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CONVEX AUTOMORPHISMS OF PARTIAL MONOUNARY ALGEBRAS

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Convex isomorphisms of lattices were investigated in [5], [6] and [4]. This concept was studied also for ordered sets [1] and for q-lattices [2].

The notion of a convex subset of a partial monounary algebra was introduced and studied in [3].

In the present paper we shall deal with convex automorphisms of partial monounary algebras which are defined as follows. Let (A, f) be a partial algebra. A bijection $h\colon A \to A$ is said to be a convex automorphism if, for each $B \subseteq A$, B is convex if and only if $h(B)$ is convex.

Several results of [3] will be applied.

Let us remark that a convex automorphism h need not be, in general, an automorphism with respect to the partial operation f.

The main topics we are interested in are as follows:

a) relations between convex automorphisms of (A, f) and connected components of (A, f) (cf. Sections 3 and 4);

b) conditions under which each convex automorphism of (A, f) is an automorphism with respect to f (cf. Section 5).

1. PRELIMINARIES

By a (partial) monounary algebra we understand a pair (A, f), where A is a nonempty set and $f\colon A \to A$ is a (partial) mapping. Let $\mathcal{U}$ be the class of all partial monounary algebras. To each $(A, f) \in \mathcal{U}$ there corresponds a directed graph $G(A, f) = (A, E)$ without loops and multiple edges which is defined as follows: an ordered pair (a, b) of distinct elements of A belongs to E iff $f(a) = b$.

In this section we suppose that $(A, f) \in \mathcal{U}$.

We recall the following definition (cf. [3]).

Definition 1.1. A subset $B \subseteq A$ will be called convex (in (A,f)) if, whenever a, b_1, b_2 are distinct elements of A such that $b_1, b_2 \in B$ and there is a path (in $G(A,f)$) going from b_1 to b_2, not containing the element b_2 twice and containing the element a, then a belongs to B as well. Let $\mathrm{Co}(A,f)$ be the system of all convex subsets of (A,f).

Remark 1.2. The system $\mathrm{Co}(A,f)$ is partially ordered by inclusion, and it is a lattice.

Notation 1.3. Let x, $y \in A$. The symbol $L_f(x,y)$ denotes the least convex subset of (A,f) which contains the elements x and y.

Definition 1.4. Let $(D,g) \in \mathcal{U}$ and let $h\colon A \to D$ be a bijection. Then h is said to be a convex isomorphism of (A,f) onto (D,g) if, whenever $B \subseteq A$, then $B \in \mathrm{Co}(A,f)$ iff $h(B) = \{h(b)\colon b \in B\} \in \mathrm{Co}(D,g)$. A convex isomorphism of (A,f) onto (A,f) is called a convex automorphism of (A,f).

Remark 1.5. Let x, $y \in A$, $n \in \mathbb{N}$. If we write $y = f^n(x)$, then we suppose that $x \in \operatorname{dom} f$, $f(x) \in \operatorname{dom} f$, ..., $f^{n-1}(x) \in \operatorname{dom} f$ and the elements y and $f^n(x)$ coincide.

Writing $y \neq f^n(x)$ we mean that either

(1) $x \in \operatorname{dom} f$, $f(x) \in \operatorname{dom} f$, ..., $f^{n-1}(x) \in \operatorname{dom} f$

and the elements y and $f^n(x)$ are distinct, or (1) fails to hold.

Remark. Let $X \subseteq A$ be such that if $x \in X \cap \operatorname{dom} f$, then $f(x) \in X$. The partial operation f reduced to the set X is denoted by the same symbol f; in the case when it is necessary to emphasize the set X under consideration we apply the notation $f \restriction X$.

Definition 1.6. Let x, $y \in A$. If there are m, $n \in \mathbb{N} \cup \{0\}$ such that $f^n(x) = f^m(y)$, then we write $x \equiv_f y$. The relation $\equiv_f$ is an equivalence relation on A. A partial monounary algebra (A,f) is said to be connected, if $A/\equiv_f$ is a one-element set. If $X \in A/\equiv_f$, then X is called a connected component of (A,f).

Further, put $f^{-1}(x) = \{z \in \operatorname{dom} f\colon f(z) = x\}$.

Notation 1.7. Let $S(A,f)$ be the set of all $x \in A$ satisfying some of the following conditions:

(1) x belongs to a connected component M of (A,f) with $\operatorname{card} M \leqslant 2$;

(2) (a) $x \in \operatorname{dom} f$, (b) $f^{-1}(x) = \emptyset$, (c) either $f(x) \in \operatorname{dom} f$, $f^2(x) = f(x)$, or $f(x) \notin \operatorname{dom} f$;

(3) (a) $x \notin \operatorname{dom} f$ or $x \in \operatorname{dom} f$, $f(x) = x$, (b) if $y \in \operatorname{dom} f$, $f(y) \in \operatorname{dom} f$, $f(y) \neq x$, then $f^2(y) \neq x$.

Example 1.8. If (A, f) is a partial monounary algebra from Fig. 1 or Fig. 2, then $S(A, f) = \{a, b, c\}$ or $S(A, f) = \{a, b_1, b_2\}$, respectively.

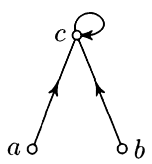

Fig. 1

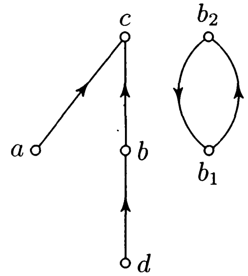

Fig. 2

Lemma 1.9. ([3], 3.1.2). *The set $A - S(A, f)$ is closed under the operation f.*

Lemma 1.10. ([3], 3.2). *Let $x \in A$. Then $x \in S(A, f)$ if and only if the following conditions are satisfied:*

(i) *$\{x\} \vee \{y\}$ covers $\{x\}$ and $\{y\}$ (in $\mathrm{Co}(A, f)$) for each $y \in A - \{x\}$;*

(ii) *if z_1, z_2 are distinct elements of $A - \{x\}$, then $\{x\} \not\leqslant \{z_1\} \vee \{z_2\}$.*

Corollary 1.10.1. *If $x \in S(A, f)$ and $y \in A$, then $L_f(x, y) = \{x, y\}$.*

Proof. Let $x \in A - S(A, f)$, $y \in A$. If $y = x$, then obviously $L_f(x, y) = \{x, y\}$. If $y \neq x$, then $L_f(x, y) = \{x\} \vee \{y\}$ (in $\mathrm{Co}(A, f)$). Assume that there is $u \in L_f(x, y) - \{x, y\}$. Then $\{u\} \leqslant \{x\} \vee \{y\}$ and we obtain

$$\{y\} \leqslant \{u\} \vee \{y\} \leqslant \{x\} \vee \{y\}.$$

Thus 1.10 (i) implies

$$\{u\} \vee \{y\} = \{x\} \vee \{y\}.$$

Hence

$$\{x\} \leqslant \{x\} \vee \{y\} = \{u\} \vee \{y\},$$

a contradiction to 1.10 (ii). □

Lemma 1.11. *Let x, $y \in A - S(A, f)$. The following conditions are equivalent:*

(i) *x and y belong to the same connected component of (A, f);*

(ii) *x and y belong to the same connected component of $(A - S(A, f), f)$.*

Proof. Let (i) hold. Since $A - S(A, f)$ is a closed under f in view of 1.9 and x, $y \in A - S(A, f)$, we obtain that (ii) is valid. The implication (ii) $\Rightarrow$ (i) is obvious. □

The following notion was introduced in [3].

Definition 1.12. (A, f) is said to be coherent, if it is connected and either $\operatorname{card} A = 1$ or $S(A, f) = \emptyset$.

2. A new partial operation f'

In 2.1–2.10 let (A, f) be a partial monounary algebra and let h be a convex automorphism of (A, f).

If $x \in A$, $B \subseteq A$, then we will also write $x' = h(x)$, $B' = h(B)$.

Notation 2.1. Let us define a partial operation f' on A as follows. If $x \in A$, then there is a unique element $a \in A$ with $x = a'$. We put

(i) if $a \notin \operatorname{dom} f$, then $x \notin \operatorname{dom} f'$;

(ii) if $a \in \operatorname{dom} f$, then $x \in \operatorname{dom} f'$ and $f'(x) = \big(f(a)\big)'$.

Lemma 2.2. *The mapping h is an isomorphism from (A, f) onto (A, f').*

Proof. Since h is a convex automorphism, h is a bijection. Let $a \in A$, $a \in \operatorname{dom} f$. We are to prove

(1) $h(a) \in \operatorname{dom} f'$, $h\big(f(a)\big) = f'\big(h(a)\big)$.

The relation (1) follows from 2.1 (ii), where $x = h(a)$.

Now let $a \in A - \operatorname{dom} f$, $x = h(a)$. By 2.1 (i), $x \notin \operatorname{dom} f'$. Therefore h is an automorphism of (A, f) onto (A, f'). □

Lemma 2.3. $\operatorname{Co}(A, f) = \operatorname{Co}(A, f')$.

Proof. Let us consider the relation

(1) $B \in \operatorname{Co}(A, f)$.

Since h is an isomorphism from (A, f) onto (A, f') by 2.2, the relation (1) is equivalent to

(2) $B' \in \operatorname{Co}(A, f')$.

Further, h is a convex automorphism of (A, f), thus (1) is equivalent to the relation

(3) $B' \in \operatorname{Co}(A, f)$.

Thus (2) and (3) are equivalent for each subset B of A, i.e. for each subset B' of A. Therefore $\operatorname{Co}(A, f) = \operatorname{Co}(A, f')$. □

Corollary 2.4. *$L_f(x, y) = L_{f'}(x, y)$ for each x, $y \in A$.*

Proof. $L_f(x, y)$ is the least set $B \in \operatorname{Co}(A, f)$ such that $\{x, y\} \subseteq B$, hence 2.3 yields that $L_f(x, y)$ is the least set $B \in \operatorname{Co}(A, f')$ such that $\{x, y\} \subseteq B$, i.e., it coincides with $L_{f'}(x, y)$. □

Remark 2.5. In view of 2.4 we will use the symbol L instead of L_f or $L_{f'}$. Since h is a convex automorphism, then we get

$$L(x,y) = B \Leftrightarrow L(x',y') = B'$$

for each x, $y \in A$, $B \subseteq A$.

Corollary 2.6. $S(A,f) = S(A,f')$.

Proof. The assertion follows from 1.10 and 2.3, since the conditions (i) and (ii) of 1.11 depend only on the lattices $\mathrm{Co}(A,f)$ and $\mathrm{Co}(A,f')$. □

Lemma 2.7. *The partial monounary algebras* $(A - S(A,f), f)$ *and* $(A - S(A,f'), f')$ *have the same partition into connected components.*

Proof. By 2.6, $S(A,f) = S(A,f')$. Next, 1.9 implies that $A - S(A,f)$ is closed under f and $A - S(A,f')$ is closed under f'. Now 2.3 and [3], 6.3 imply what was required. □

Corollary 2.8. *Let* $M \subseteq A$. *Then* M *is a connected component of* $(A - S(A,f), f)$ *if and only if* M *is a connected component of* $(A - S(A,f'), f')$.

Proof. Immediately from 2.7. □

Lemma 2.9. *Let* $M \subseteq A$. *Then* M *is a connected component of* $(A - S(A,f), f)$ *if and only if* M' *is a connected component of* $(A - S(A,f), f)$.

Proof. The following conditions are equivalent by 1.10 and 2.2:
(1) M is a connected component of $(A - S(A,f), f)$,
(2) M' is a connected component of $(A - S(A,f'), f')$.
By 2.8 (with M' instead of M), (2) is equivalent to
(3) M' is a connected component of $(A - S(A,f), f)$.
Thus (1) $\Leftrightarrow$ (3), which is the desired equivalence. □

Corollary 2.10. *Assume that each connected component of* (A,f) *is coherent. Let* $M \subseteq A$. *Then* M *is a connected component of* (A,f) *if and only if* M' *is a connected component of* (A,f).

Proof. Let M be a coherent connected component of (A,f). By 1.12, either $\operatorname{card} M = 1$ or $S(M, f \restriction M) = \emptyset$. If $M \subseteq A - S(A,f)$, then M' is a connected component of $(A - S(A,f), f)$ according to 2.9, thus M' is a connected component of (A,f). Assume that $M \subseteq S(A,f)$. Then $S(M, f \restriction M) = M \neq \emptyset$. We obtain that $\operatorname{card} M = 1$, $M = \{m\}$. By 2.2, $m' \in S(A,f)$ and then 2.6 implies

(1) $m' \in S(A, f)$.

Let K be a connected component of (A, f) containing m'. Then K is coherent, i.e. either card $K = 1$ or $S(K, f \restriction K) = \emptyset$. The second relation contradicts (1), therefore

(2) $K = \{m'\} = M'$.

Conversely, let M' be a connected component of (A, f). There is a connected component K of (A, f) such that $K \cap M \neq \emptyset$. Then K' is a connected component of (A, f) with $K' \cap M' \neq \emptyset$, therefore $K' = M'$ and $K = M$. □

3. Convex automorphisms and connected components

Lemma 3.1. *Let $(A, f) \in \mathcal{U}$, let $h\colon A \to A$ be a bijection. Then h is a convex automorphism of (A, f) if and only if*

(i) $h(S(A, f)) = S(A, f)$,

(ii) $h \restriction (A - S(A, f))$ *is a convex automorphism of* $(A - S(A, f), f)$.

Proof. Let h be a convex automorphism of (A, f). By 1.9, $A - S(A, f)$ is closed under f, thus $(A - S(A, f), f)$ is a partial monounary algebra. Assume that

(1) $x \in S(A, f)$

is valid. Since h is an isomorphism of (A, f) onto (A, f') according to 2.2, we obtain that (1) is equivalent to

(2) $h(x) \in S(A, f')$

and then 2.6 implies that it is equivalent to

(3) $h(x) \in S(A, f)$.

Therefore $h(S(A, f)) \subseteq S(A, f)$. Let $y \in S(A, f)$. The mapping h is a bijection, thus there exists $x \in A$ such that $h(x) = y$. Then $h(x) \in S(A, f)$ and the equivalence (1) $\Leftrightarrow$ (3) implies that $x \in S(A, f)$, i.e. $y = h(x) \in h(S(A, f))$. We obtain $S(A, f) \subseteq h(S(A, f))$ and hence (i) is valid.

Further, (i) implies that $h(A - S(A, f)) = A - S(A, f)$ and by the assumption that h is a convex automorphism of (A, f) we get that $h \restriction (A - S(A, f))$ is a convex automorphism of the partial monounary algebra $(A - S(A, f), f)$.

Conversely, let (i) and (ii) hold. Assume that B is a convex subset of (A, f). Let us show that $h(B)$ is convex, i.e.

(4) $L_f(x, y) \subseteq h(B)$ for each x, $y \in h(B)$.

Let $x = h(a)$, $y = h(b)$, a, $b \in B$. In view of (i) we have

(5) $x \in S(A, f) \Leftrightarrow a \in S(A, f)$,

(6) $y \in S(A, f) \Leftrightarrow b \in S(A, f)$.

If $x \in S(A, f)$, then $L_f(x, y) = \{x, y\}$ by 1.10.1 and then $L_f(x, y) \subseteq h(B)$. Analogously, if $y \in S(A, f)$, then $L_f(x, y) = \{x, y\} \subseteq h(B)$. Suppose that $\{x, y\} \subseteq A - S(A, f)$. If x and y do not belong to the same connected component of $(A -$

$S(A,f),f\big)$, then $L_f(x,y) = \{x,y\}$; thus we can assume that there is a connected component M such that $\{x,y\} \subseteq M$ and $L_f(x,y) \neq \{x,y\}$. By virtue of (5) and (6), $\{a,b\} \subseteq A - S(A,f)$. Since (ii) is valid, we obtain

(7) $h\big(L_f(a,b)\big) = L_f(x,y)$

and therefore

$$L_f(x,y) = h\big(L_f(a,b)\big) \subseteq h(B).$$

Now assume that $B \subseteq A$ is such that $h(B) \in \mathrm{Co}(A,f)$. We will show

(9) $L_f(u,v) \subseteq B$ for each u, $v \in B$.

Let u, $v \in B$. Put $c = h(u)$, $d = h(v)$. Then $\{c,d\} \subseteq h(B) \in \mathrm{Co}(A,f)$, thus

(10) $L_f(c,d) \subseteq h(B)$.

By (i) we have

(11) $u \in S(A,f) \Leftrightarrow c \in S(A,f)$,

(12) $v \in S(A,f) \Leftrightarrow d \in S(A,f)$.

If either $u \in S(A,f)$ or $v \in S(A,f)$, then 1.10.1 implies that $L_f(u,v) = \{u,v\} \subseteq B$. Let $\{u,v\} \subseteq A - S(A,f)$. By (ii) we obtain

(13) $h\big(L_f(u,v)\big) = L_f(c,d)$.

Then (10) and (13) yield that $h\big(L_f(u,v)\big) \subseteq h(B)$, therefore (since h is a bijection) the relation (9) is valid. □

Theorem 3.2. *Let $(A,f) \in \mathcal{U}$, let $h\colon A \to A$ be a bijection. Then h is a convex automorphism of (A,f) if and only if*

(i) $h\big(S(A,f)\big) = S(A,f)$,

(ii) *M is a connected component of $\big(A - S(A,f),f\big)$ if and only if $h(M)$ is a connected component of $\big(A - S(A,f),f\big)$,*

(iii) *if M is a connected component of $\big(A - S(A,f),f\big)$, then $h \restriction M$ is a convex isomorphism of (M,f) onto $\big(h(M),f\big)$.*

P r o o f. Let h be a convex automorphism of (A,f). By 3.1 and 2.9, (i) and (ii) hold. The condition (iii) can be proved analogously as (ii) of 3.1.

Suppose that (i)–(iii) are satisfied. Let us show (ii) of 3.1, i.e. that, for each $B \subseteq A - S(A,f)$, $B \in \mathrm{Co}\,\big(A - S(A,f),f\big)$ if and only if $h(B) \in \mathrm{Co}\,\big(A - S(A,f),f\big)$. Assume that $B \subseteq A - S(A,f)$. There exist distinct connected components M_i of (A,f), $i \in I$, such that

(1) $B = \bigcup\limits_{i\in I}(B \cap M_i)$.

Put $B_i = B \cap M_i$ for $i \in I$. Since h is a bijection and (i)–(iii) hold, we have

(2) $h(B) = \bigcup\limits_{i\in I} h(B) \cap h(M_i)$.

Consider the following conditions:

(3.1) B is convex,

(3.2) B_i is convex for each $i \in I$,

(3.3) $h(B_i)$ is convex for each $i \in I$,

(3.4) $h(B)$ is convex.

The equivalences (3.1) $\Leftrightarrow$ (3.2) and (3.3) $\Leftrightarrow$ (3.4) are obvious.

Let (3.2) hold and $i \in I$. Since $B_i \subseteq A - S(A, f)$, by (i) we get that $h(B_i) \subseteq A - S(A, f)$. The set $M_i - S(A, f)$ is a connected component of $\big(A - S(A, f), f\big)$ in view of 1.9, thus (iii) and (3.2) yield that $h(B_i)$ is a convex subset of $\big(h(M_i - S(A, f)), f\big)$. Thus (3.3) is valid.

Now let (3.3) be satisfied and $i \in I$. If $K \subseteq A - S(A, f)$ is a connected component of $\big(A - S(A, f), f\big)$ such that $h(B_i) \subseteq K$, then $K = h(M)$ for some $M \subseteq A$. By (i), $M \subseteq A - S(A, f)$. Then (ii) implies that M is a connected component of $\big(A - S(A, f), f\big)$. We have

$$h(B_i) \subseteq K = h(M),$$

thus $B_i \subseteq M$. Since a connected component of (A, f) containing B_i is M_i, we get

$$M = M_i - S(A, f).$$

The relation (iii) yields

(4) $B_i \in \mathrm{Co}(M, f) \Leftrightarrow h(B_i) \in \mathrm{Co}(K, f)$,

thus we obtain that the set B_i is convex.

Therefore (3.2) $\Leftrightarrow$ (3.3) is valid, hence the relation (3.1) $\Leftrightarrow$ (3.4) implies that the condition (ii) of 3.1 is satisfied. □

Lemma 3.3. *Let $(A, f) \in \mathcal{U}$ and let h be a convex automorphism of (A, f). If M is a connected component of $\big(A - S(A, f), f\big)$, then $\mathrm{Co}(M, f) = \mathrm{Co}(M, f')$.*

Proof. By 2.3, $\mathrm{Co}(A, f) = \mathrm{Co}(A, f')$. Let M be a connected component of $\big(A - S(A, f), f\big)$. Assume that $B \in \mathrm{Co}(M, f)$. Obviously $B \in \mathrm{Co}(A, f)$, thus $B \in \mathrm{Co}(A, f')$. Since $B \subseteq M$, the relation $B \in \mathrm{Co}(A, f')$ implies that $B \in \mathrm{Co}(M, f')$ (M is a connected component of $\big(A - S(A, f), f'\big)$ in view of 2.6 and 2.8).

Now let $C \in \mathrm{Co}(M, f')$. Then $C \in \mathrm{Co}(A, f') = \mathrm{Co}(A, f)$. We have $C \subseteq M$, where M is a connected component of $\big(A - S(A, f), f\big)$, thus $C \in \mathrm{Co}(M, f)$. □

Corollary 3.4. *Let $(A, f) \in \mathcal{U}$ and let h be a convex automorphism of (A, f). If M is a connected component of $\big(A - S(A, f), f\big)$, then $M' = h(M)$ is a connected component of $\big(A - S(A, f), f\big)$ and $\mathrm{Co}(M', f) = \mathrm{Co}(M', f')$.*

Proof. The assertion follows from 2.9 and 3.3. □

4. On connected components of $(A - S(A,f), f)$

Again, let (A, f) be a partial monounary algebra. According to 3.2, we can describe all convex automorphisms of (A, f) if for each connected components M and K of $(A - S(A,f), f)$ we can describe all convex isomorphisms of M onto K.

In the present section we shall investigate convex isomorphisms with the mentioned property.

In other words, our aim is to sharpen the condition (iii) from Theorem 3.2, i.e., to give a more detailed characterization of the mappings $h \restriction M$ from 3.2.

We have to distinguish several cases with respect to the structure of M. (Notice that $\operatorname{card} M > 1$ since M is a connected component of $(A - S(A,f), f)$.) We introduce the conditions (Ci) (i=1,2,3,4,8) and (Ci.j) (where either i $= 5, 6, 7$ and j $= 1, 2, 3$ or i $= 9$ and j $= 1, 2$) for M.

These conditions are as follows:

(C1) (M, f) contains a 2-element cycle $\{c_1, c_2\}$ such that $f^{-1}(c_1) \neq \{c_2\}$ and $f^{-1}(c_2) \neq \{c_1\}$.

(C2) (M, f) contains no cycle, $M \subseteq \operatorname{dom} f$ and $f \restriction M$ is non-injective.

(C3) $M = \{a_i : i \in \mathbb{Z}\}$, $a_i \neq a_j$ for $i \neq j$, $f(a_i) = a_{i+1}$ for each $i \in \mathbb{Z}$.

(C4) $M = \{a_i : i \in \mathbb{N}\}$, $a_i \neq a_j$ for $i \neq j$, $f(a_i) = a_{i+1}$ for each $i \in \mathbb{N}$.

(C5) $M = \{a_1, a_2, \dots, a_k\}$, $k > 1$, $a_i \neq a_j$ for $i \neq j$, $f(a_i) = a_{i-1}$ for each $i = 2, \dots, k$.

(C5.1) (C5) holds and $f(a_1) = a_1$.

(C5.2) (C5) holds and $f(a_1) = a_2$.

(C5.3) (C5) holds and $a_1 \notin \operatorname{dom} f$.

(C6) $M = \{a_i : i \in \mathbb{N}\}$, $a_i \neq a_j$ for $i \neq j$, $f(a_i) = a_{i-1}$ for each $i \in \mathbb{N}$, $i > 1$.

(C6.1) (C6) holds and $f(a_1) = a_1$.

(C6.2) (C6) holds and $f(a_1) = a_2$.

(C6.3) (C6) holds and $a_1 \notin \operatorname{dom} f$.

(C7) There is $c \in M$ such that $\operatorname{card}\big(f^{-1}(c) - \{c\}\big) = 1$.

(C7.1) (C7) holds, $f(c) = c$, and (C5.1) and (C6.1) fail to hold.

(C7.2) (C7) holds, $\{f(c)\} = f^{-1}(c)$, and (C5.2) and (C6.2) fail to hold.

(C7.3) (C7) holds, $c \notin \operatorname{dom} f$, and (C5.3) and (C6.3) fail to hold.

(C8) There is $C \subseteq M$ such that C is a cycle with more than two elements.

(C9) There is $c \in M$ such that $\operatorname{card}\big(f^{-1}(c) - \{c\}\big) > 1$.

(C9.1) (C9) holds and $f(c) = c$.

(C9.2) (C9) holds and $c \notin \operatorname{dom} f$.

Lemma 4.0. *Let M be a connected component of $(A - S(A,f), f)$. Then (M,f) satisfies exactly one of the conditions* (C1)–(C4), (C5.1)–(C5.3), (C6.1)–(C6.3), (C7.1)–(C7.3), (C8), (C9.1)–(C9.2).

Proof. The proof can be performed by successive elimination of the other cases. $\square$

In the following lemmas 4.1–4.9 assume that $h\colon A \to A$ is a bijection and that M is a connected component of $(A - S(A,f), f)$ such that $h(M)$ is a connected component of $(A - S(A,f), f)$.

We use the symbols x', M' and f' as in Section 2.

Lemma 4.1. *Let* (C1) *hold. Then h is a convex isomorphism of (M,f) onto (M',f) is and only if h is an isomorphism of (M,f) onto (M',f).*

Proof. Let h be a convex isomorphism of (M,f) onto (M',f). By 2.2 we get

(1) h is an isomorphism of (M,f) onto (M',f').

Thus

(2) (M',f') contains a 2-element cycle $\{c_1', c_2'\}$ such that

$$(f')^{-1}(c_1') \neq \{c_2'\} \quad \text{and} \quad (f')^{-1}(c_2') \neq \{c_1'\}.$$

Applying 3.3 (with M' instead of M) we obtain

(3) $\mathrm{Co}(M',f) = \mathrm{Co}(M',f')$.

Then [3], 5.6.2 implies, in view of (2) and (3), that $f \restriction M' = f' \restriction M'$. Thus (1) yields that h is an isomorphism of (M,f) onto (M',f).

The converse implication is obvious. $\square$

Lemma 4.2. *Let* (C2) *hold. Then h is a convex isomorphism of (M,f) onto (M',f) if and only if h is an isomorphism of (M,f) onto (M',f).*

Proof. Let h be a convex isomorphism of (M,f) onto (M',f). Analogously as in the proof of 4.1, $\mathrm{Co}(M',f) = \mathrm{Co}(M',f')$ is valid. Then [3], 5.2 (i) implies that $f \restriction M' = f' \restriction M'$. Therefore h, an isomorphism of (M,f) onto (M',f'), is an isomorphism of (M,f) onto (M',f).

The converse implication is obvious. $\square$

Lemma 4.3. *Let* (C3) *be satisfied. Then h is a convex isomorphism of (M,f) onto (M',f) if and only if either*

(D3.1) $f(a_i') = a_{i+1}'$ *for each* $i \in \mathbb{Z}$,

or

(D3.2) $f(a_i') = a_{i-1}'$ *for each* $i \in \mathbb{Z}$.

P r o o f. Let h be a convex isomorphism of (M, f) onto (M', f). Then 2.2 yields

(1) h is an isomorphism of (M, f) onto (M', f').

Therefore

(2) $M' = \{a'_i : i \in \mathbb{Z}\}$, $a'_i \neq a'_j$ for $i \neq j$, $f'(a'_i) = a'_{i+1}$ for each $i \in \mathbb{Z}$. Since $\mathrm{Co}(M', f) = \mathrm{Co}(M', f')$ by virtue of 3.3, we obtain (with respect to [3], 5.2 (ii)) that either $f(a'_i) = a'_{i+1}$ for each $i \in \mathbb{Z}$ or $f(a'_i) = a'_{i-1}$ for each $i \in \mathbb{Z}$.

It is obvious that if (D3.1) or (D3.2) holds, then h is a convex isomorphism of (M, f) onto (M', f). □

Lemma 4.4. *Let* (C4) *hold. Then h is a convex isomorphism of (M, f) onto (M', f) if and only if some of the following conditions is satisfied:*

(D4.1) $f(a'_i) = a'_{i+1}$ *for each* $i \in \mathbb{N}$,

(D4.2) $f(a'_i) = a'_{i-1}$ *for each* $i \in \mathbb{N}$, $i > 1$, $f(a_1) = a_1$,

(D4.3) $f(a'_i) = a'_{i-1}$ *for each* $i \in \mathbb{N}$, $i > 1$, $f(a_1) = a_2$,

(D4.4) $f(a'_i) = a'_{i-1}$ *for each* $i \in \mathbb{N}$, $i > 1$, $a_1 \notin \operatorname{dom} f$.

P r o o f. If h is convex isomorphism of (M, f) onto (M', f), then analogously as above we can apply [3]. Now the validity of some of the conditions (D4.1)–(D4.4) follows from [3], 5.3.2.

The converse implication is obvious. □

Lemma 4.5. *Let* (C5.j) *be valid for some* j $\in \{1, 2, 3\}$. *Consider the conditions*

(D5) $f(a_i) = a'_{i-1}$ *for each* $i = 2, \ldots, k$,

(E5) $f(a'_i) = a'_{i+1}$ *for each* $i = 1, 2, \ldots, k-1$.

Then h is a convex isomorphism of (M, f) onto (M', f) if and only if some of the following conditions is satisfied:

(D5.1) (D5) *holds and* $f(a'_1) = a'_1$,

(D5.2) (D5) *holds and* $f(a'_1) = a'_2$,

(D5.3) (D5) *holds and* $a'_1 \notin \operatorname{dom} f$,

(D5.4) (E5) *holds and* $f(a'_k) = a'_k$,

(D5.5) (E5) *holds and* $f(a'_k) = a'_{k-1}$,

(D5.6) (E5) *holds and* $a'_k \notin \operatorname{dom} f$.

P r o o f. The assertion can be obtained with respect to [3], 5.3.3 analogously as above. □

Lemma 4.6. *Let* (C6.j) *be valid for some* j$\in \{1, 2, 3\}$. *Then h is a convex isomorphism of (M, f) onto (M', f) if and only if some of the conditions* (D4.1)–(D4.4) *is satisfied.*

P r o o f. We get the assertion analogously as above, applying [3], 5.3.2. □

Corollary 4.6.1. *Let* $X = \{(\mathrm{C4}), (\mathrm{C6.1}), (\mathrm{C6.2}), (\mathrm{C6.3})\}$. *If* (M, f) *satisfies a condition* $X_1 \in X$, *then there is* $X_2 \in X$ *such that* (M', f) *satisfies* X_2.

Lemma 4.7. *Let* (C7.j) *hold for some* j $\in \{1,2,3\}$. *Consider the condition*
(D7) $f(a') = (f(a))'$ *for each* $a \in M - \{c\}$ *and* $\mathrm{card}\,(f^{-1}(c') - \{c'\}) = 1$.
Then h *is a convex isomorphism of* (M, f) *onto* (M', f) *if and only if some of the following conditions is satisfied:*
(D7.1) (D7) *holds and* $f(c') = c'$,
(D7.2) (D7) *holds and* $\{f(c')\} = f^{-1}(c')$,
(D7.3) (D7) *holds and* $c' \notin \mathrm{dom}\, f$.

Proof. By means of [3], 5.4.2. □

Corollary 4.7.1. *Let* m$\in \{5,7\}$, j $\in \{1,2,3\}$. *If* (M, f) *satisfies* (Cm.j), *then* (M', f) *satisfies* (Cm.i) *for some* i $\in \{1,2,3\}$.

Lemma 4.8. *Let* (C8) *hold. Then* h *is a convex isomorphism of* (M, f) *onto* (M', f) *if and only if*
(D8) C' *is a cycle of* (M', f) *and* $f(a') = (f(a))'$ *for each* $a \in M - C$.

Proof. The required assertion can be shown by applying [3], 5.6.2 analogously as above. □

Corollary 4.8.1. *If* (M, f) *satisfies* (C8), *then* (M', f) *satisfies* (C8) *as well; moreover, the cycles of* (M, f) *and of* (M', f) *have the same cardinality.*

Lemma 4.9. *Let* (C9.1) *or* (C9.2) *hold. Consider the condition*
(D9) $f(a') = (f(a))'$ *for each* $a \in M - \{c\}$ *and* $\mathrm{card}\,(f^{-1}(c') - \{c'\}) > 1$.
Then h *is a convex isomorphism of* (M, f) *onto* (M', f) *if and only if some of the following conditions is valid:*
(D9.1) (D9) *holds and* $f(c') = c'$,
(D9.2) (D9) *holds and* $c' \notin \mathrm{dom}\, f$.

Proof. Analogously as above, in view of [3], 5.5.2. □

Corollary 4.9.1. *If* (M, f) *satisfies* (C9.1) *or* (C9.2), *then* (M', f) *satisfies* (C9.1) *or* (C9.2).

The above investigation can be summarized as follows. Let (A, f) be a partial monounary algebra and let $h\colon A \to A$ be a bijection. According to 3.2, the mapping h is a convex automorphism of (A, f) if and only if h can be "divided" into parts $h \restriction K$, where either

(a) $K = S(A, f)$

or

(b) K is a connected component of $\big(A - S(A, f), f\big)$.

If $K = S(A, f)$, then $h \restriction K$ is a permutation of K in view of 3.2. Thus if we want to describe all convex automorphisms of (A, f), then (by 3.2) it suffices to investigate, when the mapping h is a convex isomorphism of (M, f) onto (M', f), where M satisfies one of the conditions (C1)–(C4), (C5.1)–(C5.3), (C6.1)–(C6.3), (C7.1)–(C7.3), (C8), (C9.1)–(C9.2). Hence from 3.2 and from 4.1–4.9 we obtain

Theorem 4.10. *Let (A, f) be a partial monounary algebra, let $h\colon A \to A$ be a bijection. Then h is a convex automorphism of (A, f) if and only if*

(i) *The conditions* (i) *and* (ii) *from* 3.2 *are satisfied;*

(ii) *if M is a connected component of $\big(A - S(A, f), f\big)$, then $h \restriction M$ satisfies the corresponding conditions from* 4.1–4.9.

5. Automorphisms of (A, f)

Let (A, f) be a partial monounary algebra.

In the present section we investigate the question under which conditions each convex automorphism of (A, f) is an automorphism with respect to f.

It turns out that this occurs iff certain situations in the structure of (A, f) are excluded. In order to characterize these "bad" situations we shall apply the conditions which were used in Section 4.

In the following Lemma 5.1–5.12 suppose that each convex automorphism of (A, f) is an automorphism of (A, f).

Lemma 5.1. *If there are distinct elements a, $b \in S(A, f)$ with $f(a) = b$, then $f(b) = a$.*

Proof. Let the assumption hold. Put $h(a) = b$, $h(b) = a$ and $h(x) = x$ for each $x \in A - \{a, b\}$. The mapping h is a convex automorphism of (A, f) in view of 3.2. Then h is an automorphism of (A, f) and we obtain

$$f(b) = f\big(h(a)\big) = h\big(f(a)\big) = h(b) = a.$$

□

Notation 5.2. Let M be a connected component of $\big(A - S(A,f), f\big)$ and assume that (M,f) satisfies either one of the conditions (C5.1)–(C5.3) or one of the conditions (C7.1)–(C7.3), (C8.1)–(C8.2). A partial monounary algebra $(M, \overline{f})$ is said to be a tail of (M,f), if

$$\overline{f}(x) = f(x) \quad \text{for each } x \in M - \{a_1\} \quad \text{and} \quad a_1 \notin \operatorname{dom} \overline{f},$$

or

$$\overline{f}(x) = f(x) \quad \text{for each } x \in M - \{c\} \quad \text{and} \quad c \notin \operatorname{dom} \overline{f},$$

respectively.

Lemma 5.3. *At lest one of the following conditions is satisfied:*
(i) $S(A,f) = \{c_1, c_2\}$ *and it is a cycle;*
(ii) $f(x) = x$ *for each* $x \in S(A,f)$*;*
(iii) $S(A,f) \subseteq A - \operatorname{dom} f$*;*
(iv) $S(A,f) \neq \emptyset$ *and there is* $c \in A - S(A,f)$ *with* $f(x) = c$ *for each* $x \in S(A,f)$*.*

Proof. Suppose that none of the conditions (i)–(iv) is satisfied. Then $\operatorname{card} S(A,f) > 1$. Next, assume that the following condition is valid:

(1) There are distinct elements c_1, $c_2 \in S(A,f)$ such that $\{c_1, c_2\}$ is a cycle.

Let $d \in S(A,f) - \{c_1, c_2\}$ (this set is nonempty) and put $h(c_1) = d$, $h(d) = c_1$, $h(x) = x$ otherwise. According to 3.2, h is a convex automorphism of (A,f). Thus h is an automorphism of (A,f) and

(2) $c_2 = h(c_2) = h\big(f(c_1)\big) = f\big(h(c_1)\big) = f(d)$.

Then the elements d, c_1, c_2 belong to the same connected component of (A,f), $\{d, c_1, c_2\} \subseteq S(A,f)$. By (1), this connected component has more than 2 elements and contains a 2-element cycle, which contradicts the definition of $S(A,f)$.

Now let (1) fail to hold. Suppose that

(3) there is $b \in S(A,f) - \operatorname{dom} f$.

The assumption that (iii) does not holds yields that there is $a \in S(A,f) \cap \operatorname{dom} f$. Put $g(a) = b$, $g(b) = a$ and $g(x) = x$ otherwise. Obviously, g is not an automorphism of (A,f) and g is a convex automorphism of (A,f) with respect to 3.2, a contradiction.

Assume that (3) is not valid. Then 5.1 yields that if $w \in S(A,f) \cap \operatorname{dom} f$, then $f(w) \notin S(A,f)$. Thus (cf. (iv)) we get

(4) there are v, $w \in S(A,f)$ and $u \in A - S(A,f)$ with

$$f(v) = u, \quad f(w) \neq u.$$

Put $t(v) = x$, $t(w) = v$, $t(x) = x$ otherwise. By 3.2, t is a convex automorphism of (A, f), but t is not an automorphism of (A, f), since

$$t(f(v)) = t(u) = u \neq f(w) = f(t(w)).$$

We have shown if we suppose that none of the conditions (i)–(iv) is satisfied, then we get a contradiction. □

Lemma 5.4. *Let* (iv) *of* 5.3 *hold. Then either* $f(c) = c$ *or* $c \notin \operatorname{dom} f$.

P r o o f. The assertion follows from 5.3 (iv) and 1.7. □

Lemma 5.5. *Let* (iv) *of* 5.3 *hold. Suppose that* M, K *are distinct connected components of* $(A - S(A, f), f)$ *such that* $c \in M$ *and* (K, f) *satisfies either* (C5.j) *or* (C7.j) *for some* j$\in \{1, 2, 3\}$ *or* (C9.j) *for some* j $\in \{1, 2\}$. *Then the tails* $(M, \overline{f})$ *and* $(K, \overline{f})$ *are non-isomorphic.*

P r o o f. By way of contradiction, let h be an isomorphism of $(M, \overline{f})$ onto $(K, \overline{f})$. Put

$$g(y) = \begin{cases} h(y) & \text{if } y \in M, \\ h^{-1}(y) & \text{if } y \in K, \\ y & \text{if } y \in A - (M \cup K). \end{cases}$$

By 3.2, 4.5, 4.7 and 4.9, g is a convex automorphism of (A, f). Further, if $x \in S(A, f)$. then

$$g(f(x)) = g(c) \in K,$$
$$f(g(x)) = f(x) = c \notin K,$$

thus g is not an automorphism of (A, f), which is a contradiction. □

Lemma 5.6. *If* M *is a connected component of* $(A - S(A, f), f)$, *then* (M, f) *does not satisfy the condition* (C3).

P r o o f. By way of contradiction, suppose that (M, f) satisfies (C3). Then $M = \{a_i \colon i \in \mathbb{Z}\}$, $a_i \neq a_j$ for $i \neq j$, $f(a_i) = a_{i+1}$. Put $h(a_i) = a_{-i}$ for each $i \in \mathbb{Z}$, $h(x) = x$ otherwise. By 4.3, the mapping $h \restriction M$ is a convex automorphism of (M, f) onto (M, f), since

$$f(a_i') = f(h(a_i)) = f(a_{-i}) = a_{-i+1} = h(a_{i-1}) = a_{i-1}'.$$

Then 3.2 yields that h is a convex automorphism of (A, f). Further, we have

$$f(h(a_0)) = f(a_0) = a_1 \neq a_{-1} = h(a_1) = h(f(a_0)),$$

thus h is not an automorphism of (A, f). □

Lemma 5.7. *Let M and K be connected components of $(A - S(A,f), f)$. Next, let $X = \{$*(C4), (C6.1), (C6.2), (C6.3)*$\}$ and let X_1, X_2 be distinct elements of X such that (M,f) satisfies X_1. Then (K,f) does not satisfy X_2.*

Proof. Assume e.g. that K fulfils (C4) and M fulfils (C6.1). (The remaining cases can be investigated analogously.) Then

$K = \{a_i \colon i \in \mathbb{N}\}$, $a_i \neq a_j$ for $i \neq j$, $f(a_i) = a_{i+1}$ for each $i \in \mathbb{N}$,

$M = \{b_i \colon i \in \mathbb{N}\}$, $b_i \neq b_j$ for $i \neq j$, $f(b_i) = b_{i-1}$ for each $i \in \mathbb{N}$, $i > 1$, $f(b_1) = b_1$.

Put $h(a_i) = b_i$, $h(b_i) = a_i$ for each $i \in \mathbb{N}$, $h(x) = x$ otherwise. Then h is a convex automorphism of (A,f) by 3.2, 4.4 and 4.6, but it is not an automorphism of (A,f), a contradiction. □

Lemma 5.8. *Let M and K be connected components of $(A - S(A,f), f)$, $\operatorname{card} M = \operatorname{card} K$. If (M,f) satisfies* (C5.i) *and (K,f) satisfies* (C5.j) *for some* i, j $\in \{1,2,3\}$, *then* i = j.

Proof. Analogously as in 5.5. □

Remark 5.8.1. The following example shows that the assertion which we obtain from 5.8 if the conditions (C5.i), (C5.j) are replaced by (C7.i), (C7.j) is not valid. Put $A = K \cup M$,

$$M = \{x, y, u, c\}, \quad f(x) = f(y) = u, \quad f(u) = f(c) = c,$$
$$K = \{x', y', u', c'\}, \quad f(x') = f(y') = u', \quad f(u') = c' \notin \operatorname{dom} f.$$

Then M and K are connected components of $(A,f) = (A - S(A,f), f)$, $\operatorname{card} M = \operatorname{card} K$, (M,f) satisfies (C7.1) and (K,f) satisfies (C7.3). Consider the mapping $h\colon A \to A$ such that $h(x) = x'$, $h(x') = x$, $h(y) = y'$, $h(y') = y$, $h(u) = u'$, $h(u') = u$, $h(c) = c'$, $h(c') = c$. Then h is not an automorphism, but it is a convex automorphism by 4.9 and 4.7.

Lemma 5.9. *Let M and K be connected components of $(A - S(A,f), f)$. If (M,f) satisfies* (C7.i) *and (K,f) satisfies* (C7.j) *for some* i, j $\in \{1,2,3\}$, i $\neq$ j, *then they have non-isomorphic tails.*

Proof. Let the assumption hold and suppose that $(M,\overline{f}) \simeq (K,\overline{f})$, where $(M,\overline{f})$ and $(K,\overline{f})$ are the corresponding tails. Then there are c, $a \in M$, d, $b \in K$ such that

(1) $f^{-1}(c) - \{c\} = \{a\}$, $f^{-1}(d) - \{d\} = \{b\}$

and

(2.1) either $f(c) = c$ or $f(c) = a$ or $c \neq \operatorname{dom} f$, (2.2) either $f(d) = d$ or $f(d) = b$ or $d \notin \operatorname{dom} f$.

Let $g\colon M \to K$ be an isomorphism of $(M, \overline{f})$ onto $(K, \overline{f})$. Since i $\neq$ j, the mapping g is not an isomorphism of (M, f) onto (K, f). Further g is a convex isomorphism of (M, f) onto (K, f) in view of 4.7. If we put

$$h(x) \begin{cases} g(x) & \text{if } x \in M, \\ g^{-1}(x) & \text{if } x \in K, \\ x & \text{otherwise,} \end{cases}$$

then h is a convex automorphism of (A, f) according to 3.2 and it is not an automorphism of (A, f). □

Notation 5.9.1. Let M be a connected component of $\big(A - S(A, f), f\big)$ such that (M, f) satisfies the condition (C8), i.e., there is $C = \{c_1, \dots, c_n\} \subseteq M$ such that C is an n-element cycle of (M, f), $n > 1$. For $i = 1, \dots, n$ put

$$A_i = \big\{x \in M\colon (\exists k \in \mathbb{N})\big(f^k(x) = c_i, f^{k-1}(x) \notin C\big)\big\}$$

and consider the monounary algebra $(A_i \cup \{c_i\}, \overline{f})$, where

$$\overline{f}(x) = f(x) \quad \text{for each } x \in A_i,\ c_i \notin \operatorname{dom} \overline{f}.$$

Then the partial monounary algebra $(A_i \cup \{c_i\}, \overline{f})$ is said to be a tail of (M, f).

By the system of the tails of (M, f) we understand the set of all tails of (M, f).

Lemma 5.10. *If M is a connected component of $\big(A - S(A, f), f\big)$ with a cycle possessing more than two elements, then no two distinct tails of (M, f) are isomorphic.*

Proof. Let $n \in \mathbb{N}$, $n > 2$ and let $C = \{c_1, \dots, c_n\}$ be the cycle of (M, f). Assume that $i, j \in \{1, \dots, n\}$, $i \neq j$ are such that $(A_i \cup \{c_i\}, \overline{f}) \simeq (A_j \cup \{c_j\}, \overline{f})$. Let $h\colon A_i \cup \{c_i\} \to A_j \cup \{c_j\}$ be the corresponding isomorphism. Put

$$g(x) \begin{cases} h(x) & \text{if } x \in A_i \cup \{c_i\}, \\ h^{-1}(x) & \text{if } x \in A_j \cup \{c_j\}, \\ x & \text{otherwise.} \end{cases}$$

By 4.8, the mapping $g \restriction M$ is a convex automorphism of (M, f) and then 3.2 implies that g is a convex automorphism of (A, f), which is a contradiction, since g is not an automorphism of (A, f). □

Lemma 5.11. *Let M and K be non-isomorphic connected components of $\big(A - S(A,f), f\big)$ with cycles C, D, $\operatorname{card} C = \operatorname{card} D = n > 2$. Let $\{T_1, \dots, T_n\}$ and $\{Q_1, \dots, Q_n\}$ be the system of the tails of (M,f) and of (K,f), respectively. If φ is a permutation of the set $\{1, 2, \dots, n\}$, then there exists $i \in \{1, 2, \dots, n\}$ such that T_i is not isomorphic to $Q_{\varphi(i)}$.*

Proof. Let the assumption be satisfied and let φ be a permutation of $\{1, \dots, n\}$ such that $T_i \simeq Q_{\varphi(i)}$ for each $i \in \mathbb{N}$; the corresponding isomorphism will be denoted by g_i. Put

$$g(x) = g_i(x) \quad \text{if } x \in T_i.$$

Then g is a convex isomorphism of (M,f) onto (K,f), but it is not an isomorphism of (M,f) onto (K,f) (since (M,f) and (K,f) are non-isomorphic). Put

$$h(x) \begin{cases} g(x) & \text{if } x \in M, \\ g^{-1}(x) & \text{if } x \in K, \\ x & \text{otherwise.} \end{cases}$$

Then h is a convex automorphism of (A,f) by 3.2 and 4.8, and it is not an automorphism of (A,f), which is a contradiction. □

Lemma 5.12. *Let M and K be connected components of $\big(A - S(A,f), f\big)$. If (M,f) satisfies* (C9.1) *and (K,f) satisfies* (C9.2), *then they have non-isomorphic tails.*

Proof. Analogously as 5.9. □

Theorem 5.13. *Let $(A,f) \in \mathcal{U}$. Then the following conditions are equivalent:*
(a) *Each convex automorphism of (A,f) is an automorphism of (A,f).*
(b) *The assertions of* 5.3–5.12 *hold.*

Proof. Let (a) be valid. Then according to 5.3–5.12, (b) is true. Conversely, let (b) hold. Assume that h is a convex automorphism of (A,f). Then (i)–(iii) of 3.2 are valid.

Let $x \in S(A,f)$. By 3.2 (i), $h(x) \in S(A,f)$. If $x \notin \operatorname{dom} f$, then 5.3 yields that $h(x) \notin \operatorname{dom} f$. If $x \in \operatorname{dom} f$, then 5.3 implies that some of the following conditions is satisfied:

(1.1) $S(A,f) = \{x, h(x)\}$ is a 2-element cycle of (A,f).
(1.2) $f(x) = x$, $f\big(h(x)\big) = h(x)$,
(1.3) $f(x) = f\big(h(x)\big) \notin S(A,f)$.

If (1.1) holds, then $h\big(h(x)\big) = x$ by 3.2 (i), thus

(2.1) $f\big(h(x)\big) = x = h\big(h(x)\big) = h\big(f(x)\big)$.

If (1.2) is valid, then

(2.2) $f\big(h(x)\big) = h(x) = h\big(f(x)\big)$.

Let (1.3) hold. Put $c = f(x)$ and suppose that M is the connected component of $\big(A - S(A, f), f\big)$ containing c. Assume that $u = h(c)$. By 5.4, either $f(c) = c$ or $c \notin \operatorname{dom} f$, hence (M, f) satisfies (C5.1), (C5.3), (C9.1) or (C9.2). Then 4.7.1 and 4.9.1 imply that $\big(h(M), f\big)$ satisfies (C5.i) or (C7.i) for some i $\in \{1, 2, 3\}$ or (C9.i) for some i $\in \{1, 2\}$. If $h(M) = M$, then $h(c) = c$, thus

(2.3) $f\big(h(c)\big) = f(x) = c = h(c) = h\big(f(x)\big)$.

Assume that $K = h(M) \neq M$. By 4.5, 4.7 or 4.9, if $y \in K - \{c\}$, then $h(y) \in \operatorname{dom} f$, $f\big(h(y)\big) \neq h(y)$ and $h\big(f(y)\big) = f\big(h(y)\big)$, thus $h\big(\overline{f}(y)\big) = \overline{f}\big(h(y)\big)$. The mapping $h \restriction M$ is an isomorphism of $(M, \overline{f})$ onto $(K, \overline{f})$, which contradicts 5.5. Therefore we have shown

(3) $x \in S(A, f) \cap \operatorname{dom} f \Leftrightarrow h(x) \in S(A, f) \cap \operatorname{dom} f$,

$x \in S(A, f) \cap \operatorname{dom} f \Rightarrow h\big(f(x)\big) = f\big(h(x)\big)$.

Now it suffices to verify that for each connected component M of $\big(A - S(A, f), f\big)$ the mapping h is an isomorphism of (M, f) onto $\big(h(M), f\big)$.

Suppose that M is a connected component of $\big(A - S(A, f), f\big)$. If (M, f) satisfies (C1) or (C2), then 4.1 or 4.2 implies that h is an isomorphism of (M, f) onto $\big(h(M), f\big)$.

By 5.6, (M, f) does not satisfy (C3).

Let (C4) be valid for (M, f). Then $\big(h(M), f\big)$ fulfils some of the conditions (C4), (C6.1), (C6.2), (C6.3) according to 4.6.1, hence 5.7 yields that $\big(h(M), f\big)$ fulfils (C4). Then h is an isomorphism of (M, f) onto $\big(h(M), f\big)$ by 4.4.

Let (M, f) satisfy (C5.i) (i $\in \{1, 2, 3\}$). By 4.7.1, $\big(h(M), f\big)$ satisfies (C5.j) for some j $\in \{1, 2, 3\}$, and by 5.8, j = i. Then 4.5 implies that h is an isomorphism of (M, f) onto $\big(h(M), f\big)$.

If (M, f) fulfils (C6.i) (i $\in \{1, 2, 3\}$), then $\big(h(M), f\big)$ fulfils some conditions of the set $\{$(C4), (C6.1), (C6.2), (C6.3)$\}$ with respect to 4.6.1. By 5.7, $\big(h(M), f\big)$ satisfies the condition (C6.i) and then 4.6 implies that h is an isomorphism of (M, f) onto $\big(h(M), f\big)$.

Assume that (M, f) satisfies (C7.i) (i $\in \{1, 2, 3\}$). By 4.7.1, $\big(h(M), f\big)$ satisfies (C7.j) for some j $\in \{1, 2, 3\}$. If i = j, then h is an isomorphism of (M, f) onto $\big(h(M), f\big)$ by virtue of 4.7. Let i $\neq$ j. Then $h \restriction M$ is an isomorphism of $(M, \overline{f})$ onto $\big(h(M), \overline{f}\big)$, which is a contradiction to 5.9.

Let (M, f) satisfy (C8). By 4.8.1, $\big(h(M), f\big)$ satisfies (C8), too, and $\operatorname{card} C = \operatorname{card} D$, where C, D are the cycles of (M, f) and $\big(h(M), f\big)$. Put $n = \operatorname{card} C$ and let $\{T_1, \dots, T_n\}$ and $\{Q_1, \dots, Q_n\}$ be the systems of the tails of (M, f) and of $\big(h(M), f\big)$, respectively. The mapping $h \restriction M$ is a convex isomorphism, thus 4.8 implies that there is a permutation φ on $\{1,\dots,n\}$ such that

(4) $T_i \simeq Q_{\varphi(i)}$ for each $i \in \{1,\dots,n\}$.

From 5.11 it follows that there exists an isomorphism g of (M,f) onto $(h(M),f)$. Let

(5) $C = \{c_1,\dots,c_n\} \subseteq M$, where $f(c_1) = c_2$, $f(c_2) = c_3$, ..., $f(c_n) = c_1$,

(6) $D = \{g(c_1),\dots,g(c_n)\}$.

By 4.8, $D = h(C)$. Without loss of generality we can suppose that

(7) $\{c_i\} = T_i \cap C$ for each $i \in \{1,\dots,n\}$.

Then

(8) $\{h(c_i)\} = h(T_i) \cap D$ for each $i \in \{1,\dots,n\}$.

By 5.10, $Q_k \simeq Q_\ell$ implies $k = \ell$. From 4.8 and from the definition of the system of tails in 5.10.1 it follows that $h(T_i) \simeq T_i$ for each $i \in \{1,\dots,n\}$. Then (4) yields $h(T_i) = Q_{\varphi(i)}$. Thus

(9) $\{h(c_i)\} = Q_{\varphi(i)} \cap D$ for each $i \in \{1,\dots,n\}$.

Further, $g(T_i) \simeq T_i$ and hence we get analogously

(10) $\{g(c_i)\} = Q_{\varphi(i)} \cap D$ for each $i \in \{1,\dots,n\}$,

i.e. $g(c_i) = h(c_i)$ for each $i \in \{1,\dots,n\}$. Let $i \in \{1,\dots,n-1\}$. The mapping g is an isomorphism, hence

$$h(f(c_i)) = h(c_{i+1}) = g(c_{i+1}) = g(f(c_i)) = f(g(c_i)) = f(h(c_i)),$$
$$h(f(c_n)) = h(c_1) = g(c_1) = g(f(c_n)) = f(g(c_n)) = f(h(c_n)).$$

Moreover, in view of 4.8 we obtain that $h(f(a)) = f(h(a))$ for each $a \in A - C$, thus the mapping $h \restriction M$ is an isomorphism of (M,f) onto $(h(M),f)$.

Let (M,f) satisfy (C9.1) or (C9.2). By 4.9.1, $(h(M),f)$ fulfils either (C9.1) or (C9.2). Further, 4.9 implies that $h \restriction M$ is an isomorphism of $(M,\overline{f})$ onto $(h(M),\overline{f})$, which contradicts 5.12 and completes the proof. □

References

[1] *P. Emanovský*: Convex isomorphic ordered sets. Math. Bohemica *118* (1993), 29–35.

[2] *P. Emanovský*: Conve: isomorphism of q-lattices. Math. Bohemica *118* (1993), 37–42.

[3] *D. Jakubíková-Studenovská*: Convex subsets of partial monounary algebras. Czech. Math. J. *38 (113)* (1988), 655–672.

[4] *M. Kolibiar, J. Lihová*: Convex automorphisms of a lattice. Math. Slovaca. To appear.

[5] *V. I. Marmazeev*: The lattice of convex sublattices of a lattice. Ordered sets and lattices, No 9. Saratov Gos. Univ., Saratov, 1986, pp. 50–58. (In Russian.)

[6] *V. I. Marmazeev*: A group of automorphisms of the lattice of convex sublattices of a lattice. Vestsi Akad. Nauk BSSR, Ser. fiz. mat. navuk, 1988, pp. 110–112. (In Russian, English summary.)

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