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REDUCING REAL ALMOST-LINEAR SECOND-ORDER PARTIAL DIFFERENTIAL OPERATORS IN TWO INDEPENDENT VARIABLES TO A CANONICAL FORM

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1. INTRODUCTION

This paper deals with the classical method of reducing real almost-linear second-order partial differential operators in two independent variables to a canonical form. A standard presentation of the method involves a good deal of calculations which usually are obscure. In the present article, we intend to illuminate the geometrical character of those calculations. We first show that reducing a real almost-linear second-order partial differential operator to a canonical form amounts to reducing a suitable symmetric 2-contravariant tensor field to a canonical form. Next we show that any symmetric 2-contravariant tensor field of locally constant type can locally be reduced to a canonical form. More specifically, given an almost-linear second-order partial differential operator P on an open region Ω of $\mathbb{R}^2$

$$Pu = a_{11}\frac{\partial^2 u}{\partial x_1^2} + 2a_{12}\frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{22}\frac{\partial^2 u}{\partial x_2^2} + f\Big(x, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\Big)$$
$$(u \in C^\infty_{\mathbb{R}}(\Omega),\ x \in \Omega),$$

where $a_{11}, a_{12}, a_{22} \in C^\infty_{\mathbb{R}}(\Omega)$ and $f \in C^\infty_{\mathbb{R}}(\Omega \times \mathbb{R}^3)$, we associate with P a symmetric 2-contravariant tensor field

$$\sigma_{\overset{\circ}{P}} = a_{11}\frac{\partial}{\partial x_1} \otimes_s \frac{\partial}{\partial x_1} + 2a_{12}\frac{\partial}{\partial x_1} \otimes_s \frac{\partial}{\partial x_2} + a_{22}\frac{\partial}{\partial x_2} \otimes_s \frac{\partial}{\partial x_2}.$$

We show that a canonical form of P can be found by reducing $\sigma_{\overset{\circ}{P}}$ to a canonical form, and that the latter reduction can always be done locally in Ω provided the type of $\sigma_{\overset{\circ}{P}}$ is locally constant.

Let V be an n-dimensional real vector space, q be a quadratic form on V, and b be the associated symmetric bilinear form. q is said to take a canonical form in a basis $\{e_i\colon i = 1, \dots, n\}$ of V if, letting $b(e_i, e_j) = a_{ij}$ $(1 \leqslant i, j \leqslant n)$, we have $a_{ij} = 0$ for $i \neq j$ and all the a_{ii} that are different from zero are equal in modulus. Sylvester's theorem guarantees that any quadratic form takes a canonical form in some basis. The numbers r_0, r_+, r_- of those i's for which $a_{ii} = 0$, $a_{ii} > 0$, and $a_{ii} < 0$, respectively, are determined uniquely. $r = r_+ + r_-$ is the rank of q. The form q is called:

(i) elliptic if $r = n$, and either $r_+ = n$ or $r_- = n$,
(ii) parabolic if $r < n$,
(iii) hyperbolic if $r = n$, and either $r_+ = 1$ or $r_- = 1$.

In the case $n = 2$, if, given a basis $\{e_1,\ e_2\}$, we let $\Delta = a_{12}^2 - a_{11}a_{22}$, then q is:

(i) elliptic if and only if $\Delta < 0$,
(ii) parabolic if and only if $\Delta = 0$,
(iii) hyperbolic if and only if $\Delta > 0$.

We will adhere to the convention according to which, in the case $n = 2$, q takes in a given basis $\{e_1,\ e_2\}$:

(i) a canonical elliptic form if $a_{11} = a_{22} \neq 0$ and $a_{12} = 0$,
(ii) a canonical parabolic form if $a_{12} = a_{22} = 0$,
(iii) a canonical hyperbolic form if $a_{11} = a_{22} = 0$ and $a_{12} \neq 0$.

Let M be a C^∞ manifold of dimension n. We denote by $C^\infty_{\mathbb{R}}(M)$ ($C^\infty_{\mathbb{C}}(M)$) the space of all real-valued (complex-valued) C^∞ functions on M. If M is real-analytic, then $C^\omega_{\mathbb{R}}(M)$ ($C^\omega_{\mathbb{C}}(M)$) will denote the space of all real-valued (complex-valued) real-analytic functions on M. If M is a complex manifold (of complex dimension n), then $A(M)$ will denote the space of all holomorphic functions on M. For $a \in M$, we denote by $T_a(M)$ the tangent space of M at a, and by $T_a^*(M)$ we denote the cotangent space of M at a. $T_a(M) \otimes_{\mathbb{R}} \mathbb{C}$ and $T_a^*(M) \otimes_{\mathbb{R}} \mathbb{C}$ will stand for the complexification of $T_a(M)$ and $T_a^*(M)$, respectively. $\Gamma^\infty(T(M))$ denotes the space of all C^∞ vector fields on M and $\Gamma^\infty(T^*(M))$ denotes the space of all vector C^∞ 1-forms on M. If M is real-analytic, then $\Gamma^\omega(T(M) \otimes_{\mathbb{R}} \mathbb{C})$ will denote the space of all complex-valued real-analytic vector fields on M and $\Gamma^\omega(T^*(M) \otimes_{\mathbb{R}} \mathbb{C})$ will denote the space of all complex-valued real-analytic 1-forms on M, and if M is complex, then $A(T(M) \otimes_{\mathbb{R}} \mathbb{C})$ will denote the space of all holomorphic vector fields on M and $A(T^*(M) \otimes_{\mathbb{R}} \mathbb{C})$ will denote the space of all holomorphic 1-forms on M. For a vector space V and a positive integer p, we denote by $\bigotimes_s^p V$ the corresponding space of symmetric p-contravariant tensors, and by $\bigotimes_s^p V^*$ we denote the corresponding space of symmetric p-covariant tensors. $\Gamma^\infty(\bigotimes_s^p T(M))$ will stand for the space of C^∞ symmetric p-contravariant tensor fields on M, and $\Gamma^\infty(\bigotimes_s^p T^*(M))$ will stand for the space of C^∞ symmetric p-covariant tensor fields on M.

If V is an n-dimensional real vector space, then a tensor $\delta \in \bigotimes_s^2 V$ is said to be elliptic (parabolic, hyperbolic) if δ treated as a quadratic form on the dual space V^* of V is elliptic (parabolic, hyperbolic). If M is a C^∞ manifold of dimension n, then a tensor field $\sigma \in \Gamma^\infty(\bigotimes_s^2 T(M))$ is called elliptic (parabolic, hyperbolic) if $\sigma(a)$ is elliptic (parabolic, hyperbolic) for each $a \in M$. Let (U, φ) be a coordinate system in M with $\varphi = (x_1, \ldots, x_n)$. A tensor field $\sigma \in \Gamma^\infty(\bigotimes_s^2 T(M))$ is said to take a canonical form in (U, φ) if, for each $a \in M$, $\sigma(a)$ treated as a quadratic form on $T_a^*(M)$ takes a canonical form in the basis $\{(\mathrm{d}x_i)_a\colon 1 \leqslant i \leqslant n\}$. According to the convention adopted, in the case $n = 2$ a tensor field $\sigma \in \Gamma^\infty(\bigotimes_s^2 T(M))$ takes in (U, φ):

(i) a canonical elliptic form if, for some $f \in C^\infty_{\mathbb{R}}(U)$ with $f \neq 0$ on U,

$$\sigma = f\left(\frac{\partial}{\partial x_1} \otimes_s \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \otimes_s \frac{\partial}{\partial x_2}\right),$$

(ii) a canonical parabolic form if, for some $f \in C^\infty_{\mathbb{R}}(U)$,

$$\sigma = f\frac{\partial}{\partial x_1} \otimes_s \frac{\partial}{\partial x_1},$$

(iii) a canonical hyperbolic form if, for some $f \in C^\infty_{\mathbb{R}}(U)$ with $f \neq 0$ on U,

$$\sigma = f\frac{\partial}{\partial x_1} \otimes_s \frac{\partial}{\partial x_2}.$$

3. Direct images of operators

Let Ω be an open region in $\mathbb{R}^n$, and P be a real almost-linear second-order partial differential operator on Ω of the form

$$Pu = \sum_{1 \leqslant i,j \leqslant n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + f\left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) \qquad (u \in C^\infty_{\mathbb{R}}(\Omega),\ x \in \Omega),$$

where $a_{ij} \in C^\infty_{\mathbb{R}}(\Omega)$ satisfy $a_{ij} = a_{ji}$ $(1 \leqslant i, j \leqslant n)$ and $f \in C^\infty_{\mathbb{R}}(\Omega \times \mathbb{R}^{n+1})$. The *principal part* $\overset{\circ}{P}$ of P is the operator on Ω defined by

$$\overset{\circ}{P}u = \sum_{1 \leqslant i,j \leqslant n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} \qquad (u \in C^\infty_{\mathbb{R}}(\Omega)).$$

The *principal symbol* $\sigma_{\overset{\circ}{P}}$ of P at $x \in \Omega$ is the element of $\bigotimes_s^2 T_x(\Omega)$ defined as

$$\sigma_{\overset{\circ}{P}}(x) = \sum_{1 \leqslant i,j \leqslant n} a_{ij}(x) \left(\frac{\partial}{\partial x_i} \right)_x \otimes_s \left(\frac{\partial}{\partial x_j} \right)_x .$$

P is said to be elliptic (parabolic, hyperbolic) at $x \in \Omega$ if $\sigma_{\overset{\circ}{P}}(x)$ is elliptic (parabolic, hyperbolic). P is called elliptic (parabolic, hyperbolic) on Ω if $\sigma_{\overset{\circ}{P}}(x)$ is elliptic (parabolic, hyperbolic) at each point of Ω. P is said to take a canonical elliptic (parabolic, hyperbolic) form on Ω if $\sigma_{\overset{\circ}{P}}$ takes a canonical elliptic (parabolic, hyperbolic) form in the canonical coordinate system on Ω.

Let φ be a C^∞ diffeomorphism from Ω into $\mathbb{R}^n$. For $x \in \Omega$, we denote by $\varphi_{*,x}$ the differential of φ at x and also, for any positive p, the pth tensor power of this differential. We denote by $\varphi_*(P)$ the direct image of P by φ, which is the operator on $\varphi(\Omega)$ acting by the rule

$$(\varphi_*(P))u = (P(u \circ \varphi)) \circ \varphi^{-1} \qquad (u \in C_{\mathbb{R}}^\infty(\varphi(\Omega))).$$

We have the fundamental theorem as follows (cf. [4, Section II.11.3]):

Theorem 1. *If P is a real almost-linear second-order partial differential operator on an open region Ω of $\mathbb{R}^n$ and φ is a C^∞ diffeomorphism from Ω into $\mathbb{R}^n$, then, for each $x \in \Omega$,*

$$\sigma_{\overset{\circ}{\varphi_*(P)}}(\varphi(x)) = \varphi_{*,x}(\sigma_{\overset{\circ}{P}}(x)).$$

In view of Theorem 1, P is elliptic (parabolic, hyperbolic) at $a \in \Omega$ if and only if $\varphi_*(P)$ is elliptic (parabolic, hyperbolic) at $\varphi(a)$.

A diffeomorphism φ is said to reduce P to a canonical elliptic (parabolic, hyperbolic) form if $\varphi_*(P)$ takes a canonical elliptic (parabolic, hyperbolic) form on $\varphi(\Omega)$ with respect to the canonical coordinate system in $\varphi(\Omega)$. In view of Theorem 1, φ reduces P to a canonical elliptic (parabolic, hyperbolic) form if and only if $\sigma_{\overset{\circ}{P}}$ takes a canonical elliptic (parabolic, hyperbolic) form in the coordinate system (Ω, φ).

This section presents our main results on reduction to a canonical form. We begin by stating two auxiliary theorems. The first of them is a theorem on rectification of a vector field (cf. [1, Proposition 8.3.2]; see also [5, Theorem 2.11.8]), that can be proved by applying a theorem on solvability of the Cauchy problem for ordinary differential equations. The other is a holomorphic analogue of the first one, and can be established by utilising a suitable theorem on solvability of the Cauchy problem for ordinary differential equations in the complex domain (cf. [5, Theorem 1.8.10]).

Theorem 2. *Let M be a C^∞ manifold of dimension n and let $X \in \Gamma^\infty(T(M))$. Then for each $a \in M$ with $X(a) \neq 0$ there exists a coordinate system (U, φ) at a with $\varphi = (x_1, \ldots, x_n)$ such that $X = \frac{\partial}{\partial x_1}$ on U.*

Theorem 3. *Let M be a complex manifold of complex dimension n and let $X \in A(T(M) \otimes_{\mathbb{R}} \mathbb{C})$. Then for each $a \in M$ with $X(a) \neq 0$ there exists a coordinate system (U, φ) at a with $\varphi = (z_1, \ldots, z_n)$ such that $X = \frac{\partial}{\partial z_1}$ on U.*

We now use the above theorems to establish the following result:

Theorem 4. *If M is a two-dimensional C^∞ manifold and $\omega \in \Gamma^\infty(T^*(M))$, then for each $a \in M$ with $\omega(a) \neq 0$ there exists an open neighbourhood U of a and $f, g \in C^\infty_{\mathbb{R}}(U)$ such that $f \neq 0$, $\mathrm{d}g \neq 0$, and*

$$\omega = f\,\mathrm{d}g \tag{1}$$

on U. If M is a two-dimensional real analytic manifold and $\omega \in \Gamma^\omega(T^*(M) \otimes_{\mathbb{R}} \mathbb{C})$, then for each $a \in M$ such that $\omega(a) \neq 0$ there exists an open neighbourhood U of a and $f, g \in C^\omega_{\mathbb{C}}(U)$ such that $f \neq 0$, $\mathrm{d}g \neq 0$, and (1) holds on U.

Proof. Let M be a two-dimensional C^∞ manifold, let $\omega \in \Gamma^\infty(T^*(M))$, and let $a \in M$ be such that $\omega(a) \neq 0$. Then there exists an open neighbourhood $U \subset M$ of a and $X \in \Gamma^\infty(T(U))$ such that $\omega(m) \neq 0$ and $X(m) \neq 0$ for each $m \in U$, and $\omega(X) = 0$ on U. In view of Theorem 2, by shrinking U if necessary, one can find a one-to-one C^∞ mapping φ from U into $\mathbb{R}^2$ with $\varphi = (x_1, x_2)$ such that $X = \frac{\partial}{\partial x_1}$ on U. Now ω can be represented in U as

$$\omega = a_1\,\mathrm{d}x_1 + a_2\,\mathrm{d}x_2$$

for some $a_1, a_2 \in C^\infty_{\mathbb{R}}(U)$. Since $\omega(\frac{\partial}{\partial x_1}) = 0$, it follows that

$$\omega = a_2\,\mathrm{d}x_2.$$

Taking a_2 and x_2 for f and g, respectively, we obtain (1).

Now let M be a two-dimensional real analytic manifold, let $\omega \in \Gamma^\omega(T^*(M) \otimes_{\mathbb{R}} \mathbb{C})$, and let $a \in M$ be such that $\omega(a) \neq 0$. Then there exists a coordinate system (V, ψ) with $a \in V$ and $X \in \Gamma^\omega(T(V) \otimes_{\mathbb{R}} \mathbb{C})$ such that $\omega(m) \neq 0$ and $X(m) \neq 0$ for each $m \in V$, and $\omega(X) = 0$ on V. Choose an open neighbourhood $\tilde{V} \subset \mathbb{C}^2$ of $\psi(a)$ such that: 1° $\tilde{V} \cap \mathbb{R}^2 = \psi(V)$; 2° the push-forward $\psi_*(X)$ of X by ψ (i.e. the unique vector field on $\tilde{V}$ that is ψ-related to X) has an extension to a holomorphic vector field $\tilde{X}$ on $\tilde{V}$; 3° the pull-back $(\psi^{-1})^*(\omega)$ of ω by ψ^{-1} has an extension to a holomorphic 1-form $\tilde{\omega}$ on $\tilde{V}$. By Theorem 3, there exists a holomorphic one-to-one map ζ from an open neighbourhood $\tilde{V}' \subset \tilde{V}$ of $\psi(a)$ into $\mathbb{C}^2$ with $\zeta = (z_1, z_2)$ such that $\tilde{X} = \frac{\partial}{\partial z_1}$ in $\tilde{V}'$. Reasoning as before, we see that $\tilde{\omega}$ can be represented as $\tilde{\omega} = a_2 dz_2$ for some $a_2 \in A(\tilde{V}')$. Now letting $U = \psi^{-1}(\tilde{V}' \cap \mathbb{R}^2)$, $f = a_2 \circ \psi$, and $g = z_2 \circ \psi$, we obtain (1). □

Now we are ready to state our main result.

Theorem 5. *Let M be a two-dimensional C^∞ manifold, $\sigma \in \Gamma^\infty(\bigotimes_s^2 T(M))$, and $a \in M$ be such that $\sigma(a)$ is either elliptic or hyperbolic, or there exists an open neighbourhood of a on which σ is parabolic. In the elliptic case, M and σ are additionally assumed to be real-analytic. Then there exists a coordinate system (U, φ) at a in which σ takes a canonical form.*

Proof. Let (U, ψ) be a coordinate system at a with $\psi = (x_1, x_2)$ such that if

$$\sigma = a_{11}\frac{\partial}{\partial x_1} \otimes_s \frac{\partial}{\partial x_1} + 2a_{12}\frac{\partial}{\partial x_1} \otimes_s \frac{\partial}{\partial x_2} + a_{22}\frac{\partial}{\partial x_2} \otimes_s \frac{\partial}{\partial x_2} \tag{2}$$

for some $a_{11}, a_{12}, a_{22} \in C^\infty_{\mathbb{R}}(M)$, then Δ (defined, let us recall, as $a_{12}^2 - a_{11}a_{22}$) is either negative, or null, or positive on U. Let $\omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2$. Being non-degenerate (and in fact symplectic), the 2-form ω induces an isomorphism I_ω between the spaces $\Gamma^\infty(T(U))$ and $\Gamma^\infty(T^*(U))$ defined by

$$(I_\omega(X))(Y) = \omega(X, Y) \qquad (X, Y \in \Gamma^\infty(T(U))).$$

If $(U, \tilde{\psi})$ is another coordinate system on U with $\tilde{\psi} = (y_1, y_2)$ in which ω takes the form

$$\omega = f\,\mathrm{d}y_1 \wedge \mathrm{d}y_2 \tag{3}$$

for some $f \in C^\infty_{\mathbb{R}}(U)$ with $f \neq 0$ on U, then, as one can easily verify,

$$\begin{aligned} I_\omega\Big(\frac{\partial}{\partial y_1}\Big) &= f\,\mathrm{d}y_2, \\ I_\omega\Big(\frac{\partial}{\partial y_2}\Big) &= -f\,\mathrm{d}y_1. \end{aligned} \tag{4}$$

Let $I_\omega \otimes I_\omega$ be the tensor square of I_ω mapping isomorphically $\Gamma^\infty(\bigotimes_s^2 T(U))$ onto $\Gamma^\infty(\bigotimes_s^2 T^*(U))$. The last identities imply that

$$I_\omega \otimes I_\omega\Big(\frac{\partial}{\partial y_1} \otimes_s \frac{\partial}{\partial y_1}\Big) = f^2\, \mathrm{d}y_2 \otimes_s \mathrm{d}y_2, \tag{5}$$

$$I_\omega \otimes I_\omega\Big(\frac{\partial}{\partial y_1} \otimes_s \frac{\partial}{\partial y_2}\Big) = -f^2\, \mathrm{d}y_1 \otimes_s \mathrm{d}y_2, \tag{6}$$

$$I_\omega \otimes I_\omega\Big(\frac{\partial}{\partial y_2} \otimes_s \frac{\partial}{\partial y_2}\Big) = f^2\, \mathrm{d}y_1 \otimes_s \mathrm{d}y_1. \tag{7}$$

In particular, (2) together with (5), (6), and (7) yields

$$I_\omega \otimes I_\omega(\sigma) = a_{22}\, \mathrm{d}x_1 \otimes_s \mathrm{d}x_1 - 2a_{12}\, \mathrm{d}x_1 \otimes_s \mathrm{d}x_2 + a_{11}\, \mathrm{d}x_2 \otimes_s \mathrm{d}x_2. \tag{8}$$

We now consider the following three cases.

A. *Hyperbolic type*: $\Delta > 0$ on U. By shrinking U if necessary, we may assume that at least one of the functions a_{11} and a_{22} does not vanish in U (for otherwise σ already takes a hyperbolic canonical form on U). Suppose that $a_{11} \neq 0$ on U. Using (8), it is readily verified that

$$a_{11} I_\omega \otimes I_\omega(\sigma) = \omega_1 \otimes_s \omega_2, \tag{9}$$

where

$$\begin{aligned} \omega_1 &= (a_{12} + \sqrt{\Delta})\, \mathrm{d}x_1 - a_{11}\, \mathrm{d}x_2, \\ \omega_2 &= (a_{12} - \sqrt{\Delta})\, \mathrm{d}x_1 - a_{11}\, \mathrm{d}x_2. \end{aligned} \tag{10}$$

By Theorem 4, upon shrinking U if necessary, one can choose $\kappa, \lambda, \mu, \nu \in C^\infty_{\mathbb{R}}(U)$ so that $\kappa \neq 0$, $\lambda \neq 0$, $\mathrm{d}\mu \neq 0$, $\mathrm{d}\nu \neq 0$, and

$$\begin{aligned} \omega_1 &= \kappa\, \mathrm{d}\mu, \\ \omega_2 &= \lambda\, \mathrm{d}\nu \end{aligned} \tag{11}$$

on U. Let $\varphi: U \to \mathbb{R}^2$ be the map given by $\varphi = (\mu, \nu)$. Since, by (10) and (11),

$$\kappa\lambda\, \mathrm{d}\mu \wedge \mathrm{d}\nu = \omega_1 \wedge \omega_2 = -2a_{11}\sqrt{\Delta}\omega, \tag{12}$$

it follows from (6) that

$$I_\omega \otimes I_\omega\Big(\frac{\partial}{\partial \mu} \otimes_s \frac{\partial}{\partial \nu}\Big) = -\frac{\kappa^2\lambda^2}{4a_{11}^2\Delta}\, \mathrm{d}\mu \otimes_s \mathrm{d}\nu. \tag{13}$$

Moreover, (12) in conjunction with the inverse function theorem implies that φ is a diffeomorphism if U is sufficiently small. Now comparison of (9), (11), and (13) shows that

$$\sigma = -\frac{4a_{11}\Delta}{\kappa\lambda}\frac{\partial}{\partial\mu}\otimes_s\frac{\partial}{\partial\nu}.$$

We see that σ takes a canonical hyperbolic form in the coordinate system (U,φ).

B. *Parabolic type*: $\Delta = 0$ on U. As previously, we may assume that $a_{11} \neq 0$ on U. Using (8), it is easy to verify that

$$a_{11} I_\omega \otimes I_\omega(\sigma) = \theta \otimes_s \theta, \tag{14}$$

where

$$\theta = a_{12}\,\mathrm{d}x_1 - a_{11}\,\mathrm{d}x_2. \tag{15}$$

By Theorem 4, upon shrinking U if necessary, one can choose $\kappa, \mu \in C^\infty_{\mathbb{R}}(U)$ so that $\kappa \neq 0$, $\mathrm{d}\mu \neq 0$, and

$$\theta = \kappa\,\mathrm{d}\mu \tag{16}$$

on U. Let $\varphi\colon U \to \mathbb{R}^2$ be the map given by $\varphi = (\eta, \mu)$, where $\eta \in C^\infty_{\mathbb{R}}(U)$ is chosen so that $\mathrm{d}\eta \wedge \mathrm{d}\mu \neq 0$ in a neighbourhood of a. It follows from the inverse function theorem that U can be contracted so that φ is a local diffeomorphism on U. Writing ω in the form

$$\omega = f\,\mathrm{d}\eta \wedge \mathrm{d}\mu$$

with $f \in C^\infty_{\mathbb{R}}(U)$ nowhere vanishing and using (5), we see that

$$I_\omega \otimes I_\omega\Big(\frac{\partial}{\partial\eta}\otimes_s\frac{\partial}{\partial\eta}\Big) = f^2\,\mathrm{d}\mu \otimes_s \mathrm{d}\mu.$$

Comparing the last equality with (14) and (16), we obtain

$$\sigma = \frac{\kappa^2}{a_{11}f^2}\frac{\partial}{\partial\eta}\otimes_s\frac{\partial}{\partial\eta}.$$

We see that σ takes a canonical parabolic form in the coordinate system (U,φ).

C. *Elliptic type*: $\Delta < 0$ on U. As previously, we may assume that $a_{11} \neq 0$ on U. Using (8), it is easy to verify that

$$a_{11} I_\omega \otimes I_\omega(\sigma) = \theta \otimes_s \overline{\theta}, \tag{17}$$

where

$$\theta = (a_{12} + \sqrt{-\Delta}\mathrm{i})\,\mathrm{d}x_1 - a_{11}\,\mathrm{d}x_2, \qquad (18)$$
$$\overline{\theta} = (a_{12} - \sqrt{-\Delta}\mathrm{i})\,\mathrm{d}x_1 - a_{11}\,\mathrm{d}x_2$$

are elements of $\Gamma^{\omega}(T^*(U) \otimes_{\mathbb{R}} \mathbb{C})$. By Theorem 4, upon shrinking U if necessary, one can choose $\lambda, \mu \in C^{\omega}_{\mathbb{C}}(U)$ so that $\lambda \neq 0$, $\mathrm{d}\mu \neq 0$, and

$$\theta = \lambda\,\mathrm{d}\mu, \qquad (19)$$
$$\overline{\theta} = \overline{\lambda}\,\mathrm{d}\overline{\mu}$$

on U. Letting

$$\lambda = \lambda_1 + \lambda_2\mathrm{i}, \qquad (20)$$
$$\mu = \mu_1 + \mu_2\mathrm{i},$$

we have, by (17),

$$a_{11} I_{\omega} \otimes I_{\omega}(\sigma) = (\lambda_1^2 + \lambda_2^2)(\,\mathrm{d}\mu_1 \otimes_s \mathrm{d}\mu_1 + \mathrm{d}\mu_2 \otimes_s \mathrm{d}\mu_2). \qquad (21)$$

Let $\varphi: U \to \mathbb{R}^2$ be the map given by $\varphi = (\mu_1, \mu_2)$. In view of (18) and (19),

$$\mathrm{d}\mu \wedge \mathrm{d}\overline{\mu} = -\frac{2a_{11}\sqrt{-\Delta}\mathrm{i}}{\lambda_1^2 + \lambda_2^2}\omega,$$

and so

$$\mathrm{d}\mu_1 \wedge \mathrm{d}\mu_2 = -\frac{1}{2\mathrm{i}}\,\mathrm{d}\mu \wedge \mathrm{d}\overline{\mu} = \frac{a_{11}\sqrt{-\Delta}}{\lambda_1^2 + \lambda_2^2}\omega \neq 0. \qquad (22)$$

Hence, by (5) and (7),

$$I_{\omega} \otimes I_{\omega}\left(\frac{\partial}{\partial \mu_1} \otimes_s \frac{\partial}{\partial \mu_1} + \frac{\partial}{\partial \mu_2} \otimes_s \frac{\partial}{\partial \mu_2}\right) = -\frac{(\lambda_1^2 + \lambda_2^2)^2}{a_{11}^2 \Delta}(\,\mathrm{d}\mu_1 \otimes_s \mathrm{d}\mu_1 + \mathrm{d}\mu_2 \otimes_s \mathrm{d}\mu_2). \qquad (23)$$

Moreover, (22) together with the inverse function theorem shows that U can be contracted so that φ is a local diffeomorphism in U. Now, by (21) and (23),

$$\sigma = -\frac{a_{11}\Delta}{\lambda_1^2 + \lambda_2^2}\left(\frac{\partial}{\partial \mu_1} \otimes_s \frac{\partial}{\partial \mu_1} + \frac{\partial}{\partial \mu_2} \otimes_s \frac{\partial}{\partial \mu_2}\right).$$

We see that σ takes a canonical elliptic form in the coordinate system (U, φ). □

As an immediate corollary, we obtain the following result:

Theorem 6. *Let P be a real almost-linear second-order partial differential operator on an open region Ω of $\mathbb{R}^2$ and $a \in \Omega$ be such that P is either elliptic or hyperbolic at a, or there exists an open neighbourhood of a on which P is parabolic. In the elliptic case, P is additionally assumed to have real-analytic coefficients. Then there exists an open neighbourhood $U \subset \Omega$ of a and a C^∞ diffeomorphism φ from U into $\mathbb{R}^n$ reducing P to a canonical elliptic (parabolic, hyperbolic) form on $\varphi(U)$.*

In closing, we remark that the above result can be formulated in terms of geometric objects defined invariantly. As such a formulation would require a heavy machinery of jet bundles, we do not present it here. We refer the interested reader to [2] and [3] for a suitable material concerning differential equations on jet bundles.

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