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EDGE-DOMATIC NUMBERS OF DIRECTED GRAPHS

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In [1] E. J. Cockayne and S. T. Hedetniemi introduced the domatic number of an undirected graph G as the maximum number of classes of a partition of the vertex set of G into dominating sets. Many variants of this number have been later studied, among them the edge-domatic number of an undirected graph [2]. Here we will study an analogous concept for directed graphs. The adjacency of edges in a directed graph will be introduced analogously to the paper [3].

We consider finite directed graphs (shortly digraphs) without loops in which two vertices may be joined by two edges only if these edges are oppositely directed.

Two edges of a digraph G will be called adjacent, if the terminal vertex of one of them is the initial vertex of the other. A subset D of the edge set $E(G)$ of G is called edge-dominating, if for each edge $e \in E(G) - D$ there exists an edge $f \in D$ adjacent to e. A partition of $E(G)$ is called an edge-domatic partition of G, if all of its classes are edge-dominating sets in G. The maximum number of classes of an edge-domatic partition of G is called the edge-domatic number of G and denoted by $\operatorname{ed}(G)$.

Sometimes it is more convenient to speak about edge-domatic colourings instead of edge-domatic partitions. A colouring of edges of a digraph G is called edge-domatic, if each edge is adjacent in G to edges of all colours different from its own. (Two adjacent edges may be coloured by the same colour.) Then the edge-domatic number of G is equal to the maximum number of colours of an edge-domatic colouring of G. Equivalence of this definition with the previous one is evident.

The edge-domatic number of a directed graph G is evidently equal to the domatic number of the graph $L(G)$ whose vertex set is the edge set of G and in which two vertices are adjacent if and only if they are adjacent as edges in G. Thus the following proposition follows directly from the results of E. J. Cockayne and S. T. Hedetniemi.

Proposition 1. *Let C_n be the directed cycle of length n. If $n \equiv 0 \pmod 3$, then $\operatorname{ed}(C_n) = 3$, otherwise $\operatorname{ed}(C_n) = 2$.*

Now by C_n^2 we denote the graph obtained from an undirected circuit of length n by replacing each undirected edge by a pair of oppositely directed edges.

Theorem 1. *Let n be an integer, $n \geqslant 3$. If $n \equiv 0 \pmod 4$, then $\operatorname{ed}(C_n^2) = 4$, otherwise $\operatorname{ed}(C_n^2) = 3$.*

Proof. First we shall construct an edge-domatic colouring of any graph C_n^2 by 3 colours. Let the vertices of C_n^2 be $v_0, \ldots, v_{n-1}$ and let the notation of edges be as usual. Let $p = [\frac{1}{3}n] - 1$. For $i = 0, \ldots, p$ the edges $v_{3i}v_{3i+1}$ and $v_{3i+3}v_{3i+2}$ will be coloured by 1, the edges $v_{3i+1}v_{3i}$ and $v_{3i+2}v_{3i+1}$ by 2 and the edges $v_{3i+1}v_{3i+2}$ and $v_{3i+2}v_{3i+1}$ by 3. If $n \equiv 0 \pmod 3$, then all edges are already coloured. If $n \equiv 1 \pmod 3$, we colour the edges $v_{n-1}v_0$ and v_0v_{n-1} by 3. If $n \equiv 2 \pmod 3$, we colour $v_{n-1}v_0$ by 1, v_0v_{n-1} by 2 and $v_{n-2}v_{n-1}$ and $v_{n-1}v_{n-2}$ by 3. The reader may verify that this colouring is edge-domatic and thus $\operatorname{ed}(C_n^2) \geqslant 3$ for each $n \geqslant 3$.

Now suppose that $n \equiv 0 \pmod 4$. We shall construct an edge-domatic colouring of C_n^2 by 4 colours. Let $p = [\frac{1}{4}n] - 1$. For $i = 0, \ldots, p$ the edges $v_{4i}v_{4i+1}$ and $v_{4i+3}v_{4i+2}$ will be coloured by 1, the edges $v_{4i+1}v_{4i+2}$ and $v_{4i+4}v_{4i+3}$ by 2, the edges $v_{4i+2}v_{4i+3}$ and $v_{4i+1}v_{4i}$ by 3 and the edges $v_{4i+3}v_{4i+4}$ and $v_{4i+2}v_{4i+1}$ by 4. Again we have an edge-domatic colouring and thus $\operatorname{ed}(C_n^2) \geqslant 4$ for $n \equiv 0 \pmod 4$. As each edge is adjacent only to three other edges, this number cannot be greater and therefore $\operatorname{ed}(C_n^2) = 4$ for $n \equiv 0 \pmod 4$.

Now suppose that there exists an edge-domatic colouring of C_n^2 by 4 colours for some n. As each edge is adjacent only to three others, no two adjacent edges may have the same colour. Neither may two edges having a common adjacent edge have the same colour. Without loss of generality let v_0v_1 be coloured by 1. The edges v_1v_0 and v_1v_2 are adjacent to v_0v_1 and therefore they cannot be coloured by 1. Without loss of generality let v_1v_2 be coloured by 2. As both v_1v_2, v_1v_0 are adjacent to v_0v_1, the edge v_1v_0 cannot have the colour 2; without loss of generality let it have the colour 3. The edge v_2v_1 is adjacent to v_1v_0 coloured by 3 and to v_1v_2 coloured by 2 and thus it cannot be coloured by 2 or 3. Further, both v_2v_1 and v_0v_1 are adjacent to v_1v_0; v_0v_1 is coloured by 1 and thus v_2v_1 must be coloured by 4. The edge v_1v_2 is coloured by 2 and adjacent to v_0v_1 coloured by 1, to v_2v_1 coloured by 4 and to v_2v_3; this implies that v_2v_3 must be coloured by 3. The edge v_3v_2 is adjacent to v_2v_3 coloured by 3 and to v_2v_1 coloured by 4. Further, both v_3v_2 and v_1v_2 coloured by 2 are adjacent to v_2v_1; hence v_3v_2 must be coloured by 1. In an analogous way we prove that v_3v_4 must be coloured by 4 and v_4v_3 by 2. If we continue in this way, everything is cyclically repeated. This implies that n is divisible by 4 and the above described edge-domatic colouring of C_n^2 for $n \equiv 0 \pmod 4$ is the single (up to the change of notation of colours) edge-domatic colouring of C_n^2 by 4 colours. □

Now we shall consider a complete digraph with n vertices; we denote it by DK_n. First we prove a lemma.

Lemma 1. *Let G be a directed graph. let D be an edge-dominating set in G. Let u be a vertex of G which is not an initial vertex of any edge from D. Then the number of elements of D is greater than or equal to the indegree of u.*

Proof. By $\Gamma^{-1}(u)$ denote the set of the initial vertices of all edges of G whose terminal vertex is u. Let $v \in \Gamma^{-1}(u)$. Then either $vu \in D$ or v is the terminal vertex of at least one edge from D. We define a mapping $f\colon \Gamma^{-1}(u) \to D$ in the following way. If $v \in \Gamma^{-1}(u)$ and $vu \in D$, then $f(v) = vu$. If $v \in \Gamma^{-1}(u)$ and $vu \notin D$, then $f(v)$ is an arbitrary chosen edge which is in D and whose terminal vertex is v. It is evident that f is an injection of $\Gamma^{-1}(u)$ into D and this implies $|\Gamma^{-1}(u)| \leqslant |D|$. As $|\Gamma^{-1}(u)|$ is the indegree of u, the assertion is proved. □

Dually we can prove the following lemma.

Lemma 1′. *Let G be a directed graph, let D be an edge-dominating set in G. Let u be a vertex of G which is not a terminal vertex of any edge from D. Then the number of elements of D is greater than or equal to the outdegree of u.*

Now we prove a theorem.

Theorem 2. *For every integer $n \geqslant 2$ we have $\operatorname{ed}(DK_n) = n$.*

Proof. Let D be an edge-dominating set in DK_n. If every vertex of DK_n is the initial vertex of an edge from D and simultaneously also the terminal vertex of an edge from D, then the subgraph of DK_n formed by the edges from D is a spanning subgraph of DK_n in which all indegrees and all outdegrees are non-zero. The number of edges of such a graph is at least n and thus $|D| \geqslant n$. If there exists a vertex u of DK_n which is not the initial vertex of an edge from D, we use Lemma 1. The indegree of u is $n-1$ and thus $|D| \geqslant n-1$. If there exists a vertex u of DK_n which is not the terminal vertex of an edge from D, we use Lemma 1' and obtain again $|D| \geqslant n-1$. Therefore an edge-domating set in DK_n has at least $n-1$ elements. The graph DK_n has $n(n-1)$ edges and thus $\operatorname{ed}(DK_n) \leqslant n$. A partition of $E(DK_n)$, each of whose classes is the set of all edges outgoing from a vertex, is an edge-domatic partition of DK_n having n classes; hence $\operatorname{ed}(DK_n) = n$. □

Now we will study a special case of edge-domatic partitions. An edge-domatic partition D of a digraph G will be called a CM-partition, if all classes of D are complete matchings of G (considered regardless of the orientation).

The next theorem will concern cube graphs. The cube graph of dimension n, where n is a positive integer, is an undirected graph Q_n whose vertex set $V(Q_n)$ is the set of all n-dimensional Boolean vectors (i.e. vectors having all coordinates from the set $\{0,1\}$) and in which two vertices are adjacent if and only if they differ in exactly one coordinate.

Theorem 3. *For every positive integer n the cube graph Q_n can be directed in such a way that the resulting digraph has a CM-partition. If n is even then, moreover, the resulting digraph is regular (as a digraph) of degree $\frac{1}{2}n$.*

Proof. Consider Q_n for n even. For $i = 1, \ldots, n$ let D_i be the set of all edges of Q_n whose end vertices differ in the i-th coordinate. Evidently the sets D_i for $i = 1, \ldots, n$ are pairwise disjoint and they all have the same number of elements. Now we shall direct the edges of Q_n. Denote $p = \frac{1}{2}n - 1$. For $i = 0, 1, \ldots, p$ each edge from $D_{2i+1} \cup D_{2i+2}$ has exactly one end vertex $(v_1, \ldots, v_n)$ such that $v_{2i+1} = v_{2i+2}$. If an edge belongs to D_{2i+1} (or to D_{2i+2}), then it will be directed in such a way that such a vertex will be its initial (or terminal, respectively) vertex. The digraph thus obtained is regular, because each vertex has the property that for any $i \in \{0, \ldots, p\}$ it is either the initial vertex of an edge from D_{2i+1} and the terminal vertex of an edge from D_{2i+2}, or the terminal vertex of an edge from D_{2i+1} and the initial vertex of an edge from D_{2i+2}.

Now take an edge e of Q_n. If $e \in D_{2i+1}$ for some $i = 0, \ldots, p$, then it is adjacent to two edges from D_{2i+2}; one of them comes into its initial vertex, the other goes from its terminal vertex. For each $k \in \{1, \ldots, n\}$ different from $2i + 1$ and $2i + 2$ the end vertices of e are either both initial, or both terminal vertices of edges from D_k; in both cases e is adjacent to an edge from D_k. Therefore D_{2i+1} is an edge-dominating set for any $i \in \{0, \ldots, p\}$. Analogously we can prove that so is D_{2i+2} for any $i \in \{0, \ldots, p\}$. Therefore $\{D_1, \ldots, D_n\}$ is an edge-domatic partition of the digraph obtained by the just described directing of edges of Q_n. We have proved the assertion for n even. If n is odd, then we direct the edges of $E(Q_n) - D_n$ in the above described way. Then we direct the edges from D_n in such a way that for any of them the initial vertex has the last coordinate equal to zero. It is easy to see that we obtain the required digraph. □

Obviously not every graph having a complete matching can be directed to have a CM-partition.

Theorem 3. *A tournament with n vertices has a CM-partition if and only if $n = 2$.*

Proof. For $n = 2$ the assertion holds trivially. If n is odd, then a complete graph with n vertices cannot have a complete matching. Consider $n = 4$. Take the undirected complete graph K_4 and try to direct its edges to obtain a tournament with a CM-partition. Let v_1, v_2, v_3, v_4 be the vertices of K_4. All decompositions of K_4 into complete matchings are isomorphic; therefore without loss of generality we may colour the edges v_1v_2, v_3v_4 by 1, v_1v_4, v_2v_3 by 2, v_1v_3, v_2v_4 by 3. Again without loss of generality we may choose the orientation of edges coloured by 1 from v_1 to v_2 and from v_3 to v_4. The edge joining v_2, v_3 must be directed from v_2 to v_3; otherwise it would not be adjacent to any edge coloured by 1. Analogously there is an edge from v_4 to v_1. Now suppose that there is an edge from v_1 to v_3 (coloured by 3). If there is an edge from v_2 to v_4 (also coloured by 3), then the edge v_2v_3 is adjacent to no edge coloured by 3; it there is an edge from v_4 to v_2, then v_1v_2 is adjacent to no edge coloured by 3. The case when there is an edge from v_4 to v_2 is analogous. Therefore the assertion holds for $n = 4$. Further we proceed by induction. If $n \geqslant 3$, then $n = d \cdot 2k$, where k is a non-negative integer and either d is odd and $d \geqslant 3$, or $d = 4$. If $k = 0$, then n is odd or $n = 4$; for these cases the proof has been already done. Let $n = d \cdot 2^k$ for $k \geqslant 1$ and suppose that for $n = d \cdot 2^{k-1}$ the assertion is true. Let T be a tournament with n vertices, let v be a vertex of T. Let R be the set of all terminal vertices of edges outgoing from v, let $|R| = r$. Let these edges be coloured by the colours $1, \ldots, r$. Let e be an edge joining two vertices from R, let w be its terminal vertex. The edge e cannot be coloured by the same colour as vw, because then the edges coloured by this colour would not form a matching. If e is coloured by any other colour than that by which an edge outgoing from v is coloured, then vw is adjacent to no edge coloured by this colour. Therefore all edges of the subtournament T_0 induced by R must be coloured by the $n - r - 1$ colours $r + 1, \ldots, n - 1$ and for each of these colours the set of edges which are coloured by it must form a matching of this subtournament. If $r \geqslant \frac{1}{2}n + 1$, then $n - r - 1 \leqslant \frac{1}{2}n - 2 \leqslant r - 3$ and this is evidently impossible. If $r = \frac{1}{2}n$, then $r = d \cdot 2^{k-1}$. Suppose that there exists a decomposition of T_0 into $n - r - 1 = \frac{1}{2}n - 1$ matchings. Then all these matchings are complete. If $i \in \{r + 1, \ldots, n - 1\}$ and an edge e of T_0 is not coloured by it, then it must be adjacent to an edge f coloured by i. This edge f must be in T_0, because the edges coloured by i form a matching of T_0 and thus no vertex of T_0 can be incident with an edge coloured by i and not belonging to T_0. This implies that T_0 must have a CM-partition, which contradicts the induction hypothesis. If $r \leqslant \frac{1}{2}n - 1$, then there are at least $\frac{1}{2}n$ edges incoming into v and the proof can be done dually. $\square$

There is another interesting case of edge-domatic partitions. Let G be an undirected graph regular of degree $2k$ and let there exist a partition of the edge set of

G into $2k+1$ matchings. It would be interesting to find a condition under which G could be directed in such a way it becomes a regular digraph of degree k and the partition becomes an edge-domatic partition of the resulting digraph; we will call it an MM-partition. The importance of the MM-partition is in the fact that in a regular digraph G of degree k every edge is adjacent to $2k$ edges and therefore the edge-domatic number of G cannot be greater that $2k+1$.

Problem. Does there exist a graph of this kind for every positive integer k?

We will show only two examples.

Example 1. For $k=1$ the k-regular digraphs with MM-partitions are all directed cycles whose lengths are divisible by 3.

Example 2. For $k=2$ a k-regular digraph with an MM-partition is given by the following matrix:

$$\begin{bmatrix}
0 & +2 & 0 & -5 & +1 & 0 & 0 & 0 & 0 & -4 \\
-2 & 0 & +3 & 0 & 0 & +1 & 0 & 0 & -5 & 0 \\
0 & -3 & 0 & +4 & 0 & 0 & +1 & 0 & 0 & -2 \\
+5 & 0 & -4 & 0 & 0 & 0 & 0 & +1 & -3 & 0 \\
-1 & 0 & 0 & 0 & 0 & +2 & 0 & -5 & 0 & +3 \\
0 & -1 & 0 & 0 & -2 & 0 & +3 & 0 & +4 & 0 \\
0 & 0 & -1 & 0 & 0 & -3 & 0 & +4 & 0 & +5 \\
0 & 0 & 0 & -1 & +5 & 0 & -4 & 0 & +2 & 0 \\
0 & +5 & 0 & +3 & 0 & -4 & 0 & -2 & 0 & 0 \\
+4 & 0 & +2 & 0 & -3 & 0 & -5 & 0 & 0 & 0
\end{bmatrix}.$$

This matrix describes already the digraph and its edge-domatic colouring. If $k \in \{1,2,3,4,5\}$, then the symbol $+k$ (or $-k$) in the i-th row and the j-th column means that there exists an edge from the i-th to the j-th vertex (or from the j-th to the i-th vertex, respectively) coloured by the colour k. The symbol 0 means that these vertices are not adjacent.

At the end we will prove a theorem concerning tournaments.

Theorem 4. *A tournament T has an MM-partition if and only if it is a directed cycle of length* 3.

Proof. If T is a directed cycle of length 3, then the assertion is true (see Example 1 and Proposition 1). Let n be the number of vertices of T. The tournament T can be a regular digraph of degree k only if $n=2k+1$; therefore n must be odd. For $k=1$ we have $n=3$; then T is either a directed cycle of length 3, or the acyclic tournament with 3 vertices. The former case was already mentioned, in the latter T is not regular. The case $n = 4$ is impossible, because n must be odd. Let $n \geqslant 5$; then $k \geqslant 2$. Suppose that there exists an MM-partition $\mathscr{P}$ of T and let its classes be coloured by the colours $1, \ldots, 2k + 1$. There exist at least two edges coloured by 1; let e_1, e_2 be two of them. Let the initial vertices of e_1, e_2 be u_1, u_2, their terminal vertices v_1, v_2. As T is a tournament, there exists either the edge u_1v_2, or the edge v_2u_1. In the first case the edge u_1v_2 is adjacent to no edge coloured by 1. In the other the edge v_2u_1 is adjacent to two edges coloured by 1, namely e_1 and e_2; therefore it is adjacent to edges of at most $2k - 1$ colours. As there are $2k + 1$ colours, there exists a colour by which neither v_2u_1, not any edge adjacent to it is coloured. In both cases we have a contradiction with the assumption that $\mathscr{P}$ is an MM-partition. $\square$

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