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# A HAKE-TYPE PROPERTY FOR THE $\nu_1$-INTEGRAL AND ITS RELATION TO OTHER INTEGRATION PROCESSES

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## INTRODUCTION

Specializing our abstract concept of non-absolutely convergent integrals (cf. [Ju-No 1]), we introduced in [Ju-No 2] the relatively simple $\nu_1$-integral over $n$-dimensional compact intervals. This integral not only shows all the usual properties but also yields a very general divergence theorem including points of unboundedness of the involved vector function. In [Ju-No 3] this result is used as a basic part for a geometrically improved version of the divergence theorem.

The studies in this paper are devoted to a further development of the $\nu_1$-theory. In Section 1 we extend the notion of $\nu_1$-integrability to point functions $f$ defined on a bounded measurable set $A \subseteq \mathbb{R}^n$, and we then establish a Hake-type theorem involving both a point function $f$ and an interval function $F$ (the associated indefinite integral). In particular, it is shown how the integrability on $A$ can be deduced from the integrability on any interval contained in the interior of $A$. Of course here the main difficulties arise at the boundary of $A$, and we found a characteristic null condition for $F$ to be the relevant property. Indeed we do not require this condition along the topological but only along a 'reduced' boundary of $A$ which will be important for further applications.

In [Ju-No 2] the $\nu_1$-integral was shown to extend the $M_1$-integral (cf. [JKS]), and in Section 2 we prove that it also extends the variational integral as defined in [Pf].

$\mathbb{R}$ and $\mathbb{R}^+$ denote the set of all real and all positive real numbers respectively, $n$ is a fixed positive integer, and we work in $\mathbb{R}^n$ with the usual inner product and the associated norm. For $x = (x_i) \in \mathbb{R}^n$ and $r > 0$ we set $B(x,r) = \{y = (y_i) \in \mathbb{R}^n : |x_i - y_i| < r, 1 \leqslant i \leqslant n\}$.

Given a set $E \subseteq \mathbb{R}^n$ we denote by $E^\circ$, $\operatorname{cl} E$, $\partial E$ and $d(E)$ the interior, closure, boundary and the diameter of $E$, respectively.

The $n$-dimensional outer Lebesgue measure in $\mathbb{R}^n$ is denoted by $|\cdot|_n$, and terms like measurable and almost everywhere (a.e.) always refer to this measure if the contrary is not stated explicitly. By $|\cdot|_{n-1}$ we denote the $(n-1)$-dimensional outer Hausdorff measure in $\mathbb{R}^n$ which coincides on $\mathbb{R}^{n-1}$ ($\subseteq \mathbb{R}^n$) with the $(n-1)$-dimensional outer Lebesgue measure ($|\cdot|_0$ being the counting measure). A set $E \subseteq \mathbb{R}^n$ is said to be $\sigma_{n-1}$-finite if it can be expressed as a countable union of sets with finite $(n-1)$-dimensional outer Hausdorff measure.

Let $E \subseteq \mathbb{R}^n$ be measurable, $x \in \mathbb{R}^n$. Then we call $x$ a density or a dispersion point of $E$ if, respectively,

$$\liminf_{r \to 0} \frac{|E \cap B(x,r)|_n}{(2r)^n} = 1 \quad \text{or} \quad \limsup_{r \to 0} \frac{|E \cap B(x,r)|_n}{(2r)^n} = 0.$$

We denote the set of all density points of $E$ by $\operatorname{int}_e E$, and $\operatorname{cl}_e E$ denotes the complement of the set of all dispersion points of $E$. By [Saks] the sets $E$, $\operatorname{int}_e E$, $\operatorname{cl}_e E$ differ at most by sets of $|\cdot|_n$-measure zero, and we obviously have the inclusions $E^\circ \subseteq \operatorname{int}_e E \subseteq \operatorname{cl}_e E \subseteq \operatorname{cl} E$. We set $\partial_e E = \operatorname{cl}_e E - \operatorname{int}_e E$, $\operatorname{cl}_r E = \operatorname{cl} \operatorname{cl}_e E$, $\partial_r E = \operatorname{cl}_r E - E^\circ$, and we see that $\partial_e E \subseteq \partial_r E \subseteq \partial E$.

An interval in $\mathbb{R}^n$ is always assumed to be compact and non-degenerate, and a family of intervals in $\mathbb{R}^n$ is said to be non-overlapping if they have pairwise disjoint interiors. A cube is an interval with all sides having equal length, and the support of a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is the closure of the set of points where $f$ is different from zero.

## 1. A HAKE-TYPE PROPERTY FOR THE $\nu_1$-INTEGRAL

We begin this section by recalling the basic definitions concerning $\nu_1$-integration, cf. [Ju-No 1,2].

An *interval function* $F$ (on $\mathbb{R}^n$) associates with every interval $I$ ($\subseteq \mathbb{R}^n$) a real number $F(I)$. The interval function $F$ is said to be *additive* if for every interval $I$ and any *decomposition* $\{I_k\}$ *of* $I$ (i.e. a finite sequence of non-overlapping intervals $I_k$ whose union is $I$) the equality $F(I) = \sum F(I_k)$ holds true.

We call an interval function $F$ *differentiable at* $x \in \mathbb{R}^n$ if $F$ is derivable in the ordinary sense at $x$ (according to [Saks]), and in that case $F'(x)$ denotes the ordinary derivative of $F$ at $x$.

Let $F$ be additive and suppose $I$ to be an interval such that $F(J) = 0$ for each interval $J \subseteq \mathbb{R}^n - I^\circ$ (we say that $F$ has *compact support*). Then a standard argument yields that the real number $F(I)$ is independent of $I$ and in what follows this unique number will be denoted by $F(\mathbb{R}^n)$.

**Remark 1.1.** By $\mathcal{B}$ we denote the system of all compact subsets $I$ of $\mathbb{R}^n$ which are of the form $I = \bigtimes_{i=1}^{n} [a_i, b_i]$ where $a_i, b_i \in \mathbb{R}$, and with the understanding that $I = \emptyset$ if $a_i > b_i$ for one $i$. Whenever an interval function $F$ is given, we can extend $F$ to the whole of $\mathcal{B}$ by setting $F(I) = 0$ if $I$ is not an interval. If $F$ is an additive interval function then its extension is additive in the sense of [Ju-No 1, Sec. 3].

A *control condition* $C$ associates with any positive numbers $K$ and $\Delta$ a class $C(K, \Delta)$ of finite sequences $\{I_k\}$ with $I_k \in \mathcal{B}$. Furthermore, with $C$ we associate a system $\mathcal{E}(C)$ of subsets of $\mathbb{R}^n$, and the control conditions $C^\alpha_{1,2}$ $(0 \leqslant \alpha < n)$, $C^n$ we use in the concept of $\nu_1$-integration are explicitly defined in [Ju-No 2, Sec. 1]. We set $\Gamma = \{C^\alpha_i : 0 \leqslant \alpha \leqslant n-1, i = 1, 2\}$, $\dot\Gamma = \{C^n\} \cup \{C^\alpha_i : n-1 < \alpha < n, i = 1, 2\}$.

Given $E \subseteq \mathbb{R}^n$ and $\delta \colon E \to \mathbb{R}^+$, a finite sequence of pairs $\{(x_k, I_k)\}$ is called *$(E, \delta)$-fine* if the $I_k$ are non-overlapping intervals, $x_k \in E \cap I_k$ and $d(I_k) < \delta(x_k)$.

Let $F$ be an interval function, $C \in \Gamma \cup \dot\Gamma$ and $E \subseteq \mathbb{R}^n$. Then $F$ satisfies the *null condition corresponding to $C$ on $E$*, in short $F$ satisfies $\mathcal{N}(C, E)$, if the following is true: $\forall \varepsilon > 0, K > 0\ \exists \Delta > 0,\ \delta \colon E \to \mathbb{R}^+$ such that $\sum |F(I_k)| \leqslant \varepsilon$ holds for any $(E, \delta)$-fine sequence $\{(x_k, I_k)\}$ with $\{I_k\} \in C(K, \Delta)$.

**Remark 1.2.** Let $E \subseteq \mathbb{R}^n$ and let $F$ be an additive interval function which is differentiable at each point $x \in E$. Then $F$ satisfies $\mathcal{N}(C^*, E)$ with $C^* = C^{n-1}_1$. Indeed, if $\varepsilon > 0$ and $K > 0$ are given then set $\Delta = 1$ and let $x \in E$. By the differentiability of $F$ at $x$ there exist positive numbers $K(x)$ and $\delta(x)$ such that $|F(I)| \leqslant K(x) d(I)^n$ holds for any interval $I$ containing $x$ and having diameter less than $\delta(x)$ (cf., e.g., [Ku-Jar, Cor. 1]). We obviously may assume $\delta(x) \leqslant \varepsilon / K K(x)$ for $x \in E$, and thus we conclude for any $(E, \delta)$-fine sequence $\{(x_k, I_k)\}$ with $\{I_k\} \in C^*(K, \Delta)$ (i.e. $\sum d(I_k)^{n-1} \leqslant K$):

$$\sum |F(I_k)| \leqslant \sum K(x_k) \delta(x_k) d(I_k)^{n-1} \leqslant \varepsilon$$

as desired.

A *division of a set* $M \subseteq \mathbb{R}^n$ consists of a set $E'$ and a sequence of pairs $(E_i, C_i)_{i \in \mathbb{N}}$ such that $M$ is the disjoint union of all the sets $E_i$ and $E'$, $|M - E'|_n = 0$, $C_i \in \Gamma \cup \dot\Gamma$ and $E_i \in \mathcal{E}(C_i)$.

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function with compact support. Then $f$ is said to be $\nu_1$-*integrable* if there exists an additive interval function $F$ with $F' = f$ a.e. and a division $E'$, $(E_i, C_i)_{i\in\mathbb{N}}$ of $\mathbb{R}^n$ such that $F$ is differentiable on $E'$ and satisfies $\mathcal{N}(C_i, E_i)$ for $i \in \mathbb{N}$ as well as $\mathcal{N}(C^*, E_i)$ if $C_i \in \dot{\Gamma}$. In this case $F$ is uniquely determined, has compact support, and we write ${}^{\nu_1}\!\!\int f = F(\mathbb{R}^n)$, cf. Remark 1.1, 1.2 and [Ju-No 1, Sec. 5].

Suppose $A$ to be a bounded measurable subset of $\mathbb{R}^n$ and $f\colon A \to \mathbb{R}$. Then we define the function $f_A\colon \mathbb{R}^n \to \mathbb{R}$ by $f_A(x) = f(x)$ if $x \in A$ and zero elsewhere. We say that $f$ is $\nu_1$-*integrable on* $A$ if $f_A$ is $\nu_1$-integrable, and in this case we write ${}^{\nu_1}\!\!\int_A f = {}^{\nu_1}\!\!\int f_A$.

**Remark 1.3.** For the properties of the $\nu_1$-integral we refer the reader to [Ju-No 1, Sec. 5] and [Ju-No 2]. In particular, the $\nu_1$-integral extends the Lebesgue integral, and therefore the definition of $\nu_1$-integrability also applies to functions which are defined only a.e. Furthermore, if $f$ is $\nu_1$-integrable and if $F$ denotes the corresponding interval function then $f$ is $\nu_1$-integrable on any interval $I$ and $F(I) = {}^{\nu_1}\!\!\int_I f$.

**Remark 1.4.** Let $I$ be an interval in $\mathbb{R}^n$, $x \in I^\circ$, $C \in \Gamma \cup \dot{\Gamma}$, $E \subseteq \mathbb{R}^n$. Then it is clear what we mean by an (additive) interval function on $I$, which is differentiable at $x$ or which satisfies $\mathcal{N}(C, E)$ (just require all intervals occuring in the definitions given above to lie in $I$). By [Ju-No 1, Remark 5.1 (iii)] we see that a function $f\colon I \to \mathbb{R}$ is $\nu_1$-integrable on $I$ iff there exists an additive interval function $F$ on $I$ with $F' = f$ a.e. on $I$ and a division $E'$, $(E_i, C_i)_{i\in\mathbb{N}}$ of $I$ with $E' \subseteq I^\circ$ and such that $F$ is differentiable on $E'$ and satisfies $\mathcal{N}(C_i, E_i)$ for $i \in \mathbb{N}$ as well as $\mathcal{N}(C^*, E_i)$ if $C_i \in \dot{\Gamma}$. Furthermore, in that case $F$ is uniquely determined and $F(J) = {}^{\nu_1}\!\!\int_J f$ for each subinterval $J$ of $I$, cf. also [Ju-No 2, Sec. 1].

Suppose $G$ to be an open subset of $\mathbb{R}^n$, $f\colon G \to \mathbb{R}$ and let $F$ be an interval function on $\mathbb{R}^n$. Then $F$ and $f$ are said to be $\nu_1$-*associated in* $G$ if $f$ is $\nu_1$-integrable on any interval $I$ contained in $G$ and $F(I) = {}^{\nu_1}\!\!\int_I f$.

Let $M$ be an arbitrary subset of $\mathbb{R}^n$ and $F$ an interval function on $\mathbb{R}^n$. We say that $F$ satisfies a *condition* $(N)$ *on* $M$ if there exists a division $E'$, $(E_i, C_i)_{i\in\mathbb{N}}$ of $M$ such that $F$ is differentiable on $E'$ and satisfies $\mathcal{N}(C_i, E_i)$ for $i \in \mathbb{N}$ as well as $\mathcal{N}(C^*, E_i)$ if $C_i \in \dot{\Gamma}$.

**Theorem 1.** *Suppose $A$ to be a bounded measurable subset of $\mathbb{R}^n$, $f\colon A \to \mathbb{R}$ and let $F$ be an interval function on $\mathbb{R}^n$. Then $f$ is $\nu_1$-integrable on $A$ and $F(I) = {}^{\nu_1}\!\!\int_I f_A$ for each interval $I$ in $\mathbb{R}^n$ iff the following conditions are satisfied:*

(i) *$F$ is additive and $F(I) = 0$ for each interval $I \subseteq \mathbb{R}^n - \mathrm{cl}_r A$,*

(ii) *$F$ and $f$ are $\nu_1$-associated in $A^\circ$,*

(iii) *$F$ satisfies a condition $(N)$ on $\partial_r A$ and $F' = f_A$ a.e. on $\partial_r A$.*

*In any case $F'$ exists a.e. on $A$, $F'$ is $\nu_1$-integrable on $A$ and*

$$F(\mathbb{R}^n) = {}^{\nu_1}\!\!\int_A F' = {}^{\nu_1}\!\!\int_A f.$$

Proof. Assume first $f$ to be $\nu_1$-integrable on $A$ and $F(I) = {}^{\nu_1}\!\!\int_I f_A$ for each interval $I$. By definition there is an additive interval function $G$ with $G' = f_A$ a.e. and a corresponding division. Consequently, by Remark 1.3 we see that $F = G$, and $F$ and $f$ are $\nu_1$-associated in $A^\circ$. Let $I \subseteq \mathbb{R}^n - \mathrm{cl}_r\, A$ be an interval and observe that $f_A = 0$ a.e. on $I$. Thus, again by Remark 1.3, ${}^{\nu_1}\!\!\int_I f_A = 0$. To see that $F$ satisfies a condition $(N)$ on $\partial_r A$ one only has to intersect the division corresponding to $G$ with $\partial_r A$ (note that $E \in \mathcal{E}(C)$ implies $\tilde{E} \in \mathcal{E}(C)$ for any $\tilde{E} \subseteq E$, $C \in \Gamma \cup \dot{\Gamma}$). Furthermore, since $F' = f$ a.e. on $A$ the $\nu_1$-integrability of $F'$ on $A$ follows and ${}^{\nu_1}\!\!\int_A F' = {}^{\nu_1}\!\!\int_A f = {}^{\nu_1}\!\!\int f_A = F(\mathbb{R}^n)$.

Conversely, suppose the conditions (i)–(iii) to be satisfied. We express $A^\circ$ as an at most countable union of non-overlapping cubes $I_i$ $(i \geqslant 1)$. By (ii) and Remark 1.4 $f$ is $\nu_1$-integrable on $I_i$ and $F$ satisfies a condition $(N)$ on $I_i^\circ$ (intersect the division of $I_i$ according to Remark 1.4 with the interior of $I_i$). Note that in particular $F' = f$ a.e. on $I_i^\circ$, hence a.e. on $A^\circ$, and since $F' = 0$ on $\mathbb{R}^n - \mathrm{cl}_r\, A$ by (i) we have $F' = f_A$ a.e. Since any $I_i$ is contained in the interior of $A$ we can slightly enlarge $I_i$ to a cube $J_i$ still lying in the interior of $A$ and containing $I_i$ in its interior. The same argument as before yields that $F$ satisfies a condition $(N)$ on $\partial I_i$ and consequently on $\partial I_i - \bigcup\limits_{j=1}^{i-1} \partial I_j$, $i \geqslant 1$. Now, taking into account the set $\mathbb{R}^n - \mathrm{cl}_r\, A$ and all divisions according to the conditions $(N)$ satisfied by $F$ on $\partial_r A$, $I_i^\circ$ and $\partial I_i - \bigcup\limits_{j=1}^{i-1} \partial I_j$ $(i \geqslant 1)$, we see that $F$ satisfies a condition $(N)$ on $\mathbb{R}^n$, and thus by definition $f_A$ is $\nu_1$-integrable. Again by Remark 1.3 we have $F(I) = {}^{\nu_1}\!\!\int_I f_A$ for any interval $I$ in $\mathbb{R}^n$. □

## 2. Relations to other integrals

In this section we assume $I$ to be a fixed interval in $\mathbb{R}^n$ and $f$ to be a fixed real-valued function defined on $I$. We will prove that if $f$ is *variationally integrable on $I$* in the sense of [Pf] then $f$ is $\nu_1$-integrable on $I$ and both integrals coincide.

A bounded measurable set $A \subseteq \mathbb{R}^n$ is called a $BV$ set if $|\partial_e A|_{n-1}$ is finite (see [Pf], [Fed]), and for any $BV$ set $A$ we define its regularity by $r(A) = |A|_n / d(A)|\partial_e A|_{n-1}$ if $d(A)|\partial_e A|_{n-1} > 0$ and by $r(A) = 0$ else. We denote by $BV_I$ the system of all $BV$ sets contained in $I$, and a function $F\colon BV_I \to \mathbb{R}$ is called *continuous* if for every

$\varepsilon > 0$ there is a $\delta > 0$ such that $|F(B)| < \varepsilon$ for each $B \in BV_I$ with $|B|_n < \delta$ and $|\partial_e B|_{n-1} < 1/\varepsilon$.

A function $F\colon BV_I \to \mathbb{R}$ is said to be *superadditive* if $\sum F(B_k) \leqslant F(B)$ for any $B \in BV_I$ and any finite sequence of disjoint $BV$ sets $B_k$ whose union is $B$. $F$ is called *additive* if $F$ and $-F$ are both superadditive.

Let $F\colon BV_I \to \mathbb{R}$, $\varepsilon > 0$ and a $\sigma_{n-1}$-finite set $T$ be given. Then an *$\varepsilon$-majorant* of the pair $(f, F)$ in $I \bmod T$ is a non-negative superadditive function $M\colon BV_I \to \mathbb{R}$ satisfying the following conditions: $M(I) < \varepsilon$, and for each $x \in I - T$ there exists a $\delta > 0$ such that $\big|F(B) - f(x)|B|_n\big| \leqslant M(B)$ for any $B \in BV_I$ with $x \in \operatorname{cl} B$, $d(B) < \delta$, $r(B) > \varepsilon$.

We call $f$ *$v$-integrable on $I$* if there is a continuous additive function $F\colon BV_I \to \mathbb{R}$ and a $\sigma_{n-1}$-finite set $T$ such that for any $\varepsilon > 0$ there is an $\varepsilon$-majorant of $(f, F)$ in $I \bmod T$. In this case $F$ is uniquely determined, and we write ${}^{v}\!\!\int_I f = F(I)$, cf. [Pf, Def. 5.1, Cor. 5.5].

**Proposition 1.** *Suppose $f$ to be $v$-integrable on $I$. Then $f$ is $\nu_1$-integrable on $I$ and ${}^{v}\!\!\int_I f = {}^{\nu_1}\!\!\int_I f$.*

P r o o f. We assume $f$ to be $v$-integrable, and we denote by $F$ the corresponding continuous additive function $F$ on $BV_I$ and by $T$ a corresponding $\sigma_{n-1}$-finite set. Note that $F(B) = 0$ for any $B \in BV_I$ with $|B|_n = 0$, hence $F$ is an additive interval function on $I$ (in the sense of Section 1).

First we show that $F$ satisfies $\mathcal{N}(C_1^{n-1}, I)$: let $\varepsilon > 0$, $K > 0$ be given, set $\Delta = 1$, $\varepsilon' = \frac{1}{2}\min(\varepsilon, 1/2nK)$ and determine for $\varepsilon'$ a $\delta' > 0$ in virtue of the continuity of $F$. We set $\delta(\cdot) = \delta'/2K$ on $I$, and we assume $\{(x_k, I_k)\}$ to be an $(I, \delta)$-fine sequence with $I_k \subseteq I$ and $\{I_k\} \in C_1^{n-1}(K, \Delta)$ (i.e. $\sum d(I_k)^{n-1} \leqslant K$). Then $|\bigcup I_k|_n \leqslant \sum \delta(x_k) d(I_k)^{n-1} < \delta'$ and

$$\Big|\bigcup \partial I_k\Big|_{n-1} \leqslant \sum |\partial I_k|_{n-1} \leqslant 2nK < 1/\varepsilon',$$

thus

$$\sum |F(I_k)| = F\Big(\bigcup_{F(I_k) \geqslant 0} I_k\Big) - F\Big(\bigcup_{F(I_k) < 0} I_k\Big) \leqslant 2\varepsilon' \leqslant \varepsilon.$$

Let $E \subseteq I$ with $|E|_n = 0$ and $\varepsilon > 0$ be given. Then we can determine $\delta\colon E \to \mathbb{R}^+$ such that $\sum |f(x_k)||I_k|_n \leqslant \varepsilon$ holds for any $(E, \delta)$-fine sequence $\{(x_k, I_k)\}$ with $I_k \subseteq I$. Indeed, write $E = \bigcup_{j \in \mathbb{N}} E_j$ with $E_j = \{x \in E\colon j - 1 \leqslant |f(x)| < j\}$, let $\varepsilon > 0$ be given and determine open sets $G_j \supseteq E_j$ with $|G_j|_n \leqslant \varepsilon / j\, 2^j$. For $x \in E_j$ we choose a

$\delta(x) > 0$ such that $B(x, \delta(x)) \subseteq G_j$, which defines the function $\delta$. Consequently, for any $(E, \delta)$-fine sequence $\{(x_k, I_k)\}$ with $I_k \subseteq I$ we get

$$\sum |f(x_k)||I_k|_n \leqslant \sum_{j\in\mathbb{N}} \sum_{x_k\in E_j} j|I_k|_n \leqslant \sum_{j\in\mathbb{N}} j|G_j|_n \leqslant \varepsilon.$$

To prove that $f$ is $\nu_1$-integrable on $I$ with ${}^{\nu_1}\!\!\int_I f = F(I) \quad (= {}^{v}\!\!\int_I f)$, we verify the constructive definition of our $\nu_1$-integral, see [Ju-No 2, Thm. 3.1]. Express $I \cap (T \cup \partial I)$ as a disjoint countable union of sets $E_i$ with $|E_i|_{n-1} < \infty \quad (i \in \mathbb{N})$, and note that $E' = I - (T \cup \partial I)$, $(E_i, C_1^{n-1})_{i\in\mathbb{N}}$ is a division of $I$ with $E' \subseteq I^\circ$. Now let, according to [Ju-No 2, Thm. 3.1], $\varepsilon > 0$, $K > 0$, $K_i > 0$ $(i \in \mathbb{N})$ be given, set $\Delta_i = 1$, $\varepsilon' = \frac{1}{5}\min(\varepsilon, 1/nK)$ and choose an $\varepsilon'$-majorant $M$ of $(f, F)$ in $I \bmod T$ which, by definition, yields a function $\delta\colon E' \to \mathbb{R}^+$. Since $|I - E'|_n = 0$ we can also determine a $\delta\colon I - E' \to \mathbb{R}^+$ such that $\sum |f(x_k)||I_k|_n \leqslant \varepsilon/5$ for any $(I - E', \delta)$-fine sequence $\{(x_k, I_k)\}$ with $I_k \subseteq I$. Obviously we may assume $\delta(\cdot) \leqslant \varepsilon/5K(1 + |f(\cdot)|)$ on $I$, and since $F$ satisfies $\mathcal{N}(C_1^{n-1}, E_i)$ resp. $\mathcal{N}(C_1^{n-1}, E')$ we can determine for $\varepsilon/5\, 2^i$ and $K_i$ resp. for $\varepsilon/5$ and $K$ a corresponding function $\delta_i\colon E_i \to \mathbb{R}^+$ resp. $\delta'\colon E' \to \mathbb{R}^+$, and we also may assume $\delta(\cdot) \leqslant \delta_i(\cdot)$ on $E_i$ resp. $\delta(\cdot) \leqslant \delta'(\cdot)$ on $E'$. Thus a function $\delta$ is defined on $I$, and we denote by $\{(x_k, I_k)\} \cup \{(x'_k, I'_k)\}$ an $(I, \delta)$-fine sequence with $I = \bigcup I_k \cup \bigcup I'_k$ fulfilling the conditions

(i) if $x_k \in E'$ then $d(I_k)^n \leqslant K|I_k|_n$; $\quad \{I_k : x_k \in E_i\} \in C_1^{n-1}(K_i)$ $(i \in \mathbb{N})$

(ii) $\{I'_k\} \in C_1^{n-1}(K)$ and $x'_k \in E'$ for all $k$.

Observing that $r(I_k^\circ) \geqslant 1/2nK > \varepsilon'$ for $x_k \in E'$ we conclude:

$$\begin{aligned}
&\Big|F(I) - \Big(\sum f(x_k)|I_k|_n + \sum f(x'_k)|I'_k|_n\Big)\Big| \\
&\quad\leqslant \sum_{x_k\in E'} |F(I_k^\circ) - f(x_k)|I_k^\circ|_n| + \sum_{i\in\mathbb{N}} \sum_{x_k\in E_i} |F(I_k)| \\
&\qquad + \sum |F(I'_k)| + \sum_{x_k\in I-E'} |f(x_k)||I_k|_n + \sum |f(x'_k)|\delta(x'_k)d(I'_k)^{n-1} \\
&\quad\leqslant \sum_{x_k\in E'} M(I_k^\circ) + \sum_{i\in\mathbb{N}} \frac{\varepsilon}{5\cdot 2^i} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5K}\sum d(I'_k)^{n-1} \\
&\quad\leqslant M(I) + \frac{4}{5}\varepsilon \leqslant \varepsilon,
\end{aligned}$$

which completes the proof. □

**Remark 2.1.** In [Jar-Ku] a further $n$-dimensional non-absolutely convergent integral is introduced, the so called $PU$-integral. Assume the support of a function $g\colon \mathbb{R}^n \to \mathbb{R}$ to be contained in $I$, and suppose $g$ to be $PU$-integrable. Then according to [Jar-Ku, Thm. 6.1], [JKS] $g$ is $M_1$-integrable on $I$, consequently by [Ju-No 2, Prop. 4.1] $g$ is $\nu_1$-integrable on $I$ and all integrals coincide. For a comparison of related integration processes see [No].

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