 2 above to similar results of other authors, some care must be taken in comparing individual hypotheses. For example, conditions (15) and (16) imply that

$$\int_{t_0}^{\infty} [r(s)/q(s)]\,\mathrm{d}s < \infty, \tag{19}$$

but do not imply that

$$\int_{t_0}^{\infty} [r(s)/a(s)]\,\mathrm{d}s < \infty. \tag{20}$$

On the other hand, conditions (19) and (20) together imply (15), and (16) and (20) imply (19). Other such subtle interrelationships also exist. In this same spirit, if $k(t) \equiv 0$ and $b(t) \equiv 0$, then it is possible to drop condition (6), i.e., $g(y) \geqslant C_1$. In this case, condition (15) in Theorem 2 would be replaced by (20). Thus, the conditions on $r(t)$ would not be quite as good as those in Theorem 2, but (6) would be dropped. The verification of this follows from the the fact that (7) implies that

$$|y|/g(y) \leqslant \overline{K}_1 G(y) + \overline{K}_2,$$

holds for all y, and then replacing (10) by the estimate

$$\begin{aligned} r(t)|y(t)|/q(t)g(y(t)) &\leqslant r(t)[1+\overline{K}_1 G(y(t))+\overline{K}_2]/q(t) \\ &\leqslant \overline{K}_1 r(t)G(y(t))/q(t)+(1+\overline{K}_2)r(t)/q(t). \end{aligned}$$

Remark. Due to the generality of the form of the damping term $h(t,x,x')$ and the perturbation term $e(t,x,x')$ as well as the form of the conditions imposed on the coefficient functions, the continuability and boundedness results above extend many previously known results of this type for equation (1), such as those in [1–9, 11–30]. For example, the above results show that all solutions of the equation

$$(t^2x')'+t^{\frac{1}{2}}x'\sin(x')+4t^5x^3[(x')^2+1]=tx^2\tanh(x')+[t\ln((x')^2+1)]^{\frac{1}{4}}+t^{\frac{3}{2}},\quad t>1,$$

are continuable and bounded, but it is not possible to conclude this fact from previously known results.

3. Convergence to zero

We now impose additional conditions on the functions in equation (1) that are sufficient to ensure that all continuable solutions of (1) tend to zero as $t\to\infty$. Assume that there exist nonnegative continuous functions $\alpha, w_1\colon [t_0,\infty)\to\mathbb{R}$ and a positive constant C_2 such that for all bounded x and every constant $B_1>1$

$$|h(t,x,y)|\leqslant w_1(t)|y|, \tag{21}$$

$$I(t)=\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\to\infty \text{ as } t\to\infty, \tag{22}$$

$$\int_{t_0}^{\infty}[2q'(s)/q(s)-(a(s)q(s))'/a(s)q(s)-B_1\alpha(s)/I(s)]_-\,\mathrm{d}s<\infty, \tag{23}$$

$$\alpha(t)\,[a(t)/q(t)]^{\frac{1}{2}}=o(I(t)),\quad t\to\infty, \tag{24}$$

$$\int_{t_0}^{t}|(\alpha(s)/q(s))'|\,(a(s)q(s))^{\frac{1}{2}}\,\mathrm{d}s=o(I(t)),\quad t\to\infty, \tag{25}$$

$$\begin{aligned} \int_{t_0}^{t}\Big\{&[w_1(s)+b(s)(q(s)/a(s))^{\frac{m}{2}}]/(a(s)q(s))^{\frac{1}{2}} \\ &+[a(s)+k(s)+r(s)]/q(s)\Big\}\,\alpha(s)\,\mathrm{d}s=o(I(t)),\quad t\to\infty, \end{aligned} \tag{26}$$

and

$$g(y)\leqslant C_2. \tag{27}$$

Theorem 4. *If conditions* (3)–(6), (11)–(17), *and* (21)–(27) *are satisfied, then every solution* $x(t)$ *of* (1) *satisfies* $x(t) \to 0$ *as* $t \to \infty$.

Proof. Let $x(t)$ be a solution of (1) and let $\varepsilon > 0$ be given. First observe that conditions (6) and (27) imply (7) with $K_1 = 2C_2/C_1$ and $K_2 = 0$, i.e.,

$$y^2/g(y) \leqslant K_9 G(y) \tag{28}$$

for all y where $K_9 = 2C_2/C_1$. Condition (3), together with the arguments given by Karsai [14] or Scott [23], shows that there exists a positive constant $\overline{E}$ such that

$$F(x(t)) - \overline{E}x(t)f(x(t)) < \varepsilon \tag{29}$$

for $t \geqslant t_0$. Let $E = C_2\overline{E}/C_1$; then $V(t)$, as defined in the proof of Theorem 1, can be rewritten in the form

$$\begin{aligned} V(t) = a(t)G(y(t))/q(t) &+ Ey^2(t)a(t)/q(t)g(y(t)) \\ &- Ey^2(t)a(t)/q(t)g(y(t)) + F(x(t)). \end{aligned}$$

By (28), there exists a positive constant E_1 such that

$$Ey^2(t)a(t)/q(t)g(y(t)) \leqslant Ea(t)K_9G(y(t))/q(t) \leqslant E_1a(t)G(y(t))/q(t).$$

Also, it follows from (27) that

$$-Ey^2(t)a(t)/q(t)g(y(t)) \leqslant -Ey^2(t)a(t)/C_2q(t),$$

so

$$V(t) \leqslant (1+E_1)a(t)G(y(t))/q(t) - Ey^2(t)a(t)/C_2q(t) + F(x(t)).$$

Then, from the identity $(a(t)x(t)y(t))' = a(t)y^2(t) + x(t)(a(t)y(t))'$,

$$\begin{aligned} V(t) &\leqslant (1+E_1)a(t)G(y(t))/q(t) + F(x(t)) \\ &\quad - E[(a(t)x(t)y(t))' - x(t)(a(t)y(t))']/C_2q(t) \\ &= (1+E_1)a(t)G(y(t))/q(t) - E(a(t)x(t)y(t))'/C_2q(t) \\ &\quad + Ex(t)[e(t,x(t),y(t)) - h(t,x(t),y(t))]/C_2q(t) \\ &\quad + F(x(t)) - Ex(t)f(x(t))g(y(t))/C_2. \end{aligned}$$

By (6), (29), and the definition of E,

$$\begin{aligned} V(t) \leqslant (1+E_1)a(t)G(y(t))/q(t) &- E_2(a(t)x(t)y(t))'/q(t) \\ &+ E_2x(t)[e(t,x(t),y(t)) - h(t,x(t),y(t))]/q(t) + \varepsilon \end{aligned} \tag{30}$$

where $E_2 = E/C_2$. Also, from the proof of Theorem 1, we have

$$V'(t) = -a'(t)y^2(t)/q(t)g(y(t)) + G(y(t))(a(t)/q(t))' \\ + y(t)[e(t,x(t),y(t)) - h(t,x(t),y(t))]/q(t)g(y(t)).$$

Observe that

$$G(y(t))(a(t)/q(t))' = a(t)G(y(t))[(a(t)q(t))'/a(t) - 2q'(t)]/q^2(t),$$

and that (28) implies

$$-a'(t)y^2(t)/q(t)g(y(t)) \leqslant K_9V(t)a'(t)_-/a(t).$$

Thus, we have

$$\text{(31)} \quad V'(t) \leqslant K_9V(t)a'(t)_-/a(t) + a(t)G(y(t))[(a(t)q(t))'/a(t) - 2q'(t)]/q^2(t) \\ + y(t)[e(t,x(t),y(t)) - h(t,x(t),y(t))]/q(t)g(y(t)).$$

Now define $H(t) = V(t)I(t)$ so that

$$H'(t) = V'(t)I(t) + V(t)\alpha(t).$$

Then from (30) and (31) we have

$$\text{(32)} \quad \begin{aligned} H'(t) \leqslant & -[2q'(t)/q(t) - (a(t)q(t))'/a(t)q(t) \\ & - (1+E_1)\alpha(t)/I(t)]a(t)G(y(t))I(t)/q(t) \\ & - E_2(a(t)x(t)y(t))'\alpha(t)/q(t) + \varepsilon\alpha(t) + K_9V(t)I(t)a'(t)_-/a(t) \\ & + I(t)y(t)[e(t,x(t),y(t)) - h(t,x(t),y(t))]/q(t)g(y(t)) \\ & + E_2\alpha(t)x(t)[e(t,x(t),y(t)) - h(t,x(t),y(t))]/q(t). \end{aligned}$$

Next, observe that: (5) and (28) imply

$$-y(t)h(t,x(t),y(t))/q(t)g(y(t)) \leqslant K_9w(t)V(t)/a(t);$$

(4), (6), and (28) imply

$$\begin{aligned} & |y(t)e(t,x(t),y(t))|/q(t)g(y(t)) \\ & \quad \leqslant (K_9+1)k(t)V(t)/[a(t)q(t)g(y(t))]^{\frac{1}{2}} \\ & \qquad + [b(t)/(a(t)q(t))^{\frac{1}{2}}](q(t)/a(t))^{\frac{m}{2}}(K_9+1)V(t)/C_1^{\frac{1}{2}} \\ & \qquad + [b(t)/(a(t)q(t))^{\frac{1}{2}}](q(t)/a(t))^{\frac{m}{2}}/C_1^{\frac{1}{2}} + K_9r(t)V(t)/(a(t)q(t))^{\frac{1}{2}} \\ & \qquad + r(t)/(a(t)q(t))^{\frac{1}{2}}g(y(t)); \end{aligned}$$

(21), (27), and (28) imply

$$
\begin{aligned}
|h(t,x(t),y(t))|/q(t) &\leqslant |y(t)|w_1(t)/q(t) \\
&\leqslant w_1(t)[a(t)y^2(t)/q(t)+1]/(a(t)q(t))^{\frac{1}{2}} \\
&\leqslant C_2 w_1(t)[a(t)y^2(t)/q(t)g(y(t))] + w_1(t)/(a(t)q(t))^{\frac{1}{2}} \\
&\leqslant K_9 C_2 w_1(t)V(t)/(a(t)q(t))^{\frac{1}{2}} + w_1(t)/(a(t)q(t))^{\frac{1}{2}};
\end{aligned}
$$

and (4) implies

$$
\begin{aligned}
|e(t,x(t),y(t))|/q(t) &\leqslant k(t)F^{\frac{1}{2}}(x(t))/q(t) + b(t)G^{\frac{m}{2}}(y(t))/q(t) + r(t)/q(t) \\
&\leqslant k(t)F^{\frac{1}{2}}(x(t))/q(t) + (b(t)/q(t))V^{\frac{m}{2}}(t)(q(t)/a(t))^{\frac{m}{2}} + r(t)/q(t).
\end{aligned}
$$

As noted in the proof of Theorem 2, $a(t) \geqslant a_1 > 0$ and $q(t) \geqslant K_4 a(t) \geqslant K_4 a_1 > 0$. Hence, we have

$$
\begin{aligned}
&|y(t)e(t,x(t),y(t))|/q(t)g(y(t)) \\
&\quad\leqslant (K_9+1)k(t)V(t)/(a(t)q(t)C_1)^{\frac{1}{2}} + K_{10}[b(t)/(a(t)q(t))^{\frac{1}{2}}](q(t)/a(t))^{\frac{m}{2}}V(t) \\
&\qquad + K_{11}[b(t)/(a(t)q(t))^{\frac{1}{2}}](q(t)/a(t))^{\frac{m}{2}} + K_9 r(t)V(t)/(a(t)q(t))^{\frac{1}{2}} \\
&\qquad + r(t)/(a(t)q(t))^{\frac{1}{2}}C_1,
\end{aligned}
$$

and

$$
\begin{aligned}
&|e(t,x(t),y(t))|/q(t) \\
&\quad\leqslant k(t)F^{\frac{1}{2}}(x(t))/q(t) + b(t)V^{\frac{m}{2}}(t)(q(t)/a(t))^{\frac{m}{2}}/(K_4 a(t)q(t))^{\frac{1}{2}} + r(t)/q(t)
\end{aligned}
$$

where $K_{10} = (K_9+1)/C_1^{\frac{1}{2}}$ and $K_{11} = 1/C_1^{\frac{1}{2}}$. Notice, next, that (28) implies $G(y) \to \infty$ as $y \to \infty$, so $x(t)$ is continuable and $V(t)$ and $x(t)$ are bounded. Therefore, we have

$$
\begin{aligned}
&K_9 V(t)I(t)a'(t)_-/a(t) + I(t)y(t)[e(t,x(t),y(t)) - h(t,x(t),y(t))]/q(t)g(y(t)) \\
&\quad\leqslant \big\{E_3 a'(t)_-/a(t) + E_4 k(t)/(a(t)q(t))^{\frac{1}{2}} + E_5[b(t)/(a(t)q(t))^{\frac{1}{2}}](q(t)/a(t))^{\frac{m}{2}} \\
&\qquad + E_6 r(t)/(a(t)q(t))^{\frac{1}{2}} + E_7 w(t)/a(t)\big\}I(t),
\end{aligned}
$$

and

$$
\begin{aligned}
&E_2\alpha(t)x(t)[e(t,x(t),y(t)) - h(t,x(t),y(t))]/q(t) \\
&\quad\leqslant [E_8(k(t)+r(t))/q(t) + E_9[b(t)(q(t)/a(t))^{\frac{m}{2}} + w_1(t)]/(a(t)q(t))^{\frac{1}{2}}]\alpha(t)
\end{aligned}
$$

for some positive constants E_i, $i = 3, 4, \ldots, 9$. Now let

$$P_3(t) = [2q'(t)/q(t) - (a(t)q(t))'/a(t)q(t) - (1 + E_1)\alpha(t)/I(t)]_-,$$
$$P_4(t) = E_8(k(t) + r(t))/q(t) + E_9[b(t)(q(t)/a(t))^{\frac{m}{2}} + w_1(t)]/(a(t)q(t))^{\frac{1}{2}}$$

and

$$\begin{aligned} P_5(t) = {} & E_3 a'(t)_-/a(t) + [E_4 k(t) + E_5 b(t)(q(t)/a(t))^{\frac{m}{2}}]/(a(t)q(t))^{\frac{1}{2}} \\ & + E_6 r(t)/(a(t)q(t))^{\frac{1}{2}} + E_7 w(t)/a(t). \end{aligned}$$

Then, from (32) we have

$$H'(t) \leqslant P_3(t)H(t) - E_2(a(t)x(t)y(t))'\alpha(t)/q(t) + P_4(t)\alpha(t) + \varepsilon\alpha(t) + P_5(t)I(t).$$

Notice next that (11)–(15) imply that $\int_{t_0}^{\infty} P_5(s)\,\mathrm{d}s < \infty$, so there exists $T > t_0$ so that $\int_T^{\infty} P_5(s)\,\mathrm{d}s < \varepsilon$. Therefore, $\int_T^t P_5(s)I(s)\,\mathrm{d}s \leqslant I(t)\int_T^t P_5(s)\,\mathrm{d}s < \varepsilon I(t)$. Also, $\int_T^t \alpha(s)\,\mathrm{d}s \leqslant I(t)$, so

$$\begin{aligned} H(t) \leqslant {} & H(T) + \int_T^t P_3(s)H(s)\,\mathrm{d}s - E_2\int_T^t [(a(s)x(s)y(s))'\alpha(s)/q(s)]\,\mathrm{d}s \\ & + \int_T^t P_4(s)\alpha(s)\,\mathrm{d}s + 2\varepsilon I(t). \end{aligned} \tag{33}$$

An integration by parts yields

$$\begin{aligned} & \int_T^t [(a(s)x(s)y(s))'\alpha(s)/q(s)]\,\mathrm{d}s \\ & \qquad = a(t)x(t)y(t)\alpha(t)/q(t) - a(T)x(T)y(T)\alpha(T)/q(T) \\ & \qquad\quad - \int_T^t a(s)x(s)y(s)[\alpha(s)/q(s)]'\,\mathrm{d}s. \end{aligned}$$

Now by (27) and (28),

$$\begin{aligned} a(t)y(t)/q(t) & \leqslant a^{\frac{1}{2}}(t)[a(t)y^2(t)/q(t) + 1]/q^{\frac{1}{2}}(t) \\ & \leqslant a^{\frac{1}{2}}(t)[C_2 a(t)K_9 G(y(t))/q(t)]/q^{\frac{1}{2}}(t) + (a(t)/q(t))^{\frac{1}{2}} \\ & \leqslant K_9 C_2 (a(t)/q(t))^{\frac{1}{2}} V(t) + (a(t)/q(t))^{\frac{1}{2}} \\ & \leqslant (E_{10}V(t) + 1)[a(t)/q(t)]^{\frac{1}{2}} \end{aligned} \tag{34}$$

for some positive constant E_{10}. Then,

$$\begin{aligned} H(t) \leqslant {} & E_{11} + E_2|x(t)|(E_{10}V(t) + 1)\alpha(t)[a(t)/q(t)]^{\frac{1}{2}} \\ & + E_2\int_T^t |x(s)|(E_{10}V(s) + 1)|(\alpha(s)/q(s))'|(a(s)q(s))^{\frac{1}{2}}\,\mathrm{d}s \\ & + \int_T^t P_3(s)H(s)\,\mathrm{d}s + \int_T^t P_4(s)\alpha(s)\,\mathrm{d}s + 2\varepsilon I(t), \end{aligned}$$

and from the boundedness of $V(t)$ and $x(t)$ we have

$$\begin{aligned}H(t) &\leqslant E_{11} + E_{12}\alpha(t)[a(t)/q(t)]^{\frac{1}{2}} + \int_T^t E_{13}|(\alpha(s)/q(s))'|(a(s)q(s))^{\frac{1}{2}}\,\mathrm{d}s \\ &\quad + \int_T^t P_4(s)\alpha(s)\,\mathrm{d}s + 2\varepsilon I(t) + \int_T^t P_3(s)H(s)\,\mathrm{d}s = P_6(t) + \int_T^t P_3(s)H(s)\,\mathrm{d}s\end{aligned}$$

where $P_6(t)$ is the sum of the first five terms in the right member of the last inequality and E_i, $i = 11, 12, 13$, are positive constants. We then have from (22)–(26) that $\limsup_{t\to\infty} P_6(t)/I(t) \leqslant 2\varepsilon$ and $\int_T^\infty P_3(s)\,\mathrm{d}s \leqslant N < \infty$. Applying a generalized version of Gronwall's inequality (see for example [10; Lemma 6]) we obtain

$$\begin{aligned}H(t) &\leqslant P_6(t) + \int_T^t P_6(s)P_3(s)\exp\left(\int_s^t P_3(u)\,\mathrm{d}u\right)\mathrm{d}s \\ &= P_6(t) + [\exp\int_T^t P_3(s)\,\mathrm{d}s]\int_T^t P_6(s)P_3(s)\exp\left(-\int_T^s P_3(u)\,\mathrm{d}u\right)\mathrm{d}s \\ &\leqslant P_6(t) + e^N\int_T^t P_6(s)P_3(s)\,\mathrm{d}s \\ &\leqslant P_6(t) + e^N\Big[\sup_{T\leqslant s\leqslant t} P_6(s)\Big]\int_T^t P_3(s)\,\mathrm{d}s \\ &\leqslant P_6(t) + Ne^N\Big[\sup_{T\leqslant s\leqslant t} P_6(s)\Big],\end{aligned}$$

and hence

$$V(t) = H(t)/I(t) \leqslant 3\varepsilon + Ne^N 2\varepsilon$$

for all sufficiently large t. Since ε is arbitrary, this implies that $V(t) \to 0$ as $t \to \infty$. Thus, from (3) and the definition of $F(x)$, we have $x(t) \to 0$ as $t \to \infty$. □

If α and q are such that $(\alpha/q) \in C^2[t_0, \infty)$, then by modifying the proof of Theorem 4 we obtain the following result.

Theorem 5. *Let* (3)–(6), (11)–(17), (21)–(24), (26), *and* (27) *hold. If, in addition,* $(\alpha/q) \in C^2[t_0, \infty)$,

$$a(t)\,|(\alpha(t)/q(t))'| = o(I(t)),\quad t \to \infty, \tag{25'}$$

and

$$\int_{t_0}^t ([a(s)(\alpha(s)/q(s))']')_-\,\mathrm{d}s = o(I(t)),\quad t \to \infty, \tag{25''}$$

hold, then every solution $x(t)$ of (1) *satisfies $x(t) \to 0$ as $t \to \infty$.*

Proof. Let $x(t)$ be a solution of (1); then Theorem 1 implies that $x(t)$ is continuable. Proceed exactly as in the proof of Theorem 4 until inequality (33) is obtained. Then, integrating the second integral in (33) by parts as in the proof of Theorem 4, we have

$$(35)\quad H(t) \leqslant E_{14} - E_2 a(t)x(t)y(t)\alpha(t)/q(t) + E_2 \int_T^t a(s)x(s)y(s)[\alpha(s)/q(s)]'\,\mathrm{d}s + \int_T^t P_3(s)H(s)\,\mathrm{d}s + \int_T^t P_4(s)\alpha(s)\,\mathrm{d}s + 2\varepsilon I(t)$$

for some constant $E_{14} > 0$. Now integrate the first integral in the right member of (35) by parts to obtain

$$\begin{aligned} H(t) \leqslant{}& E_{15} - E_2 a(t)x(t)y(t)\alpha(t)/q(t) + \frac{1}{2}E_2 a(t)[\alpha(t)/q(t)]'x^2(t) \\ &- \frac{1}{2}E_2 \int_T^t [a(s)(\alpha(s)/q(s))']'x^2(s)\,\mathrm{d}s + \int_T^t P_3(s)H(s)\,\mathrm{d}s \\ &+ \int_T^t P_4(s)\alpha(s)\,\mathrm{d}s + 2\varepsilon I(t) \end{aligned}$$

where E_{15} is a positive constant. Then, from (34) we have

$$\begin{aligned} (36)\quad H(t) \leqslant{}& E_{15} + E_2|x(t)|(E_{10}V(t) + 1)\alpha(t)[a(t)/q(t)]^{\frac{1}{2}} \\ &+ \frac{1}{2}E_2 x^2(t)a(t)|[\alpha(t)/q(t)]'| + \frac{1}{2}E_2 \int_T^t x^2(s)[a(s)(\alpha(s)/q(s))']'_-\,\mathrm{d}s \\ &+ \int_T^t P_3(s)H(s)\,\mathrm{d}s + \int_T^t P_4(s)\alpha(s)\,\mathrm{d}s + 2\varepsilon I(t). \end{aligned}$$

The remainder of the proof is the same as the latter part of the proof of Theorem 4 except for using using (25′) and (25″) in place of (25). □

If we restrict our attention to the solutions of (1) that are not eventually monotonic, condition (24) is not needed to obtain the conclusions of Theorems 4 and 5. Specifically, we have the following result.

Theorem 6. *Let* (3)–(6), (11)–(17), (21)–(23), *and* (26)–(27) *hold and let $x(t)$ be a solution of* (1) *that is not eventually monotonic. If either* (i) (25) *holds, or* (ii) $(\alpha/q) \in C^2[t_0, \infty)$ *and* (25′) *and* (25″) *hold, then $x(t) \to 0$ as $t \to \infty$.*

Proof. Let $x(t)$ be a solution of (1) that is not eventually monotonic. First, recall that by (18) and the proof of Theorem 2,

$$V'(t) \leqslant P_1(t)V(t) + P_2(t)$$

and $\int_{t_0}^{\infty} P_i(s)\,\mathrm{d}s < \infty$ for $i = 1, 2$. But $V(t)$, $P_1(t)$, and $P_2(t)$ are nonnegative, so clearly

$$V'(t) \leqslant [P_1(t) + P_2(t)][V(t) + 1] \leqslant P(t)[V(t) + 1]$$

where $P(t) = P_1(t) + P_2(t)$. The boundedness of $V(t)$ was also established in the proof of Theorem 2, so $V(t) \leqslant p$ for some positive constant p. Hence,

$$(V(t) + 1)' \leqslant (p + 1)P(t), \quad t \geqslant t_0.$$

Therefore,

$$V'(t)_+ \leqslant (p + 1)P(t),$$

and so

$$\int_{t_0}^{t} V'(s)_+\,\mathrm{d}s \leqslant (p + 1)\int_{t_0}^{t} P(s)\,\mathrm{d}s.$$

Since $V'(t)_+ = V'(t) + V'(t)_-$,

$$\int_{t_0}^{t} V'(s)_-\,\mathrm{d}s = V(t_0) + \int_{t_0}^{t} V'(s)_+\,\mathrm{d}s - V(t) \leqslant V(t_0) + (p + 1)\int_{t_0}^{t} P(s)\,\mathrm{d}s.$$

Thus, we have

$$\int_{t_0}^{t} |V'(s)|\,\mathrm{d}s \leqslant V(t_0) + 2(p + 1)\int_{t_0}^{\infty} P(s)\,\mathrm{d}s,$$

and therefore $V(t)$ is of bounded variation. Hence, $V(t)$ has a finite limit as $t \to \infty$.

To complete the proof of the theorem, we will show that $V(t) \to 0$ as $t \to \infty$. Since $V(t)$ has a finite limit as $t \to \infty$, it suffices to show that there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ and $V(t_n) \to 0$ as $n \to \infty$. Now $x(t)$ is not eventually monotonic, so choose $\{t_n\}$ such that $y(t_n) = 0$. For the proof of (i), proceed exactly as in the proof of Theorem 4 until (33) is obtained with t replaced by $t_n \geqslant T$. Then integrate the second integral on the right hand side of (33) by parts to obtain

$$\begin{aligned}\int_{T}^{t_n} &[(a(s)x(s)y(s))'\alpha(s)/q(s)]\,\mathrm{d}s \\ &= -a(T)x(T)y(T)\alpha(T)/q(T) - \int_{T}^{t_n} a(s)x(s)y(s)[\alpha(s)/q(s)]'\,\mathrm{d}s.\end{aligned}$$

As in the proof of Theorem 4, (27) and (28) imply that

$$H(t_n) \leqslant \overline{P_6}(t_n) + \int_{T}^{t_n} P_3(s)H(s)\,\mathrm{d}s$$

where

$$\overline{P_6}(t_n) = \overline{E_{11}} + \overline{E_{12}}\alpha(t)[a(t)/q(t)]^{\frac{1}{2}} + \int_T^{t_n} E_{13}|(\alpha(s)/q(s))'|(a(s)q(s))^{\frac{1}{2}}\,\mathrm{d}s + \int_T^{t_n} P_4(s)\alpha(s)\,\mathrm{d}s + 2\varepsilon I(t_n).$$

It then follows, as in the last part of the proof of Theorem 4, that $V(t_n) \to 0$ as $n \to \infty$.

To prove (ii), proceed as in the proof of Theorem 5 obtaining

$$H(t_n) \leqslant \overline{E_{15}} + \frac{1}{2}E_2 x^2(t_n)a(t_n)|[\alpha(t_n)/q(t_n)]'| + \frac{1}{2}E_2\int_T^{t_n} x^2(s)[a(s)(\alpha(s)/q(s))']'_-\,\mathrm{d}s + \int_T^{t_n} P_3(s)H(s)\,\mathrm{d}s + \int_T^{t_n} P_4(s)\alpha(s)\,\mathrm{d}s + 2\varepsilon I(t_n) \tag{36'}$$

in place of (36). The remainder of the proof that $V(t_n) \to 0$ as $n \to \infty$ is the same as the latter part of the proof of (i) except for using (25′) and (25″) in place of (25).

□

Before continuing, it will be convenient to classify the solutions of (1) as follows. A solution $x(t)$ of (1) will be called nonoscillatory if there exists $t_1 \geqslant t_0$ such that $x(t) \neq 0$ for $t \geqslant t_1$; the solution will be called oscillatory if for any $t_1 \geqslant t_0$ there exist t_2 and t_3 such that $t_1 < t_2 < t_3$ and $x(t_2)x(t_3) < 0$; and it will be called a Z-type solution if it has arbitrarily large zeros but is eventually either nonnegative or nonpositive.

If we further restrict our consideration to only the class of oscillatory solutions of (1), then we can eliminate one of the hypotheses in Theorem 6 (ii).

Theorem 7. *Suppose that $(\alpha/q) \in C^2[t_0,\infty)$. If (3)–(6), (11)–(17), (21)–(23), (26)–(27), and (25″) hold; then every oscillatory or Z-type solution $x(t)$ of (1) satisfies $x(t) \to 0$ as $t \to \infty$.*

Proof. Let $x(t)$ be an oscillatory or Z-type solution of (1). As in the proof of Theorem 6, $\lim\limits_{t\to\infty} V(t)$ exists and is finite. Choose a sequence $\{t_n\} \to \infty$ as $n \to \infty$ such that $x(t_n) = 0$ for all n. Then as in the proof of part (ii) of Theorem 6, we obtain

$$H(t_n) \leqslant \overline{E_{15}} + \frac{1}{2}E_2\int_T^{t_n} x^2(s)[a(s)(\alpha(s)/q(s))']'_-\,\mathrm{d}s + \int_T^{t_n} P_3(s)H(s)\,\mathrm{d}s + \int_T^{t_n} P_4(s)\alpha(s)\,\mathrm{d}s + 2\varepsilon I(t_n)$$

in place of (36′). That $V(t_n) \to 0$ as $n \to \infty$ follows by an argument similar to the one used in the latter part of Theorem 6 except that (25′) is not needed. □

Remark. Results similar to the conclusions of the foregoing theorems and corollary have been obtained for special cases of (1) in [1–9, 11–30]. However, because of different hypotheses and a more general perturbation term, the results here are new even when the left hand side of (1) is specialized to the forms previously studied.

Remark. As indicated in the first remark following Corollary 3, there are some interchanges in hypotheses that can be made. For example, if $k(t) \equiv 0$ and $b(t) \equiv 0$ in Theorems 4–7, then condition (6) can be dropped provided (15) is replaced by (20) and condition (7) is added.

Consider

$$(tx')' + t^2x^5[(x')^2+1]/[(x')^2+2] = e(t,x,x') \tag{E_1}$$

for $t \geqslant 1$ with

$$\begin{aligned} e(t,x,x') = |x|^3/\sqrt{6} - |\sin^3 t|/\sqrt{6}t^3 + (\sin t - t\cos t - t^2 \sin t)/t^2 \\ + \left[t^4 + (t\cos t - \sin t)^2\right]\sin^5 t/t^3\left[2t^4 + (t\cos t - \sin t)^2\right]. \end{aligned}$$

Here $a(t) = t$, $q(t) = t^2$, $f(x) = x^5$, $h(t,x,x') \equiv 0$, and $g(x') = [(x')^2+1]/[(x')^2+2]$. It is not difficult to verify that (E_1) satisfies all the hypotheses of Corollary 3 and Theorems 1, 2, and 4 by taking $r(t) = |\sin^3 t|/\sqrt{6}t^3 + [(t^2+1)|\sin t| + t|\cos t|]/t^2 + [t^4 + (t\cos t - \sin t)^2]|\sin^5 t|/t^3[2t^4 + (t\cos t - \sin t)^2]$, $k(t) \equiv 1$, $b(t) \equiv w(t) \equiv w_1(t) \equiv 0$, and $\alpha(t) = 2\ln t/t$. Thus, we can conclude that that all solutions of (E_1) are continuable and tend to zero as $t \to \infty$. This conclusion cannot be obtained from any of the results in [1–9] or [11–30]. Notice that $x(t) = \sin t/t$ is an oscillatory solution of (E_1) on $[1,\infty)$.

4. Nonoscillatory and Z-type solutions

Notice that condition (23) cannot be satisfied if

$$\begin{aligned} (a(t)/q(t))(q(t)/a(t))' &= q'(t)/q(t) - a'(t)/a(t) \\ &= 2q'(t)/q(t) - (a(t)q(t))'/a(t)q(t) \equiv 0 \end{aligned}$$

and consequently Theorems 4–7 would not hold if this were true. In particular, this would be the case if $a(t) \equiv C_3q(t)$ for some constant $C_3 > 0$. We can avoid this difficulty if we restrict our attention to only the nonoscillatory and Z-type solutions of (1).

Theorem 8. *Suppose that* (3)–(6), (11)–(17), (21), *and* (27) *hold. If $f(x)$ is bounded away from zero whenever x is bounded away from zero and*

$$\begin{aligned}(37)\qquad &\int_{t_0}^{\infty}[1/a(s)]\int_{t_0}^{s}[C_4k(u)+C_5b(u)(q(u)/a(u))^{\frac{m}{2}}\\ &\quad+r(u)+C_6w_1(u)q(u)/a(u)-C_7q(u)]\,\mathrm{d}u\,\mathrm{d}s\\ &\quad+\int_{t_0}^{\infty}[C_8/a(s)]\,\mathrm{d}s=-\infty\end{aligned}$$

for any positive constants C_i, $i=4,\dots,8$, then every nonoscillatory or Z-type solution of (1) *tends to zero as $t\to\infty$.*

Proof. Let $x(t)$ be a nonoscillatory or Z-type solution of (1), say $x(t)\geqslant 0$ for $t\geqslant t_1\geqslant t_0$; by the proof of Theorem 2, there exists a constant $L>0$ such that $V(t)\leqslant L^2$. We first show that $\liminf_{t\to\infty} x(t)=0$. If this is not the case, there exist $t_2\geqslant t_1$ and a constant $L_1>0$ such that $f(x(t))>L_1$ for $t\geqslant t_2$. Notice first that (6) and (27) imply (28) which in turn implies $|y|/g(y)\leqslant B_1G(y)+B_2$ for some positive constants B_1 and B_2. Now from (1), (4), (6), (21) and (27), we have

$$\begin{aligned}(a(t)x'(t))'&\leqslant k(t)F^{\frac12}(x(t))+b(t)G^{\frac{m}{2}}(y(t))+r(t)+|y(t)|w_1(t)-L_1C_1q(t)\\ &\leqslant Lk(t)+L^mb(t)(q(t)/a(t))^{\frac{m}{2}}+r(t)+C_2w_1(t)|y(t)|/g(y(t))-L_1C_1q(t)\\ &\leqslant Lk(t)+L^mb(t)(q(t)/a(t))^{\frac{m}{2}}+r(t)+[C_2B_1L^2q(t)/a(t)+C_2B_2]w_1(t)\\ &\quad-L_1C_1q(t).\end{aligned}$$

Since (16) implies that $q(t)/a(t)$ is bounded from below, we have

$$(a(t)x'(t))'\leqslant Lk(t)+L^mb(t)(q(t)/a(t))^{\frac{m}{2}}+r(t)+L_2w_1(t)q(t)/a(t)-L_1C_1q(t)$$

for some $L_2>0$. Integrating the last inequality twice gives

$$\begin{aligned}x(t)\leqslant x(t_2)&+a(t_2)|x'(t_2)|\int_{t_2}^{t}[1/a(s)]\,\mathrm{d}s\\ &+\int_{t_2}^{t}[1/a(s)]\int_{t_2}^{s}[Lk(u)+L^mb(u)(q(u)/a(u))^{\frac{m}{2}}+r(u)\\ &+L_2w_1(u)q(u)/a(u)-L_1C_1q(u)]\,\mathrm{d}u\,\mathrm{d}s\end{aligned}$$

which contradicts (37). Hence, we conclude that $\liminf_{t\to\infty} x(t)=0$.

To complete the proof, notice that from the proof of Theorem 2, (18) holds and $\int_{t_0}^{\infty} P_i(s)\,\mathrm{d}s < \infty$ for $i = 1, 2$. Let $\varepsilon > 0$ be given. If $x(t)$ is nonoscillatory and not eventually monotonic, then $\liminf_{t\to\infty} x(t) = 0$ implies there exists $t_3 \geqslant t_1$ such that $y(t_3) = 0$, $F(x(t_3)) < \varepsilon$, $\int_{t_3}^{\infty} P_1(s)\,\mathrm{d}s < 1$, and $\int_{t_3}^{\infty} P_2(s)\,\mathrm{d}s < \varepsilon$. If $x(t)$ is a Z-type solution, choose t_3 so that $y(t_3) = F(x(t_3)) = 0$ and so that the other inequalities are satisfied. Then integrating (18) we have

$$V(t) \leqslant V(t_3) + \int_{t_3}^{t} P_1(s)V(s)\,\mathrm{d}s + \int_{t_3}^{t} P_2(s)\,\mathrm{d}s \leqslant 2\varepsilon + \int_{t_3}^{t} P_1(s)V(s)\,\mathrm{d}s,$$

and by Gronwall's inequality we have $V(t) \leqslant 2\varepsilon \exp(\int_{t_3}^{\infty} P_1(s)\,\mathrm{d}s) \leqslant 2\varepsilon \exp(1)$. Since ε is arbitrary, it follows that $F(x(t)) \to 0$ as $t \to \infty$ which, in view of (3), implies that $x(t) \to 0$ as $t \to \infty$. Now if $x(t)$ is eventually monotonic, then $\liminf_{t\to\infty} x(t) = 0$ implies that $x(t) \to 0$ as $t \to \infty$. This completes the proof for the case when $x(t)$ is eventually nonnegative. The proof in case $x(t)$ is eventually nonpositive in similar and will be omitted. □

Theorem 8 puts the somewhat severe restriction on $g(y)$ that it be bounded from above and from below. In the next theorem we relax the condition that $g(y)$ be bounded from above by modifying condition (21) on $h(t, x, y)$ and adding condition (7). As described in the second remark following Theorem 7, the requirement that $g(y)$ be bounded from below can be dropped in both Theorem 8 and the following theorem in case $k(t) \equiv 0$ and $b(t) \equiv 0$.

Theorem 9. *Let* (3)–(7) *and* (11)–(17) *hold,* $G(y) \to \infty$ *as* $|y| \to \infty$, *and* $f(x)$ *be bounded away from zero whenever* x *is bounded away from zero. If there is a nonnegative continuous function* $w_2 \colon [t_0, \infty) \to \mathbb{R}$ *such that*

$$h(t, x, y) \geqslant -|y|w_2(t)/g(y) \tag{38}$$

for all y *and*

$$\int_{t_0}^{\infty} [1/a(s)] \int_{t_0}^{s} [C_9 k(u) + C_{10} b(u)(q(u)/a(u))^{\frac{m}{2}} + r(u) + C_{11} w_2(u) q(u)/a(u) - C_{12} q(u)]\,\mathrm{d}u\,\mathrm{d}s + \int_{t_0}^{\infty} [C_{13}/a(s)]\,\mathrm{d}s = -\infty \tag{39}$$

for all positive constants C_i, $i = 9, \ldots, 13$, *then every nonoscillatory or* Z*-type solution* $x(t)$ *of* (1) *satisfies* $x(t) \to 0$ *as* $t \to \infty$.

The proof of Theorem 9 is the same as the proof of Theorem 8 with (38) and (39) used in place of (21) and (37) respectively. Another result in this direction is the following.

Theorem 10. *Suppose that* (3)–(6), (11)–(17), (21), *and* (27) *hold, and that* $f(x)$ *is bounded away from zero whenever* x *is bounded away from zero. If, in addition,*

$$\int_{t_0}^{\infty} |q'(s)|[a(s)/q^3(s)]^{\frac{1}{2}}\,\mathrm{d}s < \infty \tag{40}$$

and

$$\int_{t_0}^{\infty} \left\{[C_{14}k(s) + C_{15}b(u)(q(u)/a(u))^{\frac{m}{2}} + r(s)]/q(s) + C_{16}w_1(s)/a(s)\right\}\,\mathrm{d}s < \infty \tag{41}$$

for all positive constants C_i, $i = 14, \ldots, 16$, *then every nonoscillatory or* Z*-type solution of* (1) *tends to zero as* $t \to \infty$.

P r o o f. Let $x(t)$ be a nonoscillatory or Z-type solution of (1); then there exists $t_1 \geqslant t_0$ so that $x(t)$ does not change sign on $[t_1, \infty)$, say $x(t) \geqslant 0$ for $t \geqslant t_1$. If $\liminf\limits_{t\to\infty} x(t) > 0$, then there exist constants $t_2 \geqslant t_1$ and $L_3 > 0$ such that $f(x(t)) \geqslant 2L_3$ for $t \geqslant t_2$. Then, as in the proof of Theorem 8,

$$(a(t)x'(t))' \leqslant Lk(t) + L^m b(t)(q(t)/a(t))^{\frac{m}{2}} + r(t) + L_2 w_1(t)q(t)/a(t) - 2L_3C_1q(t).$$

Multiplying by $1/q(t)$ and integrating by parts gives

$$\begin{aligned} a(t)x'(t)/q(t) \leqslant a(t_2)x'(t_2)/q(t_2) &+ \int_{t_2}^{t} [a(s)|x'(s)q'(s)|/q^2(s)]\,\mathrm{d}s \\ &+ \int_{t_2}^{t} P_7(s)\,\mathrm{d}s - 2L_3C_1(t - t_2) \end{aligned}$$

where

$$P_7(t) = [Lk(t) + L^m b(t)(q(t)/a(t))^{\frac{m}{2}} + r(t)]/q(t) + L_2w_1(t)/a(t).$$

From the inequality

$$|x'(t)|[a(t)/q(t)]^{\frac{1}{2}} \leqslant [x'(t)]^2 a(t)/q(t) + 1,$$

condition (27), the fact that (16) implies $a(t)/q(t)$ is bounded from above, (28), and the boundedness of $V(t)$, we have

$$a(t)|q'(t)x'(t)|/q^2(t) \leqslant L_4[a(t)/q^3(t)]^{\frac{1}{2}}|q'(t)|$$

for some positive constant L_4. Hence (40) and (41) imply that there exists $t_3 \geqslant t_2$ so that

$$a(t)x'(t)/q(t) \leqslant -L_5 t$$

for $t \geqslant t_3$ and some $L_5 > 0$. But $q(t)/a(t)$ is bounded from below, so the last inequaltiy implies that $x'(t) \to -\infty$ as $t \to \infty$ contradicting the assumption that $x(t) \geqslant 0$. Therefore, $\liminf_{t\to\infty} x(t) = 0$. The remainder of the proof is the same as the last part of the proof of Theorem 8. □

The equation

$$(E_2)$$

$$(t^2x')' + \left\{ \frac{t^4+1}{t^4} + \left[\frac{(x')^2 + \ln[(x')^2+1]}{2[(x')^2 + \ln[(x')^2+1]+1]} \right]^{\frac{1}{4}} \right\} \frac{x'}{2[(x')^2+1]} + \frac{t^2x^3[(x')^2+1]}{(x')^2+2}$$
$$= \frac{(t^4+1)}{t(2t^4+1)} - \frac{t^2}{2(t^4+1)} \left[\frac{(x')^2 + \ln[(x')^2+1]}{2[(x')^2 + \ln[(x')^2+1]+1]} \right]^{\frac{1}{4}} - \frac{x^2}{2}$$

satisfies all the hypotheses of Theorems 1, 2, and 8 and Corollary 3. Here $G(y) = \frac{1}{2}[y^2 + \ln(y^2+1)]$ and we can take $k(t) \equiv 1$, $b(t) = \frac{1}{2}t^2(t^4+1)$, $m = \frac{1}{2}$, and $r(t) = (t^4+1)/t(2t^4+1)$, $w(t) \equiv 0$, and $w_1(t) = 3$. Notice that $(q(t)/a(t))' \equiv 0$ so (23) does not hold and therefore none of Theorems 4–7 apply to (E_2). Furthermore, Theorem 10 does not apply since (40) is not satisfied. We can assert from Theorem 1 and Corollary 3 that all solutions of (E_2) are continuable and oscillatory solutions of (E_2) are bounded, but we cannot determine if oscillatory solutions tend to zero as $t \to \infty$. However, Theorem 8 implies that all nonoscillatory and Z-type solutions of (E_2) tend to zero as $t \to \infty$. Such a solution is $x(t) = 1/t$.

The equation

$$(E_3) \qquad (t^2x')' + \frac{1}{2}(t^4+1)x'/t^4[(x')^2+1] + t^2x^3[(x')^2+1] = (t^4+1)/t^5 - \frac{1}{2}x^2$$

for $t \geqslant 1$ satisfies all the hypotheses of Theorem 9. Notice that Theorems 5–8, and 10 do not apply to (E_3) since g is not bounded from above. Also, observe that $x(t) = 1/t$ is a solution of (E_3). None of the results in [1–9] and [11–30] apply to equations (E_2) and (E_3).

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