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Paul F. Conrad; Jorge Martinez

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ON ADJOINING UNITS TO HYPER-ARCHIMEDEAN l-GROUPS

PAUL CONRAD, Lawrence and JORGE MARTINEZ, Gainesville

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0. INTRODUCTION

In this article we consider the problem of adjoining a weak order unit to a hyper-archimedean lattice-ordered group. Our main results are Theorems 6 and 7. To lay the groundwork for them and develop a motivation for the so-called complementations, we shall present some preliminary work on strongly rigid extensions.

This paper may be, reasonably, regarded as a continuation of the work in [CM2].

As is the custom, we shall employ the abbreviation *l-group* for "lattice-ordered group", *l-subgroup* for a subgroup is at once a sublattice, etc. If G is an l-group then $\mathrm{Spec}(G)$ denotes the root system of all prime convex l-subgroups. (A root system is a partially ordered set in which no two incomparable elements have a common lower bound; it is well-known that $\mathrm{Spec}(G)$ is a root system; see [AF]. In fact, [AF] will be our principal reference for most elements of the general theory of l-groups. Occasionally, we shall also refer to [BKW].)

$\mathrm{Spec}(G)$ carries a natural topology, the so-called hull-kernel topology, for which the basic open sets are $U(g) = \{P \in \mathrm{Spec}(G) \colon g \notin P\}$. One important subspace which we shall consider later, and which was extensively studied in [CM1], is $\mathrm{Min}(G)$, the space of all minimal primes. $\mathrm{Min}(G)$ is but one example of a Hausdorff subspace of $\mathrm{Spec}(G)$. Observe that a subspace X of $\mathrm{Spec}(G)$ is Hausdorff precisely when it is trivially ordered by inclusion.

$\mathrm{Val}(G)$ denotes the subspace of $\mathrm{Spec}(G)$ of all values of G. (Recall that a convex l-subgroup V of G is said to be a *value* of G if there is an element $g \in G$ so that V is maximal with respect to not containing g; we also say that V is a *value* of g. The set $Y(g)$ of all values of g with the subspace topology is the *Yosida space* of g. Clearly, a Yosida space is a Hausdorff subspace of $\mathrm{Val}(G)$; if $\{g_i \colon i \in I\}$ is a set of pairwise disjoint elements of G, then the union of the corresponding Yosida spaces is, likewise, a Hausdorff subspace of $\mathrm{Val}(G)$.)

$\text{Max}(G)$ will stand for the subspace of all maximal convex l-subgroups of G; $\text{Max}(G)$ could be empty, in general.

Next, suppose that Γ denotes a root system, and R_δ stands for a subgroup of the additive real numbers R with the standard ordering, for each $\delta \in \Gamma$. $V(\Gamma, R_\delta)$ will denote the l-group of all functions $f\colon \Gamma \to \bigcup\{R_\delta\colon \delta \in \Gamma\}$, so that $f(\delta) \in R_\delta$ and the set $\text{supp}(f) = \{\delta \in \Gamma\colon f(x) \neq 0\}$ satisfies the ascending chain condition. It is well-known that $V(\Gamma, R_\delta)$ is an l-group, in which $f > 0$ means that $f \neq 0$ and for each maximal element $\alpha \in \text{supp}(f)$, $f(\alpha) > 0$; see [AF], or else [CHH].

An l-group G is said to be *hyper-archimedean* if it is archimedean and every l-homomorphic image of G is archimedean. These l-groups are fairly well understood; the reader is referred to [C4] and [M1] for many of the principal features of this class of l-groups. Let us mention one or two items, which given impetus to the research which produced [CM2] as well as this article.

G is hyper-archimedean if and only if it can be embedded as an l-subgroup of real-valued functions defined on a set I, such that if $0 < a$, $b \in G$ then there is a natural number n so that $na_i \geqslant b_i$, whenever $a_i > 0$. If such an embedding can be produced, then every representation by real-valued functions possesses this property. If in addition G has a unit then a representation may be obtained so that for each $0 < g \in G$ the set $\{g_i\colon i \in I\}$ is bounded both above and below by positive real numbers. Long ago Conrad asked if such a representation was possible without the presence of a unit. In [CM2] we were finally able to produce an example of a hyper-archimedean l-group which does not admit such an embedding; (see Example A near the end of this article.) The point of this paper then is to (intelligently) describe when a hyper-archimedean l-group can be embedded in one with a unit. Our Theorem 6 and its immediate corollary (Theorem 7) accomplish this and more.

If G is an l-subgroup of H, we say that G is *large* in H (and also that H is an *essential extension of* G) if every non-zero convex l-subgroup of H intersects G non-trivially.

Finally, recall that if C is any convex l-subgroup of the l-group G, then $C^\perp$ stands for the *polar* of C; that is, the convex l-subgroup generated by the set $\{0 \leqslant x \in G\colon x \wedge |c| = 0, \text{ for all } c \in C\}$. If $g \in G$ and C is the convex l-subgroup generated by g, then we write $g^\perp$ for $C^\perp$; the meanings of $C^{\perp\perp}$ and $g^{\perp\perp}$ should be obvious. Recall the well-known result, first observed by Šik [Š], that the set of all polars of an l-group, under the usual inclusion, is a (complete) boolean algebra.

1. Strongly Rigid Extensions

Let us suppose that G is an l-group and that H contains G as an l-subgroup; (in these circumstances, we shall frequently say that H is an extension of G.) Recall that H is said to be a *strongly rigid* extension of G if for each $0 < h \in H$ there exists a $g \in G$ such that $g \geqslant h$ and $g^{\perp\perp} = h^{\perp\perp}$. (When we have one l-group extending another, the symbol $\perp$ should be taken to denote polars in the larger of the two groups.) We wish to examine the relationship between the prime spectrum of G and that of a strongly rigid extension of G. So suppose that H is a strongly rigid extension of G. For each $P \in \mathrm{Spec}(G)$ denote by cP the convex hull of P in $C(H)$, the lattice of all convex l-subgroups of H. Because of the strong rigidity, it is easy to demonstrate that $cP \in \mathrm{Spec}(H)$. Conversely, suppose that $Q \in \mathrm{Spec}(H)$; then it is obvious that $Q \cap G \in \mathrm{Spec}(G)$ and that $P = cP \cap G$, for each prime P of G. (Strongly rigid extensions were first introduced in [CM3].)

Before stating our first proposition, let us agree to call a map between two partially ordered sets *breadth preserving* if every pair of incomparable elements is mapped to incomparable elements.

Proposition 1. *Suppose that H is a strongly rigid extension of G. Then:*

(1) *The contraction $\mu(Q) = Q \cap G$, for $Q \in \mathrm{Spec}(H)$ is onto $\mathrm{Spec}(G)$ and breadth preserving.*

(2) *The map $c\colon \mathrm{Spec}(G) \to \mathrm{Spec}(H)$ embeds $\mathrm{Spec}(G)$ as a partially ordered subset of $\mathrm{Spec}(H)$.*

(3) *For each $P \in \mathrm{Spec}(G)$, $\mu^{-1}\{P\}$ is a chain, of which cP is the least element; (we denote the largest element by vP.)*

Proof. That μ is onto is clear. Now suppose that Q_1 and Q_2 are incomparable primes of H. We may select disjoint elements x_1 and x_2 of H, so that $x_1 \in Q_1 \setminus Q_2$ and $x_2 \in Q_2 \setminus Q_1$. By the strong rigidity of H over G, there exist g_1, $g_2 \in G$, also disjoint, so that $g_i \geqslant x_i$ and $g_i^{\perp\perp} = x_i^{\perp\perp}$. Now, $g_2 \neq Q_1 \cap G$, and therefore $g_1 \in \mu Q_1$; likewise, $g_1 \neq \mu Q_2$ and $g_2 \in \mu Q_2$, proving that μQ_1 and μQ_2 are incomparable. This establishes (1).

The rest is straightforward. □

Recall that an l-group is said to have the *stranded primes property* if every prime convex l-subgroup contains a unique minimal prime. G is *projectable* if for each $x \in G$, $G = x^{\perp\perp} + x^{\perp}$. It is well-known that projectable l-groups have the stranded primes property, and that the converse is not true. It was proved in [CM3] that projectability is preserved by strongly rigid extensions. Now we have:

Corollary 1.1. *Suppose that H is a strongly rigid extension of G. Then if G has the stranded primes property then so does H.*

Suppose that H is a strongly rigid extension of G. Observe that if P is a value of G then vP is a value of H; (if P is maximal with respect to missing $g \in G$ then vP is maximal in C(H) with respect to missing g.) The map $v \in \mathrm{Val}(G) \to \mathrm{Val}(H)$ is (by Proposition 1) an order embedding. Since $vP \cap G = P$ it follows, in particular, that a special element of G remains special in H. (We remind the reader that $x \in G$ is *special* if it has only one value in G.)

Recall that G is *pseudo-special-valued* if every strictly positive element of G has a special component. This condition is weaker that *special-valuedness*: every positive element is a supremum of pairwise disjoint special elements ([M2]). For extensive information on special-valued l-groups we refer the reader to [C1], and the excellent [BiD].

Proposition 1 then also yields the following corollary:

Corollary 1.2. *Suppose that H is a strongly rigid extension of G. If G is pseudo-special-valued then so is H.*

Suppose that G is a special-valued and that H is a strongly rigid extension of G. If, in addition, every positive element of H is also a supremum of pairwise disjoint elements of G, and arbitrary suprema in G agree with those in H, then it can be shown that H is special-valued. However, it is not the case that any strongly rigid extension of a special-valued l-group is special-valued.

Let Γ be the following root system:

$$\Gamma = \{\alpha_n : n \in \mathbb{N}\} \cup \{\beta_n : n \in \mathbb{N}\} \cup \{\alpha\},$$

where each $\alpha_n > \beta_n$ and $\alpha_n > \alpha_{n+1} > \alpha$. Let G be the finitely non-zero integer-valued functions on Γ subject to the Hahn-ordering, and $H = G + Zu$, where u is defined by $u(\alpha_n) = u(\alpha) = 0$, for each natural number n, and $u(\beta_n) = 1$, for each $n \in \mathbb{N}$. The ordering of H is defined by making u infinitely large compared to any $x \in G$ which vanishes at every α_n and β_n. Then H is a strongly rigid extension of G, but we have managed to slip in a non-special value between two special ones. G is, in fact, *finite-valued*; that is to say, every element has a finite number of values. However, H is not special-valued. Note as well that H does not lie in the lateral completion of G.

Let us now take into account the structure topologies on the prime and the value spectra, and see how these spectra are affected by a strongly rigid extension.

Since we are considering extensions here let's use $U(g,G)$ to denote the basic open set determined by $g \in G$. Observe then, that if H is a strongly rigid extension of G and μ is the contraction map from $\mathrm{Spec}(H)$ onto $\mathrm{Spec}(G)$, then since $\mu^{-1}\big(U(g,G)\big) = U(g,H)$, it follows that μ is continuous. However, μ need not be open. Let Γ_0 by the root system of our earlier example of our earlier example, but with α omitted. G is the l-group of finitely non-zero integer-valued functions on Γ_0 subject to the Hahn-ordering. Define u by $u(\alpha_n) = 0$ and $u(\beta_n) = 1$, and let H be the l-subgroup of $V(\Gamma_0, Z)$ generated by G and u. Then H is a strongly rigid extension of G; however, while $\mathrm{Min}(H)$ is open in $\mathrm{Spec}(H)$, its contradiction to $\mathrm{Spec}(G)$, namely $\mathrm{Min}(G)$, is not open.

This proves not only that μ needn't be open, but also that the map c need not be continuous. In fact, the following is true:

Lemma 2. *Suppose that H is a strongly rigid extension of G. For each $h \in H$, $c^{-1}\big(U(h,H)\big) = \mu\big(U(h,H)\big)$. Therefore, c is continuous if and only if μ is open.*

Proof. For a prime P of G, $P \in c^{-1}\big(U(h,H)\big)$ if and only if $h \notin cP$. On the other hand, $P = \mu cP$, and so $c^{-1}\big(U(h,H)\big) \subseteq \mu\big(U(h,H)\big)$. Conversely, if $P = Q \cap G$, where Q is a prime of H not containing h, then $Q \subseteq cP$ with $h \notin cP$, so that $P \in c^{-1}\big(U(h,H)\big)$. □

Recall that H is an *a-extension* of G if the contradiction map $\mu C = C \cap G$ is a lattice-isomorphism from $C(H)$, the lattice of all convex l-subgroups of H, onto $G(G)$. This is equivalent to saying that for each $0 \leqslant h \in H$ there is a $g \in G$ so that $mg \geqslant h$ and $nh \geqslant g$, for suitable natural numbers m and n. (Elements with this property are said to be *a-equivalent.*)

Our next theorem then tells us when we can expect μ to be a homeomorphism:

Theorem 3. *Suppose that H is a strongly rigid extension of G. Then the following are equivalent.*

(a) *H is an a-extension of G.*

(b) *c is onto and continuous.*

(c) *μ is an open mapping and one-to-one.*

(d) *μ is an order isomorphism and a homeomorphism.*

Proof. It should be clear that (a) implies the others, and that (b) and (c) follow from (d). The equivalence of (b) and (c) is a consequence of Lemma 2 and Proposition 1.

To see that (c) implies (a), suppose that μ is an open mapping. Observe now that the compact open subsets of $\mathrm{Spec}(H)$ are precisely the sets $U(h,H)$. Now $\mu\big(U(h,H)\big)$ is both compact and open, and so $\mu\big(U(h,H)\big) = U(g,G)$, for a suitable

$g \in G$. It is now easy to verify that $Q \in \mathrm{Spec}(H)$ is a value of h if and only if it is a value of G; that is, g and h are a-equivalent. □

Incidentally, and referring to the last example, there is the following characterization of when $\mathrm{Min}(G)$ is open in $\mathrm{Spec}(G)$. Let $E(G)$ stand for the hyper-archimedean radical of G; this is the largest convex l-subgroup of G which is hyper-archimedean. (We refer the reader to [M1] for the basic properties of this radical.) Recall also—see [CM4]—that a convex l-subgroup K of the l-group G is said to be *very large* in G if it fails to be contained in any of the minimal prime subgroups of G.

Proposition 4. *The minimal prime P of G is in the interior of* $\mathrm{Min}(G)$ *precisely when P is a value of an element in the hyper-archimedean radical of G. Thus,* $\mathrm{Min}(G)$ *is an open subspace of* $\mathrm{Spec}(G)$ *if and only if $E(G)$ is very large in G.*

Proof. To say that $P \in \mathrm{Min}(G)$ is in the interior of $\mathrm{Min}(G)$ is to say that there exists an element $a \in G$ so that $U(a, G)$ consists entirely of minimal primes and contains P. This implies that $a \in E(G)$; the converse is clear, as is the second claim in the proposition. □

To conclude this section, let us consider the effect of taking a strongly rigid extension of a subdirect product of reals. More generally, suppose that X is a dense Hausdorff subspace of $\mathrm{Spec}(G)$; suppose that H is a strongly rigid extension of G, and that G is large in H. Then cX is again a dense Hausdorff subspace of $\mathrm{Spec}(H)$: that it is Hausdorff is clear from Proposition 1, and $G \cap (\bigcap cX) = \bigcap X = \{0\}$, whence it follows that $\bigcap cX$ is also trivial.

Observe that if X is Hausdorff then X and cX are trivially ordered sets of the same cardinality. But what can we say about the respective subspace topologies? In all of the following cases the restriction of c to X is a homeomorphism; (that is to say, continuous.)

(A) X is the disjoint union of Yosida spaces.

(B) cX is the disjoint union of Yosida spaces.

(C) $X = \mathrm{Min}(G)$, in which case $cX = \mathrm{Min}(H)$.

In cases (A) and (B), c is a homeomorphism because these spaces are disjoint unions of compact spaces, and on compact Hausdorff spaces any bijection which is open is continuous. As for (C), c is a homeomorphism between the minimal prime spaces simply because H is a rigid extension of G.

Question. If $X = \mathrm{Max}(G)$, then is it homeomorphis to cX?

In any event, for any Yosida space or any Hausdorff subspace X which consists of values, we are better off using the map v, so that vX is once again a set of values. For example, if X is the Yosida space of g in G then vX is the Yosida space of g

in H; (and vX is homeomorphic to cX?)Notice also that any prime Q of H lies between cQ and $v\mu Q$. Therefore, if Q is a maximal l-ideal then $Q = v\mu Q$, and, in fact, $Q \in v(\operatorname{Max}(G))$; that is, $\operatorname{Max}(H) = v(\operatorname{Max}(G))$.

Putting all of the above discussion together we get:

Proposition 5. *Suppose that H is a strongly rigid extension of G with G large in H and that G is a subdirect product of reals. Then H is also a subdirect product of reals, and, indeed, if G is realized as a group of real-valued functions on the set I then H can also be realized as a group of real-valued functions on the set I.*

Proof. If G is a subdirect product of reals then $\operatorname{Max}(G)$ is a dense Hausdorff subspace. The same is true of $\operatorname{Max}(H)$, proving that H is also a subdirect product of reals. But more is true: for any dense subset X of $\operatorname{Max}(G)$, vX is likewise dense in $\operatorname{Max}(H)$. After observing that for each $M \in X$, $G/M = H/vM$, we're done. □

Returning to the topological point of view, we have a counterpart to Proposition 5. Note first, however, that if X is a Hausdorff subspace of $\operatorname{Spec}(G)$ and c, restricted to X, is a homeomorphism onto cX, then so is the restriction of v to X; this is so because the map which assigns cP to vP is continuous.

Proposition 5a. *Suppose that G is archimedean and that H is a strongly rigid extension of G with G large in H. Then H is also archimedean and for every disjoint union Y of Yosida spaces of G which is dense in* $\operatorname{Spec}(G)$, *the canonical Yosida embedding of G into $D(Y)$ can be lifted to a Yosida embedding of H into $D(Y)$. (Note: $D(Y)$, as usual, stands for the lattice of all continuous functions defined on Y with values in the extended real numbers, which are real-valued on a dense subset of Y. Recall that, in general, $D(Y)$ is not a group.)*

Proof. First, since G is large in H, H is archimedean. In fact, it turns out that H is in the essential hull of G.

By the remark preceding this proposition as well as our earlier comments, since Y is a disjoint union of Yosida subspaces, Y, cY and vY are homeomorphic. The point is that if $Y = U\{Y(g_i)\colon i \in I\}$, for a suitable maximal pairwise disjoint set $\{g_i\colon i \in I\}$, then the (non-faithful) representation of $G/g_i^{\perp}$ in $D(Y(g_i))$ can be lifted to $H/g_i^{\perp}$. (Caution: we are not distinguish between polars calculated in G and those from H.) □

Let us first recall the definition of complementation from [CM3]. Let G be an l-group; we say that the extension $H \subseteq G^L$, the lateral completion of G, is a *complementation* of G if

(a) H is complemented,

(b) the convex hull G^c of G in H is a strongly rigid extension of G and

(c) G^c is an intersection of minimal primes of H.

If $H = \langle G, u \rangle_l$, for some unit $u > 0$, we say that H is a *simple complementation* of G. (For a discussion of lateral completeness and the lateral completion, we refer the reader to [C2], where the subject is discussed in the context of representable l-groups, of [Be], who considers lateral completions of arbitrary l-groups.)

In the sequel we assume that G does not have a weak unit. We shall also suppose, for the time being, that G is a hyper-archimedean l-group. Recall from [CM3] that if G is projectable, and it has a simple complementation H, then H is also projectable; in fact, such an l-group has a simple complementation if and only if it has a projectable complementation (Corollary 4.1.1, [CM3]).

We shall make one change from the conventions in [CM3]: since our concern will be with complementations of archimedean l-groups, it will be convenient to word inside the essential hull eG of the l-group G. Recall—see [C3]—that every archimedean l-group G has a unique *essential hull* eG: it is maximal among all the essential extensions of G which are archimedean, and can be obtained as the l-group of all continuous functions with values in the extended reals, defined on the Stone dual of the boolean algebra of all polars of G, which take on real values on a dense subset. The reader should reflect that *no part of the theory from [CM3] is altered by allowing the extensions to be computed in eG instead of* G^L.

Recall also that if H is a complemented extension of G, and $H = \langle G, u \rangle_l$, then G^c is a max-min prime of H, and $\mathrm{Min}(H)$ is a one-point compactification of $\mathrm{Min}(G)$ (Lemma 3.3, [CM3]). Thus, if H is complemented then we can ignore condition (c) in the definition of complementation.

The main theorem of this article is the following:

Theorem 6. *For a hyper-archimedean l-group G, the following are equivalent:*

(1) *G admits a simple complementation.*

(2) *G can be regarded as a group of real-valued functions on the set I such that*

  (i) $\inf\{g_i \colon g_i \neq 0\} > 0$, *for each* $g \in G$, $g \neq 0$.

  (ii) *For each* $0 < g \in G$ *and each* $n \in \mathbb{N}$, $g = g_n + g^n$, *such that* $g_n \wedge g^n = 0$ *and* $(g_n)_i < n$, *for each* $i \in I$, *while* $(g^n)_i \geqslant n$ *whenever* $(g^n)_i > 0$.

(3) *Condition* (2) *holds, with* $I = \mathrm{Min}(G)$.

(4) *There is a dense disjoint union of Yosida spaces* Y, *and an embedding* $\theta\colon G \to D(Y)$ *so that*

   (a) $\mathrm{coz}(g)$ *is a compact-open set.*

   (b) *For each* $n \in \mathbb{N}$ *and* $0 < g \in G$ *the set*

$$\{P \in Y : g(P) < n\}$$

   is compact open.

(5) *Condition* (4) *holds, with* $Y = \mathrm{Min}(G)$.

Before giving a proof of this theorem, let us review what is known about simple complementations of projectable l-groups G in which G is convex. We distill the following from Section Four of [CM3]:

Theorem 6A. *For a projectable l-group G the following are equivalent:*
(1) *G admits a simple complementation in which it is convex.*
(2) *G admits a projectable complementation in which it is convex.*
(3) *G has a rigid Specker subgroup.*

Recall that an element $0 < s \in G$ is said to be *singular* if s and $g - s$ are disjoint. A *Specker* l-group is one which is generated as an abstract group by its singular elements. For a detailed study of Specker groups, the reader is encouraged to read [C4].

Theorem 6A has the following corollary for hyper-archimedean l-groups:

Corollary 6B. *For a hyper-archimedean l-group G the following are equivalent:*
(1) *G has a simple complementation in which it is convex.*
(2) *G has a rigid Specker subgroup.*
(3) *G has an S-kernel, that is, a Specker subgroup of which G is an a-extension.*
(4) *G can be embedded in a hyper-archimedean l-group with unit as an l-ideal.*

Proof. The equivalence of (1) and (2) here comes straight from Theorem 6A. (2) and (3) are equivalent because in a hyper-archimedean l-group a subgroup is rigid if and only if it is an a-subgroup. The equivalence of (3) and (4) is in Theorem 19 of [CM5].

Theorem 6 is more general than Theorem 6B, because it drops the stipulation of convexity. The conditions (2ii) and (4b) in Theorem 6 replace the existence of the rigid Specker subgroup in Theorem 6B.

Now to the proof of Theorem 6; we show that (1) implies (3), (2) implies (1), (5) follows from (1) and that (4) implies (2). Observe that (2) and (4) are trivial consequences of (3) and (5) respectively.

Let us suppose that $H = \langle G, u\rangle_l$ is a simple complementation of G, and that G^c is the convex hull of G in H. Recall that G^c is a max-min prime of H. Since G is hyper-archimedean we can represent it as a subgroup of real-valued functions on $I = \mathrm{Min}(G)$. Since $H \subseteq eG$ we can apply Proposition 5, and lift this representation 5, and lift this representation to G^c. (CAUTION: G^c may no longer be hyper-archimedean!) G^c is projectable and convex in the simple complementation H; therefore G^c has a rigid Specker subgroup S. Without loss of generality we can assume that in the above representation the singular elements of S only take on values 0 and 1.

Now let $0 < g \in G$ and $x = (g - u) \vee 0$; note that $x \in G^c$. By projectability, we can write $g = g_1 + g_2$ with $g_1 \in x^{\perp\perp}$ and $g_2 \in x^{\perp}$. Since G is saturated in G^c (Lemma 3.4 of [CM2]) we get that the g_i belong to G. Let s be the singular component of g_2 in S; pick a in G so that $a \geqslant s$ and $s^{\perp\perp} = a^{\perp\perp}$. Therefore, if $g_2 > 0$, then for each $i \in I$ such that $(g_2)_i > 0$ we have that $(g_2)_i = g_i < 1 \leqslant a_i$ and g_2 and a vanish at the same places. From this it should be clear that there is no sequence $g_{i_n} > 0$ converging to zero, and thus (3i) holds.

As to (3ii), for each $n \in \mathbb{N}$, consider $x_n = (g - nu) \vee 0$, and decompose $g = g^n + g_n$. Since G is saturated in G^c, it can easily be shown that this is the pair of components we want. This proves that (1) implies (3).

Now let us see that (2) implies (1). Suppose that G is represented on I according to the specifications of (2). Let $H = \langle G, u\rangle_l$, where u is the constant 1. Consider $g \in G$ and $n \in \mathbb{N}$, and let's compute $(g + nu) \vee 0$. First, notice that, by condition (2i), there is for each $a > 0$ in G a natural number m so that $(m+1)a \wedge u = ma \wedge u = u(a)$. Then $u(a)$ is a component of u, and $u(a) \leqslant ma$, while $u(a)^{\perp\perp} = a^{\perp\perp}$.

Now,

$$\begin{aligned}(g + nu) \vee 0 &= (g^+ - g^- + nu) \vee 0 = \big(g^+ - (g^-)^n - (g^-)_n + nu\big) \vee 0 \\ &= \big((g^+ + nu(g^+)) - (g^-)^n + nu((g^-)^n) - (g^-)_n + nu((g^-)_n) + nw\big) \vee 0,\end{aligned}$$

where $w = u - \big(u(g^+) + u((g^-)^n) + u((g^-)_n)\big)$. By the choices of (2ii), $(g^-)_n \leqslant n\big(u((g^-)_n)\big)$ while $(g^-)^n \geqslant nu\big((g^-)^n\big)$. Therefore,

$$(g + nu) \vee 0 = g^+ + nu(g^+) + \big(n((g^-)_n)\big) - (g^-)_n + nw = g_0 + nv,$$

where $g_0 = g^+ - (g^-)_n$ and $v = u - u\big((g^-)^n\big)$. The point of this calculation is to show that the subgroup generated by G and u is an l-subgroup of H and hence H itself.

Together with conditions (2i) and (2ii) this observation implies that these two conditions, in fact, hold for H. Thus, each $0 < x \in G^c$ determines a natural number m so that $mx \wedge u = (m+1)x \wedge u = u(x)$; as before, $u(x)$ is a component of u so that $u(x) \leqslant mx$ and $u(x)$ and x have the same polar. This shows that the subgroup generated by these components is a rigid Specker subgroup of G^c. Since G^c and u clearly generate H, we may apply Theorem 3.9 from [CM3] to conclude that H is a simple complementation of G, proving that (2) implies (1).

(1) implies (5): let H be a simple complementation of G; say $H = \langle G, u \rangle_l$, where $u > 0$ is a unit of eG. Using the arguments in the preceding paragraphs, we get, in fact, that $u \in E(H)$, the hyper-archimedean radical of H. But then we can represent H as functions on $D\big(Y(u)\big)$, where $Y(u)$ is the Yosida space of u. (Note that $Y(u)$ is homeomorphic to $\mathrm{Min}(H)$.) Thus, $Y(u)$ is a compact zero-dimensional space. This means that each element of H is represented with compact cozero sets; also $u = 1$. Then it is clear that (5a) holds, and we leave the verification of (5b) to the reader; it is similar to the argument which shows that (1) implies (3ii).

The only thing left to do is restrict the domains of these functions: let X consist of the points in $Y(h)$ which are values of components of u lying in G^c. Then $X = \mathrm{Min}(G^c)$, and the latter is homeomorphic to $\mathrm{Min}(G)$, since G^c is a strongly rigid extension of G.

Finally, (4) implies (2): if $H = \langle G, u \rangle_l$, with $u = 1$, then it should be obvious that (4a) gives us (2i), taking $I = X$. We leave it to the reader to verify that (4) implies (2ii).

The proof of the theorem is now complete. □

Theorem 6 yields an obvious corollary, which spells out exactly when a hyper-archimedean l-group G has a hyper-archimedean simple complementation. The essential thing to observe (once more) about the proof of Theorem 6 is this: in the implications which yield the simple complementation observe that the unit which is adjoined lies in the hyper-archimedean radical of the extension. Armed with this insight we can now state the final result of this section, which, in sense is a culmination of this article. The proofs that (1) imply the rest are almost verbatim restatements of the corresponding arguments in the proof of Theorem 6, and will therefore be omitted.

Theorem 7. *For a hyper-archimedean l-group G, the following are equivalent:*

(1) *G admits a simple complementation which is hyper-archimedean.*
(2) *G can be represented as an l-subgroup of bounded real-valued functions defined on a set I satisfying* (2i) *and* (2ii) *of Theorem* 6.
(3) *Condition* (2) *holds, with* $I = \mathrm{Min}(G)$.

(4) *G can be represented as an l-subgroup of $C(Y)$, for some union of Yosida spaces, so as to satisfy* (4a) *and* (4b) *in Theorem* 6.

(5) *Condition* (4) *holds, with $Y = \mathrm{Min}(G)$. (Note: As is customary, $C(Y)$ stands for the group of all real-valued continuous functions defined on Y. It is well known that $C(Y)$ is an l-group (and in fact an f-ring) under the usual pointwise operations.)*

3. Comments and examples

First of all, a corollary of our results, for hyper-archimedean f-rings.

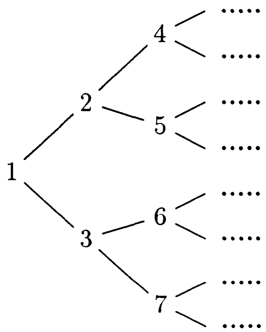
Corollary. *Suppose that G is hyper-archimedean f-ring with no non-zero nilpotent elements. Then $G^1 = \langle G, 1\rangle_l$, where 1 is the identity in eG, is a simple (hyper-archimedean) complementation of G. Moreover G^1 contains G as an l-ideal if and only if G contains a rigid Specker subring.*

P r o o f. It is easy to verify that a hyper-archimedean f-ring with no non-zero nilpotent elements satisfies condition (3) in Theorem 7. The remaining claim is obvious. □

We conclude the paper with three examples, which illustrate the distinctions implied by our results.

Example A. This is the example of [CM2]; it appears again in [CM3].

Consider the binary tree



and the family $\mathbf{M}$ of all maximal chains through the tree. Then $\mathbf{M}$ is a family of subsets of $\mathbb{N}$ which are almost disjoint—any two distinct subsets have finite intersection. $\mathbf{M}$ has the cardinality of the continuum. Let us index $\mathbf{M} = \{M_i : i \in I\}$.

Let G be the l-group generated by all the finitely non-zero sequences of integers, together with the sequences v_i defined by:

$$v_i(n) = \begin{cases} 0 & \text{if } n \notin M_i, \\ f_i(k) & \text{if } n \text{ is the } k\text{-th term of } M_i, \end{cases}$$

where $\{f_i \colon i \in I\}$ is the set of all strictly increasing sequences of natural numbers. It was shown in [CM2] that G is hyper-archimedean, but that it cannot be embedded in a hyper-archimedean l-group containing a unit. Notice, however that G does satisfy the provision of Theorem 6, and therefore does admit a simple complementation.

Example B. With $\mathbf{M} = \{M_i \colon i \in I\}$ as in the previous example, let us now consider the collection $\{g_i \colon i \in I\}$ of all strictly decreasing sequences of rational numbers which converge to 0, and define

$$w_i(n) = \begin{cases} 0 & \text{if } n \notin M_i, \\ g_i(k) & \text{if } n \text{ is the } k\text{-th term of } M_i. \end{cases}$$

Let G be the divisible l-group generated by all the finitely non-zero sequences of rational numbers, together with the w_i. As with the previous example, G is a hyper-archimedean l-group. We proved in [CM3] that G has no simple complementations at all. The reader will also readily notice that G satisfies condition (3ii) of Theorem 6, but not (3i).

In between Examples A and B we have the following item in conclusion:

Example C. $\mathbf{M} = \{M_i \colon i \in I\}$ denotes the same family as in the preceding example; the $\{g_i \colon i \in I\}$ also the same family of strictly decreasing sequences of rational numbers. Now let

$$u_i(n) = \begin{cases} 0 & \text{if } n \notin M_i, \\ g_i(k) + \pi & \text{if } n \text{ is the } k\text{-th term of } M_i. \end{cases}$$

Then let G be the divisible l-group generated by the finitely non-zero sequences of rationals, together with the u_i just defined. As with the other examples, G is hyper-archimedean. Clearly, condition (3) of Theorem 7 is satisfied, and G does have a simple complementation, namely, $H = \langle G, 1 \rangle_l$, which is hyper-archimedean. However, G does not have a rigid Specker subgroup, and therefore cannot be embedded in a hyper-archimedean l-group with unit, as an l-ideal.

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Author's address: Departement of Mathematics, University of Kansas, Lawrence, KS 66045, U.S.A.; Departement of Mathematics, University of Florida, Gainesville, FL 32611, U.S.A.