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REMARKS ON INEQUALITIES OF POINCARÉ TYPE

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1. INTRODUCTION

Let Ω be a domain in $\mathbb{R}^N$, let $m \in \mathbb{N}$, suppose that $X = X(\Omega)$ and $Y = Y(\Omega)$ are Banach function spaces in the sense of Luxemburg (see [14] and Section 2 below for details), and let $W^m = W^m(X, Y)$ be the abstract Sobolev space consisting of all $f \in X$ such that for all multi-indices α of length m, the distributional derivative $D^\alpha f$ belongs to Y. Furnished with the norm

$$f \longmapsto \|f\|_{W^m} := \|f\|_X + \|\nabla^m f\|_Y$$

where $\|\nabla^m f\|_Y := \sum\limits_{|\alpha|=m} \|D^\alpha f\|_Y$, W^m is a Banach space. The main object of this paper is to obtain useful sufficient conditions for the validity of the inequality of Poincaré type

$$\|u\|_X \leqslant K\{|F(u)| + \|\nabla^m u\|_Y\}, \quad u \in W^m. \tag{1.1}$$

Here F is a functional satisfying the conditions:

(F1) F is continuous;

(F2) $F(\lambda u) = \lambda F(u)$ for all $\lambda > 0$;

(F3) $F(u) = 0 \Rightarrow u = 0$ if u belongs to a suitable subspace of $\mathcal{P}_{m-1} \cap W^m$, where $\mathcal{P}_{m-1}$ is the class of polynomials in $\mathbb{R}^N$ of degree at most $m - 1$.

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Our methods will also occasionally apply (see Remark 4.2 below) to the derivation of the Friedrichs-type inequality

$$\|u\|_X \leqslant K\|\nabla^m u\|_Y. \tag{1.2}$$

In particular it is shown as a special case of Theorem 4.1 below (see Remark 4.1) that if there is a Banach function space Z such that $W^m(X,Y)$ is compactly embedded in Z and $W^m(Z,Y)$ is continuously embedded in X, then (1.1) holds. The proof of Theorem 4.1 is quite straightforward, and we believe that its nature is made more transparent as a result of presentation in the context of Banach function spaces. Despite the simplicity of the proof, the results are surprisingly effective and we demonstrate this by means of a variety of concrete examples of inequalities of type (1.1) or (1.2) in the setting of weighted Sobolev spaces; so far as we are aware, these examples are new. It should be emphasized that there is no assumption that the natural embedding J of $W^m(X,Y)$ in X is compact and indeed in a number of the examples given J is not compact. Should J be compact, then of course it is possible to give a more direct proof of inequalities of type (1.1) than by invoking the full machinery of Theorem 4.1. (For completeness this case is given in Corollary 4.1 below.)

Inequalities of Poincaré type are of great importance in the theory of partial differential equations; general information about them and their use may be found, for example, in the books Edmunds-Evans [7], Kufner-John-Fučík [13], Nečas [17] and Ziemer [20]. Of the many papers which have been written about such inequalities in recent years we mention especially [10] and [11]: in these a connection was established (following earlier work by Amick [1]) between these inequalities and the measure of non-compactness A of the embedding J. We mention that in many cases A is simply the distance between J and the subspace of compact linear maps from $W^m(X,Y)$ to X. What emerged was that under appropriate restrictions, (1.1) held if, and only if, $A < 1$. In a typical situation, if $A = 0$ then J is compact and the Poincaré inequality holds. In contrast to this work the present paper provides a route to (1.1) which is independent of the measure of non-compactness of the embedding map: instead of trying to prove that $A < 1$, we search for a space Z which has the properties mentioned earlier. Both approaches have their merits, and of course the connection between A and (1.1) is a striking one, but we contend that the simplicity of Theorem 4.1 and the ease by which a suitable Z may be found in diverse and interesting situations makes it highly applicable.

It is perhaps also worthwhile to point out here that under some mild restrictions it is known that inequality (1.1) has the equivalent forms:

$$\|u - L(u)\|_X \leqslant K_1 \|\nabla^m u\|_Y, \tag{1.3}$$

$$\inf_{p \in \mathscr{P}_{m-1}} \|u - p\|_X \leqslant K_2 \|\nabla^m u\|_Y. \tag{1.4}$$

Here L is a projection from $W^m(X, Y)$ to $\mathscr{P}_{m-1}$. An exhaustive list of such equivalent inequalities may be found in [11, Theorem 4.6]. Any of (1.1), (1.3) or (1.4) will be called a generalized Poincaré inequality; they are extensions of certain classical inequalities due to Poincaré. A typical example is

$$\int_\Omega \left| u - |\Omega|^{-1} \int_\Omega u \right|^q \leqslant K \left(\int_\Omega |\nabla u|^p \right)^{q/p}$$

for all u in the classical Sobolev space $W^{1,p}(\Omega)$, when $p \in [1, N)$, $q \in [p, p^*]$, $p^* = Np/(N-p)$, Ω is a bounded domain with a Lipschitz boundary and $|\Omega| := \operatorname{vol}(\Omega)$.

Ideas and notation relating to Banach function and abstract Sobolev spaces will be developed in Section 2. Section 3 summarizes an important body of recently-developed theory which links inequalities like (1.1) and (1.2) to a measure of non-compactness of a certain embedding which naturally corresponds to the inequality. Sufficient conditions for Poincaré-type inequalities (Theorem 4.1) are presented in Section 4. In Section 5 we use this result to obtain a Poincaré inequality (Theorem 5.1) in $\mathbb{R}^N$ with radial weights. The final section consists of other applications and illustrations of the main theorems in weighted Sobolev spaces. In the authors' opinion the last two sections form the core of the paper. Several of the examples appear to be both new and of theoretical interest in their own right; in one (Theorem 6.1) it has been possible to produce a higher-order Poincaré inequality which complements some results of Edmunds and Hurri [8] under very weak smoothness conditions on the boundary.

2. The Banach function space setting

We turn now to the task of outlining the essential function-space-theoretic ideas with which we shall work throughout the paper. Let $\{\Omega_n\}$ be a *fixed* sequence of bounded domains satisfying

$$\Omega_n \subset \overline{\Omega}_n \subset \Omega_{n+1} \subset \Omega \quad \text{for each } n \in \mathbb{N}. \tag{2.1}$$

We suppose that the boundary $\partial\Omega_n$ of each Ω_n is in $C^{0,1}$ (i.e., $\partial\Omega_n$ can be locally described by functions satisfying a Lipschitz condition [13, Section 5.5.6]) and that

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n. \tag{2.2}$$

For each $n \in \mathbb{N}$ we set $\Omega^n = \Omega \setminus \Omega_n$. Let $M(\Omega)$ denote the set of complex-valued Lebesgue measurable functions on Ω. Following Luxemburg [14] a *Banach function space* $X = X(\Omega) = (X(\Omega), \|\cdot\|_{X(\Omega)})$, where $X(\Omega) \subset M(\Omega)$, is a normed linear space satisfying the following axioms:

(BF1) $f \in X(\Omega)$ if, and only if, $\|f\|_{X(\Omega)} < \infty$.
(BF2) $\|f\|_{X(\Omega)} = 0$ if, and only if, $f = 0$ a.e. on Ω.
(BF3) $\|f\|_{X(\Omega)} = \big\||f|\big\|_{X(\Omega)}$ for all $f \in X(\Omega)$.
(BF4) Given any $n \in \mathbb{N}$, there is a constant $C_n = C_n(X)$ such that for all $f \in X(\Omega)$,

$$\int_{\Omega_n} |f(x)|\,\mathrm{d}x \leqslant C_n \|f\|_{X(\Omega)}.$$

(BF5) For all $n \in \mathbb{N}$, the characteristic function $\chi_{\Omega_n} \in X(\Omega)$.
(BF6) If $f, g \in M(\Omega)$ and $0 \leqslant f(x) \leqslant g(x)$ a.e. on Ω, then $\|f\|_{X(\Omega)} \leqslant \|g\|_{X(\Omega)}$.
(BF7) X is complete.

A brief discussion of the elementary properties of Banach function spaces is given in [10]. More thorough treatments may be found in [2] and [14]. In particular (because of (BF1), (BF5) and (BF6)) it turns out that $C_0^\infty(\Omega) \subset X(\Omega)$ and that the spaces $X(\Omega_n) = (X(\Omega_n), \|\cdot \chi_{\Omega_n}\|_X)$ defined by restricting the functions $f \in X(\Omega)$ to Ω_n are themselves Banach function spaces.

Now let X and Y be Banach function spaces. These give rise to the *abstract Sobolev space* $W^m(X,Y)$ of order m which is defined in the following manner: We say that $\alpha := (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$, ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) is a multi-index of dimension N. Let the symbol $|\alpha|$ stand for its *length*, that is $\sum_1^N \alpha_i$. Denote the set of all multi-indices of dimension N and length not exceeding m by $\mathscr{M}(N,m)$. By (BF4) any $f \in X$ is locally integrable. Hence given any $\alpha \in \mathscr{M}(N,m)$ it follows that the distributional derivative

$$D^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_N}},$$

exists. Accordingly let

$$W^m(X,Y) := \{f \in X : D^\alpha f \in Y \quad \text{for } |\alpha| = m\}$$

and equip this space with the norm

$$\|f\|_{W^m(X,Y)} := \|f\|_X + \|\nabla^m f\|_Y$$

where by $\|\nabla^m f\|_Y$ we mean $\sum\limits_{|\alpha|=m} \|D^\alpha f\|_Y$. By an argument similar to that of Lemma 4.1 of [10] it can be shown that $W^m(X,Y)$ is a Banach space. Since $C_0^\infty(\Omega) \subset W^m(X,Y)$ it is also useful—by analogy with ordinary Sobolev space theory—to define the closed subspace $W_0^m = W_0^m(X,Y)$ as the closure of $C_0^\infty(\Omega)$ in $W^m(X,Y)$; in general, $W_0^m(X,Y)$ is a proper subspace of $W^m(X,Y)$.

Example. Let $\mathscr{W}$ denote the class of positive a.e. measurable functions on $\mathbb{R}^N$ or weights. Additionally let $\mathscr{W}_C(\Omega) \subset \mathscr{W}$ denote the class of weights which are bounded above and below by positive constants on each compact set $Q \subset \Omega$. Note that this condition means that the members of $\mathscr{W}_C(\Omega)$ may have singularities or degeneracies only on the boundary $\partial\Omega$ of Ω and/or at ∞ if Ω is unbounded.

Given $w \in \mathscr{W}$ and $1 \leqslant r < \infty$, we define the weighted L^r space $L^r(\Omega; w)$ with the norm

$$\|u\|_{r,\Omega,w} := \left(\int_\Omega w(t)|u(t)|^r \, \mathrm{d}t\right)^{1/r}.$$

It is routine to show that the properties (BF1)–(BF7) are satisfied if $w^{-r'/r}$ (where r' is the conjugate index of r) and w are locally integrable. We shall assume throughout that both these properties hold for weights in $\mathscr{W}$. (Note that this is automatic on $\mathscr{W}_C(\Omega)$.) If $X := L^q(\Omega; v_0)$ and $Y := L^p(\Omega; v_1)$, $v_0, v_1 \in \mathscr{W}$, $1 \leqslant p, q < \infty$, we can identify $W^m(X,Y)$ with the *weighted Sobolev space* $W^{m,q,p}(\Omega; v_0, v_1)$ equipped with the norm

$$\|u\|_{m,q,p,\Omega,v_0,v_1} := \|u\|_{q,\Omega,v_0} + \|\nabla^m u\|_{p,\Omega,v_1}.$$

(If $p = q$ we shall use the abbreviated notation $\|u\|_{m,p,\Omega,v_0,v_1}$ and $W^{m,p}(\Omega; v_0, v_1)$; also in the unweighted case where $w = v_0 = v_1 = 1$ we shall write $\|u\|_{r,\Omega}$, $\|u\|_{m,q,p,\Omega}$, $W^{m,q,p}(\Omega)$, etc.) The corresponding Poincaré inequality will have the form

$$\left(\int_\Omega v_0|u|^q\right)^{1/q} \leqslant K\left\{|F(u)| + \left(\int_\Omega v_1|\nabla^m u|^p\right)^{1/p}\right\}.$$

Remark 2.1. We also note that if F satisfies (F1) we can recover a standard example in the setting of weighted Sobolev spaces. To see this suppose v_0 is integrable; then the functional $F(u) := \int_\Omega v_0 u$ is in $W^{m,q,p}(\Omega; v_0, v_1)^*$. This follows because

$$\begin{aligned}\int_\Omega v_0|u|^q &= \int_\Omega v_0^q|u|^q v_0^{1-q} \\ &\geqslant \left(\int_\Omega v_0|u|\right)^q \left(\int_\Omega v_0\right)^{1-q},\end{aligned}$$

by the Hölder inequality. Clearly this choice of F also satisfies (F2) and (F3). For additional examples see [11, Example 4.3].

3. Poincaré inequalities and the measure of non-compactness

If X and Y are Banach function spaces such that $X \subset Y$ and the natural inclusion map $J\colon X \to Y$ is continuous we say that X is *embedded* in Y and write $X \hookrightarrow Y$; similarly if J is compact we say that X is *compactly embedded* in Y and write $X \hookrightarrow\hookrightarrow Y$.

The Poincaré inequality (1.1) has recently been linked to a certain measure of non-compactness of the embedding

$$W^m(X,Y) \hookrightarrow X.$$

In order to motivate our own ideas which are developed in the next section we sketch the main features of this approach here. Proofs and additional details as well as various generalizations may be found in [9], [10] and [11]. With reference to the nested chain of subdomains $\{\Omega_n\}$ of Ω satisfying the conditions (2.1) and (2.2), set $\Omega^n = \Omega \setminus \Omega_n$ and define

$$A_n := \sup_{\|u\|_{W^m(X,Y)} \leqslant 1} \|u\chi_{\Omega^n}\|_X.$$

Clearly $0 \leqslant A_{n+1} \leqslant A_n \leqslant 1$; consequently the limit $A := \lim_{n\to\infty} A_n$ exists and $A \in [0,1]$. A similar definition applies if $u \in W_0^m(X,Y)$. The number A has many interesting properties: Suppose X, Y are Banach function spaces such that $X \hookrightarrow Y$. Let $W := W^m(X,Y)$ and let $\mathscr{K}(W,X)$ and $\mathscr{F}(W,Y)$ denote respectively the set of compact and finite rank maps from W to X, J the embedding map from W to X, and $\widetilde{\beta}(J)$ the ball measure of non-compactness of J (see [7, p.12] for the definition of this concept). Then if the norm on X has a certain absolute continuity property (see [10, Remark 5.14(ii)]) we have that

$$A = \widetilde{\beta}(J) = \inf_{K\in\mathscr{K}(W,X)} \|J-K\| = \inf_{F\in\mathscr{F}(W,X)} \|J-F\|.$$

In that case $A = 0$ if, and only if,

$$W^m(X,Y) \hookrightarrow\hookrightarrow X.$$

In the Hilbert space setting and in the particular case that $W^m = W^{m,2}(\Omega; v_0, v_1)$ and $Y = L^2(\Omega; v_0)$, W^m may be identified with the Dirichlet form associated with a symmetric differential or partial differential operator; in these circumstances $A = 0$ if, and only if, the underlying operator has *discrete* spectrum—equivalently the operator will have non-empty *continuous* spectrum if $A > 0$. (For an illustration of this point of view see Remark 6.1 below.) The connection of A with the Poincaré inequality is given by the following result [11, Theorem 3.5]:

Theorem 3.1. *Let $F \in (W^m(X,Y))^*$, and suppose that the following conditions hold:*

(H1) (F3) *holds on* $\mathscr{P}_{m-1} \cap W^m(X,Y)$.

(H2) *For each* $n \in \mathbb{N}$,

$$W^m(X(\Omega_n), Y(\Omega_n)) \hookrightarrow\hookrightarrow X(\Omega_n).$$

(H3) *If* $0 \neq u \in \mathscr{P}_{m-1} \cap X$, *there exists* $n \in \mathbb{N}$ *such that*

$$\|u\chi_{\Omega^n}\|_X < \|u\|_X.$$

Then the Poincaré inequality (1.1) *is valid on* $W^m(X,Y)$ *if, and only if,* $A < 1$.

Remark 3.1. (H2) and (H3) hold automatically in the setting of weighted Sobolev spaces when the weights belong to $\mathscr{W}_C(\Omega)$; indeed (H2) may be seen to be the restatement of an ordinary Sobolev embedding theorem. (H3) is a natural technical condition, holding in any space with absolutely continuous norm. The papers [9], [10], [11] contain numerous special cases and alternate forms of Theorem 3.1. Finally, concrete examples of Poincaré and Friedrichs inequalities in weighted Sobolev spaces, Orlicz spaces or even anisotropic spaces are given in these papers.

4. The sufficient condition

Our method of verifying the Poincaré inequality in $W^m(X,Y)$ or $W_0^m(X,Y)$ rests the following result:

Theorem 4.1. *Let W signify either $W^m(X,Y)$ or $W_0^m(X,Y)$ and assume that F satisfies* (F1), (F2) *on W and* (F3) *on $\mathscr{P}_{m-1} \cap W$. Suppose also that there exists a Banach function space Z such that*

$$W \hookrightarrow\hookrightarrow Z \tag{4.1}$$

and a (possibly nonlinear) functional G defined on W, continuous at 0 with respect to the norm

$$\|u\|_{Z,Y} := \|u\|_Z + \|\nabla^m u\|_Y,$$

and such that $G(0) = 0$. Then a sufficient condition for the validity of the Poincaré inequality (1.1) *on W is that the inequality*

$$\|u\|_X \leqslant K(\|u\|_{Z,Y} + |G(u)|) \tag{4.2}$$

holds for all $u \in W$.

Proof. Assume (1.1) is not true. Then for each $k \in \mathbb{N}$ there exists $f_k \in W$, $f_k \neq 0$ such that

$$\|f_k\|_X \geqslant k(|F(f_k)| + \|\nabla^m f_k\|_Y).$$

If we set $u_k := f_k/\|f_k\|_X$, the homogeneity of the norms and of the functional F implies that the sequence $\{u_k\}$ satisfies the conditions

$$\|u_k\|_X = 1, \tag{4.3}$$

$$F(u_k) \to 0, \tag{4.4}$$

$$\|\nabla^m u_k\|_Y \to 0 \tag{4.5}$$

as $k \to \infty$.

By (4.3) and (4.5), $\{u_k\}$ is a bounded set in W; it follows from (4.1) that there is a subsequence $\{u_{k_i}\}$ which is Cauchy in Z and therefore also (using (4.5)) with respect to $\|\cdot\|_{Z,Y}$. By (4.2) and the definition of G, $\{u_{k_i}\}$ is Cauchy in X and thus (by (4.5) again) Cauchy in W. Since W is complete, $\{u_{k_i}\}$ converges to a limit u in W. If α is a multi-index in $\mathscr{M}(N, m)$ of length m, it follows that $D^\alpha u_{k_i} \to D^\alpha u$ in Y. Then (4.5) and the uniqueness of limits imply that $D^\alpha u = 0$. Since this is true for any α, u is a polynomial p_{m-1} of degree at most $m-1$. From (4.3) we conclude that $\|u\|_X = 1$. However, $F(u) = 0$ by the continuity of F and (4.4). By (F3), $u = 0$ which is a contradiction. $\square$

Remark 4.1. In many applications $G = 0$ and the existence of the Poincaré inequality on $W^m(X, Y)$ for F satisfying (F1)–(F3) reduces to the discovery of a Banach function space Z such that the pair of embeddings

$$W^m(X, Y) \hookrightarrow\hookrightarrow Z, \tag{4.6}$$

$$W^m(Z, Y) \hookrightarrow X \tag{4.7}$$

simultaneously hold; a similar statement is true for the Poincaré inequality on $W_0^m(X, Y)$.

If $W^m(X, Y)$ embeds compactly in X, then the Poincaré inequality is true. This may be seen as we have pointed out above by appeal to Theorem 3.1 and the fact that $A = 0$. Alternatively a direct argument which is equivalent to a special case of Theorem 4.1 can be given:

Corollary 4.1. *Assume that F satisfies* (F1), (F2) *on W and* (F3) *on $\mathscr{P}_{m-1} \cap W$ where $W = W^m(X,Y)$ or $W_0^m(X,Y)$, and that the embedding*

$$W \hookrightarrow\hookrightarrow X \tag{4.8}$$

holds. Then the Poincaré inequality (1.1) *is satisfied on W.*

Proof. Taking $G = 0$ and $X = Z$, we see that (4.8) is equivalent to both (4.6) and (4.7) above. Hence the conclusion follows at once from Theorem 4.1. □

Remark 4.2. Suppose $\mathscr{P}_{m-1} \cap W^m(X,Y) = \{0\}$ or $\mathscr{P}_{m-1} \cap W_0^m(X,Y) = \{0\}$. With $F = 0$, we find that the proof of Theorem 4.1 actually establishes the Friedrichs-type inequality (1.2) on these spaces.

Remark 4.3. That shifting between $W_0^m(X,Y)$ and $W^m(X,Y)$ in Theorem 4.1 or Corollary 4.1 gives distinct results and in particular that Poincaré or Friedrichs inequalities can be valid on one of the spaces $W_0^m(X,Y)$, $W^m(X,Y)$ but not on the other can be shown as follows: Let $\Omega = (0,\infty)$ and $v_0 = v_1 = \mathrm{e}^{-t}$. Clearly $p_{m-1} \in W^{m,p}(\Omega; v_0, v_1)$ for any m. Taking $p = q = r$, and $f(t) = 1$ in Theorem 3.1 of [3], we arrive at the inequality

$$\int_\Omega \mathrm{e}^{-t}|u^{(j)}|^p \leqslant K\left\{\int_\Omega \mathrm{e}^{-t}|u|^p + \int_\Omega \mathrm{e}^{-t}|u^{(m)}|^p\right\} \tag{4.9}$$

for $j = 0, \ldots, m-1$, which is valid on $W^{m,p}(\Omega; v_0, v_1)$. On the other hand the inequality

$$\int_\Omega \mathrm{e}^{-t}|u^{(j)}|^p \leqslant K_1\left(\int_\Omega \mathrm{e}^{-t}|u|^p\right)^{(m-j)/m}\left(\int_\Omega \mathrm{e}^{-t}|u^{(m)}|^p\right)^{j/m} \tag{4.10}$$

can be established on $C_0^\infty(\Omega)$ (see [5, Example 2.2]). Suppose that $p_{m-1} \in W_0^{m,p}(\Omega; v_0, v_1)$, $p_{m-1} \neq 0$. Then there exists a sequence $\{u_k\}$ of C_0^∞ functions such that

$$\lim_{k\to\infty} \int_\Omega \mathrm{e}^{-t}|p_{m-1} - u_k|^p = 0$$

and

$$\lim_{k\to\infty} \int_\Omega \mathrm{e}^{-t}|u_k^{(m)}|^p = 0.$$

By (4.9) this implies that $u_k^{(j)} \to p_{m-1}^{(j)}$ in $L^p(\Omega; \mathrm{e}^{-t})$. It follows that p_{m-1} is in the domain of the inequality (4.10), which is impossible. Thus $\mathscr{P}_{m-1} \cap W_0^{m,p}(\Omega; v_0, v_1) = \{0\}$. In Example 6.2 below we show that the other hypotheses of Theorem 4.1 are satisfied on $W_0^{m,p}(\Omega; v_0, v_1)$. Taking $F = 0$ (cf. Remark 4.2), it follows that the inequality

$$\int_\Omega e^{-t}|u|^p \leqslant K \int_\Omega e^{-t}|u^{(m)}|^p \tag{4.11}$$

is true on $W_0^{m,p}(\Omega; v_0, v_1)$; however (4.11) cannot hold on $W^{m,p}(\Omega; v_0, v_1)$ since (as we have just seen) $\mathscr{P}_{m-1} \subset W^m$. More generally, (F3) holding on $\mathscr{P}_{m-1} \cap W^m$ is a necessary condition for the Poincaré inequality. (That the Poincaré inequality is in fact valid with weights $v_0 = v_1 = e^{-t}$ will be shown in Example 6.2.)

5. A Poincaré inequality in $\mathbb{R}^N$ for radial weights

In this section we show how Theorem 4.1 can be applied to derive a Poincaré inequality in the weighted Sobolev space setting in $\mathbb{R}^N$ for a class of radial weights. Before stating the result it will be convenient to introduce some notation appropriate to the one-dimensional setting and to modify the class $\mathscr{W}_C$: If $I = (a, b)$, $-\infty \leqslant a < b \leqslant \infty$, is an interval and v a weight on $\mathbb{R}$, define the classes:

$$\begin{aligned}
AC^m_{loc}(I) &:= \{u\colon u^{(m-1)} \text{ is locally absolutely continuous on } I\},\\
\mathscr{D}^{m,p}(I; v) &:= \{u \in AC^m_{loc}(I)\colon \int_I v|u^{(m)}|^p < \infty\},\\
\mathscr{D}^{m,p}_L(I; v) &:= \{u \in \mathscr{D}^{m,p}(I; v)\colon \lim_{t\to a} u^{(j)}(t) = 0,\ j = 0, \dots, m-1\},\\
\mathscr{D}^{m,p}_R(I; v) &:= \{u \in \mathscr{D}^{m,p}(I; v)\colon \lim_{t\to b} u^{(j)}(t) = 0,\ j = 0, \dots, m-1\}.
\end{aligned}$$

Also let $\mathscr{W}_C^*(\Omega)$ be the class of weights $v \in \mathscr{W}_C(\Omega)$ satisfying the following condition: For every $t \in \Omega$ there exists $\delta = \delta(t, v) > 0$ such that for $s \in B(t, \delta)$ where $B(t, \delta)$ is a ball of radius δ and center $t \in \mathbb{R}^N$ the inequalities

$$C_1 \leqslant \frac{v(s)}{v(t)} \leqslant C_2$$

hold with positive constants C_1, C_2 which do not depend on t.

We first require a Lemma.

Lemma 5.1. *Let $\max\{N/m, 1\} \leqslant p, q < \infty$, the first inequality being sharp if $m < N$ and let $B = B(t, r)$ be a ball of radius r and center $t \in \mathbb{R}^N$. Then there exists a constant K such that for $u \in W^{m,q,p}(B)$,*

$$\sup_{s\in\overline{B}} |u(s)| \leqslant K\{r^{-N/q}\|u\|_{q,B} + r^{m-N/p}\|\nabla^m u\|_{p,B}\}. \tag{5.1}$$

Proof. For $B = B(0,1)$ this is a consequence of a standard Sobolev embedding theorem, e.g. [13, §5.12] and Hölder's inequality. The given inequality is then obtained by dilating $B(0,1)$ to $B(t,r)$. □

Theorem 5.1. *Let $\Omega = \mathbb{R}^N$ and $1 < N < p < \infty$. Let $I = (0,\infty)$ and assume that $v_0, v_1 \in \mathscr{W}_C^*(I)$ support the one-dimensional Hardy inequality on $I = (0,\infty)$:*

$$\int_I v_0|u|^p \leqslant K \int_I v_1|u'|^p \tag{5.2}$$

on $\mathscr{D}_L^{1,p}(I;v_1)$ or on $\mathscr{D}_R^{1,p}(I;v_1)$. Let $\widetilde{v}_i(t) := |t|^{-N+1}v_i(|t|)$, $i = 0,1$. Further suppose that $v_0 \in L^1(I)$, and that F satisfies (F1), (F2) *on $W := W^{1,p}(\Omega;\widetilde{v}_0;\widetilde{v}_1)$ and* (F3) *on $\mathscr{P}_{m-1} \cap W$. Then the Poincaré inequality*

$$\int_\Omega \widetilde{v}_0|u|^p \leqslant K\left\{|F(u)|^p + \int_\Omega \widetilde{v}_1|\nabla u|^p\right\} \tag{5.3}$$

holds on W.

Proof. Let r_0 be a fixed positive number and assume that (5.2) holds on $\mathscr{D}_L^{1,p}(I;v_1)$. This is equivalent to the condition

$$\sup_{t\in(0,\infty)} \|v_0^{1/p}\|_{p,(t,\infty)}\|v_1^{-1/p}\|_{p',(0,t)} = C < \infty \tag{5.4}$$

which implies that

$$\sup_{t\in(r_0,\infty)} \|v_0^{1/p}\|_{p,(t,\infty)}\|v_1^{-1/p}\|_{p',(r_0,t)} \leqslant C < \infty.$$

Since v_0 is integrable on a right neighborhood of 0, we also get from (5.4) that

$$\sup_{t\in(0,r_0)} \|v_0^{1/p}\|_{p,(0,t)}\|v_1^{-1/p}\|_{p',(t,r_0)} < \infty.$$

From these conditions, by considering Hardy's inequality separately on $\mathscr{D}_R^{1,p}(I_1;v_1)$ with $I_1 := (0,r_0)$ and on $\mathscr{D}_L^{1,p}(I_2;v_1)$ with $I_2 := (r_0,\infty)$, we conclude that the inequality

$$\int_0^\infty v_0|u - u(r_0)|^p \leqslant K\int_0^\infty v_1|u'|^p \tag{5.5}$$

is true for $u \in \mathscr{D}^{1,p}(I;v_1)$. If (5.2) holds on $\mathscr{D}_R^{1,p}(I;v_1)$ a similar argument will also give (5.5). In either case Minkowski's inequality and the integrability of v_0 imply that the Poincaré inequality

$$\int_0^\infty v_0|u|^p \leqslant K\left\{\int_0^\infty v_1|u'|^p + |u(r_0)|^p\right\} \tag{5.6}$$

holds on $\mathscr{D}^{1,p}(I;v_1)$ and therefore also on $W^{1,p}(I;v_0,v_1)$. □

Next introduce the spherical coordinates r, φ where $\varphi := (\varphi_1, \dots, \varphi_{N-1})$ and $\varphi \in \Gamma$, $\Gamma := [0, \pi] \times \prod_{i=1}^{N-2} [0, 2\pi]$, and let $\mathrm{d}\Lambda(\varphi) := \sin^{N-2} \varphi_1 \sin^{N-3} \varphi_2 \dots \sin \varphi_{N-2} \times \mathrm{d}\varphi_1 \, \mathrm{d}\varphi_2 \dots \mathrm{d}\varphi_{N-1}$, so that $\mathrm{d}V = r^{N-1} \, \mathrm{d}r \, \mathrm{d}\Lambda(\varphi)$. Let $u \in W^{1,p}(\Omega; \widetilde{v}_0, \widetilde{v}_1)$. If we set $u_\varphi(\cdot) := u(\cdot, \varphi_1, \dots, \varphi_{N-1})$ we see from the assumptions on v_0, v_1 that $u_\varphi \in W^{1,p}(I; v_0, v_1)$ for almost every φ.

Using (5.6) and the fact that $|\nabla u| \geqslant |\partial u / \partial r|$ we have

$$
\begin{aligned}
\int_\Omega \widetilde{v}_0(t) |u(t)|^p \, \mathrm{d}t &= \int_\Gamma \int_0^\infty v_0(r) |u_\varphi(r)|^p \, \mathrm{d}r \, \mathrm{d}\Lambda(\varphi) \\
&\leqslant \int_\Gamma K \left(\int_0^\infty v_1(r) |u_\varphi'(r)|^p \, \mathrm{d}r + |u_\varphi(r_0)|^p \right) \mathrm{d}\Lambda(\varphi) \\
&\leqslant K \left\{ \int_\Omega \widetilde{v}_1(t) |\nabla u(t)|^p \, \mathrm{d}t + |G(u)|^p \right\} \qquad (5.7)
\end{aligned}
$$

where

$$
G(u) := \left(\int_\Gamma |u_\varphi(r_0)|^p \, \mathrm{d}\Lambda(\varphi) \right)^{1/p}.
$$

Below we will show that the functional G has the required continuity properties. Once this has been accomplished (5.7) will determine an inequality like (4.2), and consequently half of the work of applying Theorem 4.1 will have been done. Now, however, we turn our attention to the compact embedding (4.1). We are in fact going to "manufacture" a weight so that (4.1) holds. To begin this task, set

$$
R_1(t) := \|\widetilde{w}\|_{\infty, B_t} \|\widetilde{v}_0^{-1}\|_{\infty, B_t}
$$

and

$$
R_2(t) := f(t)^p \|\widetilde{w}\|_{\infty, B_t} \|\widetilde{v}_1^{-1}\|_{\infty, B_t},
$$

where $\widetilde{w}(t) := |t|^{-N+1} w(|t|)$, $B_t := B(t, f(t))$ and w, f are for the moment an undefined weight and an appropriate positive function. These definitions are motivated by [3, Definition 2.1 and Remark 2.1]. (In [3] cubes are used instead of balls, but the difference is inessential; we also remark that the hypothesis $p > N$ is fundamental in [3].) Further a consequence of [3, Theorem 4.1 and Remark 2.1] is that

$$
W^{1,p}(\Omega; \widetilde{v}_0, \widetilde{v}_1) \hookrightarrow\hookrightarrow L^p(\Omega; \widetilde{w}) \qquad (5.8)
$$

if

$$
\limsup_{|t| \to \infty} R_1(t) = \limsup_{t \to 0} R_1(t) = 0 \qquad (5.9)
$$

and

$$\limsup_{|t|\to\infty} R_2(t) < \infty, \qquad \limsup_{t\to 0} R_2(t) < \infty. \tag{5.10}$$

We define

$$w(s) = \begin{cases} \dfrac{1}{(s+1)(v_0(s)^{-1} + v_1(s)^{-1})} & \text{if } s \geqslant 1, \\ \dfrac{s}{v_0(s)^{-1} + v_1(s)^{-1}} & \text{if } 0 < s < 1. \end{cases}$$

Then $\widetilde{w}$ is in $\mathscr{W}_C^*(\mathbb{R}^N \setminus \{0\})$. Let $f_{\tilde{w}}\colon \mathbb{R}^N \to \mathbb{R}$ be defined by

$$f_{\tilde{w}}(t) := \min\left\{\sup\left\{\delta\colon C_1 \leqslant \frac{\widetilde{w}(s)}{\widetilde{w}(t)} \leqslant C_2 \quad \text{for } s \in B(t,\delta)\right\}, |t|, 1\right\}$$

where C_1, C_2 are the constants in the definition of the class $\mathscr{W}_C^*(\mathbb{R}^N \setminus \{0\})$. The assumptions on v_0 and v_1 also imply that $\widetilde{v}_0$ and $\widetilde{v}_1$ are in $\mathscr{W}_C^*(\mathbb{R}^N \setminus \{0\})$. Proceeding as before we construct $f_{\tilde{v}_0}$, $f_{\tilde{v}_1}$ and set

$$f(t) := \min\{f_{\tilde{w}}(t), f_{\tilde{v}_0}(t), f_{\tilde{v}_1}(t)\};$$

The function f is positive and bounded. To apply the machinery of [3] therefore it is sufficient to note first that because of the properties of f,

$$\begin{aligned} &R_1(t) = O\left(\frac{1}{|t|+1}\right), \quad R_2(t) = O\left(\frac{1}{|t|+1}\right) \quad \text{for } |t| \to \infty, \\ &R_1(t) = O(|t|), \quad R_2(t) = O(|t|) \quad \text{for } t \to 0, \end{aligned}$$

and hence both (5.9) and (5.10) hold. Thus the embedding (5.8) follows immediately from [3, Theorem 4.1].

We next (cf. Lemma 5.1) can establish the interpolation inequality

$$|u_\varphi(r_0)| \leqslant K\left\{\int_J |u_\varphi(r)|\,\mathrm{d}r + \int_J |u'_\varphi(r)|\,\mathrm{d}r\right\} \tag{5.11}$$

where J is the interval $[r_0, r_0+1]$. Applying Hölder's inequality to (5.11) and extending the range of integration on the right-hand side to the whole interval we obtain

(5.12)

$$|u_\varphi(r_0)| \leqslant K_1(r_0)\left\{\left(\int_0^\infty w(r)|u_\varphi(r)|^p\,\mathrm{d}r\right)^{1/p} + \left(\int_0^\infty v_1(r)|u'_\varphi(r)|^p\,\mathrm{d}r\right)^{1/p}\right\}.$$

Integrating (5.12) with respect to the angular part of the measure, using the Hölder inequality and transforming back into cartesian coordinates we have

$$(5.13)\qquad |G(u)| \leqslant K_1(r_0)\left\{\left(\int_\Omega \widetilde{w}(t)|u(t)|^p\,\mathrm{d}t\right)^{1/p} + \left(\int_\Omega \widetilde{v}_1(t)|\nabla u(t)|^p\,\mathrm{d}t\right)^{1/p}\right\},$$

so that G is continuous at 0 with respect to $W^{1,p}(\Omega;\widetilde{w},\widetilde{v}_1)$ norm. (A similar argument shows that G is continuous with respect to the $W^{1,p}(\Omega;\widetilde{v}_0,\widetilde{v}_1)$ norm.)

As previously noted (5.7) corresponds to (4.2). We have also derived (5.8) which corresponds to (4.1). The Poincaré inequality (5.3) now follows by Theorem 4.1.

Remark 5.1. Suppose $\Omega = \mathbb{R}^N \setminus \overline{B(0,1)}$ and the Hardy inequality (5.2) is supported by v_0 and v_1 on $(1,\infty)$. If we repeat the proof of Theorem 5.1 we find that the Poincaré inequality holds on Ω provided that $u \in W^{1,p}(\Omega;\widetilde{v}_0,\widetilde{v}_1)$.

Remark 5.2. The estimate (5.13) claims more than the continuity of G. In fact, (5.13) is an L^p-estimate for traces of functions from $W^{1,p}(\Omega;\widetilde{w},\widetilde{v}_1)$ on the sphere of radius r_0 centered at the origin. Inserting this estimate in (4.2) we can see that it is sufficient to use Theorem 4.1 with $G=0$ (cf. Remark 4.1).

6. Examples

The following list is not comprehensive but is intended to illustrate how Theorem 4.1 or Corollary 4.1 (the compact embedding case) can lead to new results which might be difficult to prove otherwise. In all examples we assume without explicit mention that F satisfies (F1), (F2) on W^m or W_0^m, and (F3) on $\mathscr{P}_{m-1}\cap W^m$ or $\mathscr{P}_{m-1}\cap W_0^m$ as the context dictates.

6.1. Let $\Omega=(a,\infty)$, $a>0$, $1\leqslant p,q<\infty$, $v_0(t)=t^\gamma$, $v_1(t)=t^\alpha$, $X=L^q(\Omega;v_0)$, $Y=L^p(\Omega;v_1)$ and $Z=L^r(\Omega;w)$, $1\leqslant r<\infty$ where $w(t)=t^\beta$ and β is to be determined.

Case 1: $1\leqslant p\leqslant q\leqslant r<\infty$. Assume

$$(6.1)\qquad \alpha=(p/q)(\gamma+mq+1)-1,$$

$$(6.2)\qquad \gamma<-1-(m-1)q.$$

In [3, Theorem 3.1(i)] take $N=g=h=\eta'=\xi'=\theta_1=\theta_2=1$, $\eta=\xi=\infty$ and $f(t)=t^\delta$ where

$$\delta := \frac{\alpha/p-\gamma/q}{m+1/q-1/p}.$$

Note that (6.1) implies that $\delta = 1$. Then according to [3, Examples 3.1, 3.5 and Theorem 3.2], the embedding $W(X,Y) \hookrightarrow Z$ holds if, and only if,

$$(\beta/r)(m - 1/p + 1/q) \leqslant (\gamma/q)(m - 1/p + 1/r) + (\alpha/p)(1/q - 1/r). \tag{6.3}$$

Further by [3, Theorem 4.1–4.2 and Remark 4.3] the embedding is compact if, and only if, this inequality is strict. Next define

$$g_u(t) := \sum_{i=0}^{m-1} \frac{(t-a)^i}{i!} u^{(i)}(a+)$$

and

$$G(u) := \left(\int_a^\infty t^\gamma |g_u(t)|^q \,\mathrm{d}t \right)^{1/q}.$$

It is clear first of all that G is defined on $W^{m,q,p}(\Omega; t^\gamma, t^\alpha)$.

Applying [4, Theorem 1.1-C] with $M = t$ (also see [4, Example 4.2]), we obtain the Hardy inequality

$$\left(\int_a^\infty t^\gamma |u(t)|^q \,\mathrm{d}t \right)^{1/q} \leqslant K(p,q) \left(\int_a^\infty t^\alpha |u^{(m)}(t)|^p \,\mathrm{d}t \right)^{1/p} \tag{6.4}$$

on $\mathscr{D}_L^{m,p}(\Omega; t^\alpha)$ if (6.1) and (6.2) are satisfied. Consequently given the definition of G we have, by Minkowski's and Hardy's inequalities, that

$$\begin{aligned} \left(\int_a^\infty t^\gamma |u|^q \right)^{1/q} &\leqslant \left(\int_a^\infty t^\gamma |u - g_u|^q \right)^{1/q} + |G(u)| \\ &\leqslant K \left(\int_a^\infty t^\alpha |u^{(m)}|^p \right)^{1/p} + |G(u)|. \end{aligned} \tag{6.5}$$

Now the inequality (4.2) follows from (6.5).

To apply Theorem 4.1 it remains to show that G is continuous at 0 on $W^{m,r,p}(\Omega; t^\beta, t^\alpha)$. According to the definition of $G(u)$, we do this by showing that h of the mappings $u \longmapsto u^{(j)}(a+)$, $j = 0, \dots, m-1$, belongs to $W^{m,r,p}(\Omega; t^\beta, t^\alpha)^*$. y choosing λ_j to be an appropriately smooth function with support on $[a,c] \subset [a,\infty)$ such that

$$\lambda_j^{(i)}(a) = \begin{cases} 1, & \text{if } i = m - j - 1 \\ 0, & \text{otherwise,} \end{cases} \qquad \lambda_j^{(i)}(c) = 0 \text{ for } i = 0, \dots, m-1,$$

repeatedly integrating by parts $\int_a^c \lambda_j u^{(m)}$, and applying Hölder's inequality, we can derive the inequality

$$|u^{(j)}(a+)| \leqslant K \left\{ \left(\int_a^c t^\beta |u|^r \right)^{1/r} + \left(\int_a^c t^\alpha |u^{(m)}|^p \right)^{1/p} \right\}.$$

The conclusion follows from the fact that t is bounded away from 0 on finite right neighborhoods of a (note that we need $a > 0$).

Case 2: $1 \leqslant q < p \leqslant r < \infty$. Applying [18, Theorem 10.11] to power weights we see that the Hardy inequality (6.4) holds on $\mathscr{D}_L^{m,p}(\Omega; t^\alpha)$ if

$$\frac{\gamma+1}{q} + m < \min\left\{\frac{\alpha+1}{p}, 1\right\} \tag{6.6}$$

holds. One can prove using [3, Theorem 4.1(i)] with $f(t) = t$ that the embedding $W^m(X, Y)$ into Z exists and is compact if

$$\frac{\beta+1}{r} < \min\left\{\frac{\gamma+1}{q}, \frac{\alpha+1}{p} - m\right\}.$$

(The details of the argument follow from Example 3.5, Remark 2.1(i) and Remark 4.3 of [3].) This can be arranged for suitable β. The rest of the argument is exactly the same as in Case 1.

In summary, when $q \geqslant p$, $\alpha = (p/q)(\gamma + mq + 1) - 1$, $\gamma < -1 - (m-1)q$, and (6.3) or when $q < p$ and the condition (6.6) is satisfied, then the inequality

$$\left(\int_a^\infty t^\gamma |u|^q\right)^{p/q} \leqslant K\left(|F(u)|^p + \int_a^\infty t^\alpha |u^{(m)}|^p\right)$$

holds on $W^{m,q,p}((a,\infty); t^\gamma, t^\alpha)$.

If in Case 1 we take $r = q$ and $\beta = \gamma$, strict inequality does not hold in (6.3). It follows that $W^m(X, Y)$ is *not* compactly embedded into X; in other words we have produced an example where $A \in (0, 1)$.

The situation in Case 2 is different and requires a more complicated analysis. Since it is the prototype for arguments in later examples in this section we present it in some detail. For $t \geqslant a$, let $I_{t,\delta} := (t, t + t^\delta)$ where $\delta > 1$. Then by Lemma 6.1 and taking $N = 1$, we obtain the interpolation inequality

$$|u(t)| \leqslant K\left\{t^{-\delta/q}\left(\int_{I_{t,\delta}} |u|^q\right)^{1/q} + t^{\delta(m-1/p)}\left(\int_{I_{t,\delta}} |u^{(m)}|^p\right)^{1/p}\right\}. \tag{6.7}$$

If $s \in I_{t,\delta}$, then

$$1 \leqslant \frac{s}{t} \leqslant 1 + t^{\delta-1}. \tag{6.8}$$

We now put the weights v_0, v_1 and w into (6.7): in the first integral of (6.7) we introduce the factor $1 = s^\gamma s^{-\gamma}$ and use (6.8) to bound $s^{-\gamma}$ in terms of a power of t;

a similar procedure is followed with respect to the second integral. If α, $\gamma \geqslant 0$ this leads to the estimate

$$
\begin{aligned}
(6.9)\quad & t^{\beta/q}|u(t)| \\
& \leqslant K\left\{t^{(\beta-\gamma-\delta)/q}\left(\int_\Omega s^\gamma|u|^q\right)^{1/q} + t^{\delta(m-1/p)-\alpha/p+\beta/q}\left(\int_\Omega s^\alpha|u^{(m)}|^p\right)^{1/p}\right\}.
\end{aligned}
$$

By an argument which is a straightforward extension of the case $W^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow C(\Omega; w)$ given in [6, Theorem 4.1, Example 5.3 and Remark 5.2] (6.9) yields a compact embedding into $C(\Omega; t^{\beta/q})$ if

$$
\beta < \begin{cases} \gamma+\delta, \\ q\alpha/p - \delta(mq - q/p). \end{cases} \tag{6.10}
$$

Specifically, let J denote the embedding of $W^{m,q,p}(\Omega; t^\gamma, t^\alpha)$ into $C(\Omega; t^{\beta/q})$ given by (6.9), and let $J_n := \chi_{I_n} \circ J$ where $I_n := [a, n]$ and $n > a$ is a positive integer. Then (6.10) implies that $\lim\limits_{n\to\infty} \|J - J_n\| = 0$. Since $v_0, v_1 \in \mathscr{W}_C(\Omega)$, standard Ascoli-type arguments show that each map J_n is compact. Therefore J is compact.

Application of Hölder's inequality then shows that $C(\Omega; t^{\beta/q})$ embeds continuously into $L^q(\Omega; v_0)$ if $\gamma - \beta < -1$. Thus $W^m(X, Y) \hookrightarrow\hookrightarrow X$ if

$$
\gamma < \begin{cases} \gamma+\delta-1, \\ q\alpha/p - \delta(mq - q/p) - 1. \end{cases} \tag{6.11}
$$

Since $\delta > 1$ the first inequality is satisfied for any γ. The argument is completed with the observation that if α, γ satisfy (6.6) they also satisfy the second inequality of (6.11) for a δ sufficiently close to 1.

The case $\gamma \geqslant 0$ and $\alpha < 0$ is inconsistent with (6.6).

Suppose $\gamma < 0$ and $\alpha < 0$. The same kind of analysis using (6.7) and (6.8) gives a compact embedding of $W^m(X, Y)$ into X if

$$
\gamma < \begin{cases} \delta(\gamma+1) - 1, \\ \delta(q\alpha/p - (mq - q/p)) - 1. \end{cases}
$$

The first inequality holds for any $\delta > 1$ if, and only if, $\gamma > -1$ and (6.6) implies the second inequality for a suitable δ.

The last possibility is $\gamma < 0$, $\alpha \geqslant 0$. Here we are led to the conditions

$$
\gamma < \begin{cases} \delta(\gamma+1) - 1, \\ q\alpha/p - \delta(mq - q/p) - 1. \end{cases}
$$

Again this requires that $\gamma > -1$; the second inequality will also hold if we choose δ close enough to 1.

Summarizing, when $q < p$ we have that $W^m(X,Y) \hookrightarrow\hookrightarrow X$ and thus $A = 0$ in all possibilities allowed by (6.6) if $\gamma > -1$. Moreover, since $\mathscr{P}_{m-1} \cap W^m(X,Y) = \{0\}$, the Friedrichs inequality (1.2) is true (cf. Remark 4.2).

Remark 6.1. Let us take Ω, v_0, v_1 as in the previous example but let $p = q = r = 2$ and omit the condition $\gamma < -1 - 2(m-1)$. Hinton and Lewis [12] have shown that the spectrum of the minimal operator $T_0(M)$ (and therefore of any self-adjoint extension) on (a, ∞) induced by the differential expression

$$M(u) := t^{-\beta}[(-1)^m(t^\alpha u^{(m)})^{(m)} + t^\gamma u]$$

is discrete and bounded below if

$$\beta < \alpha - 2m,$$
$$\beta \geqslant \gamma.$$

Let $\beta = \gamma$. If $T_{\alpha\gamma}$ is the Friedrichs extension corresponding to the form $\mathbf{t}[u,v]$ defined in $L^2(\Omega; t^\gamma)$ as

$$\mathbf{t}[u,v] := \int_a^\infty (t^\alpha u^{(m)}\overline{v}^{(m)} + t^\gamma u\overline{v})\,\mathrm{d}t,$$

we have that

$$[T_{\alpha\gamma}u, u] \geqslant \|u\|^2_{2,\Omega,w}.$$

Since the domain of $T_{\alpha\gamma}$ is a core of the domain of $\mathbf{t}$, it follows by Rellich's theorem [16, §24.5, Theorem 11] (and the fact that an embedding holds trivially) that

$$W^{m,2}(\Omega; t^\gamma, t^\alpha) \hookrightarrow\hookrightarrow L^2(\Omega; t^\gamma).$$

The Poincaré inequality (with $A = 0$)

$$\int_a^\infty t^\gamma |u|^2 \leqslant K\left\{|F(u)|^2 + \int_a^\infty t^\alpha |u^{(m)}|^2\right\}$$

now is immediate from Corollary 4.1.

6.2. Let $\Omega = (0, \infty)$, $1 \leqslant p \leqslant q < \infty$, $v_0(t) := \mathrm{e}^{-qct/p}$, $v_1(t) := \mathrm{e}^{-ct}$, $c > 0$, $w(t) := \mathrm{e}^{-\beta t}$, $\beta > qc/p$, $X = L^q(\Omega; v_0)$, $Y = L^p(\Omega; v_1)$ and $Z = L^q(\Omega; w)$. Let $g_u(t)$ be as in Example 6.1 with $a = 0$, and define

$$G(u) = \left(\int_0^\infty \mathrm{e}^{-qct/p}|g_u(t)|^q\,\mathrm{d}t\right)^{1/q}.$$

An argument similar to [4, Example 3.3] shows that the Hardy inequality

$$\left(\int_0^\infty \mathrm{e}^{-qct/p}|u(t)|^q\,\mathrm{d}t\right)^{1/q} \leqslant K\left(\int_0^\infty \mathrm{e}^{-ct}|u^{(m)}(t)|^p\,\mathrm{d}t\right)^{1/p}$$

is valid on $\mathscr{D}_L^{m,p}(\Omega; v_1)$. As in (6.5) above this leads to the inequality

$$\left(\int_0^\infty \mathrm{e}^{-qct/p}|u(t)|^q\,\mathrm{d}t\right)^{1/q} \leqslant K\left\{\left(\int_0^\infty \mathrm{e}^{-ct}|u^{(m)}(t)|^p\,\mathrm{d}t\right)^{1/p} + |G(u)|\right\}.$$

Application of [3, Theorem 4.1] (with $f(t) = 1$) shows that the embedding

$$W^{m,q,p}(\Omega; \mathrm{e}^{-qct/p}, \mathrm{e}^{-ct}) \hookrightarrow\hookrightarrow L^q(\Omega; \mathrm{e}^{-\beta t})$$

is compact. By an argument similar to that in Example 6.1 one can verify that G is continuous on $W^{m,q,p}(\Omega; \mathrm{e}^{-\beta t}, \mathrm{e}^{-ct})$. Thus by Theorem 4.1 the Poincaré inequality (1.1) follows. Use of [3, Theorem 4.2] shows that the embedding

$$W^{m,q,p}(\Omega; \mathrm{e}^{-qct/p}, \mathrm{e}^{-ct}) \hookrightarrow L^q(\Omega; \mathrm{e}^{-qct/p})$$

is not compact so that here $A \in (0, 1)$.

6.3. Let $1 < N < p < \infty$, $\Omega = \mathbb{R}^N$, $m = 1$, $v_0(s) := s^{N-1}(1+s)^\gamma$ and $v_1(s) := s^{N-1}(1+s)^\alpha$, $s \in \mathbb{R}$ where $\gamma < -N$, $\alpha > p - N$. We can verify from the assumptions on α, γ that the Muckenhoupt condition

$$\sup_{t\in I}\left(\int_0^t v_0(s)\,\mathrm{d}s\right)^{1/p}\left(\int_t^\infty v_1(s)^{-p'/p}\,\mathrm{d}s\right)^{1/p'} < \infty$$

holds. Therefore the Hardy inequality (5.2) is valid on $I = (0, \infty)$ for $u \in \mathscr{D}_R^{1,p}(I; v_1)$. Also the reader may check that $v_0 \in L^1(0, \infty)$, and $v_0, v_1 \in \mathscr{W}_C^*(I)$. Hence Theorem 5.1 may be applied to obtain the inequality

$$\int_{\mathbb{R}^N} (1 + |t|)^\gamma |u|^p \leqslant K\left\{|F(u)|^p + \int_{\mathbb{R}^N} (1 + |t|)^\alpha |\nabla u|^p\right\}.$$

on $W^{1,p}(\mathbb{R}^N; v_0, v_1)$. In view of Remark 5.1 we may also derive the inequality

$$\int_\Omega |t|^\gamma |u|^p \leqslant K\left\{|F(u)|^p + \int_\Omega |t|^\alpha |\nabla u|^p\right\}$$

where $\Omega = \mathbb{R}^N \setminus \overline{B(0,1)}$, α and γ are as above, and $u \in W^{1,p}(\Omega; |t|^\gamma, |t|^\alpha)$. In either case application of Theorem 4.2, Remark 4.3, and Example 4.1 of [3] shows that $W^{1,p}(\Omega; |t|^\gamma, |t|^\alpha)$ is not compactly embedded into $L^p(\Omega; |t|^\gamma)$ and therefore $A \in (0,1)$.

6.4. Let $1 < N < p < \infty$, $\Omega = \mathbb{R}^N$, $m = 1$, $v_0(s) = v_1(s) = s^{N-1}e^{-\gamma s}$, $\alpha > 0$. By calculations which are similar to those in the previous example the reader may verify that v_0, v_1 satisfy the hypotheses of Theorem 5.1. As a result, the inequality

$$\left(\int_\Omega e^{-\gamma|t|}|u|^p\right)^{1/p} \leqslant K\left\{|F(u)| + \left(\int_\Omega e^{-\gamma|t|}|\nabla u|^p\right)^{1/p}\right\}$$

holds on $W_0^{m,p}(\Omega; e^{-\gamma|t|}, e^{-\gamma|t|})$. An application of [3, Theorem 4.2] shows that the embedding $W_0^{m,p}(\Omega; e^{-\gamma|t|}, e^{-\gamma|t|}) \hookrightarrow L^p(\Omega; e^{-\gamma|t|})$ is not compact so that again $A \in (0,1)$.

Remark 6.2. Using the machinery of [3, Theorem 3.1] we can establish an analog of the sum inequality (4.9) on $W_0^{m,p}(\Omega; e^{-\gamma|t|}, e^{-\gamma|t|})$ for $N > 1$. If a product inequality similar to (4.10) is true on the C_0^∞ functions, the arguments of Remark 4.3 would establish the Friedrichs inequality rather than the Poincaré inequality in Example 6.4. However, whether or not such an inequality is valid for $N > 1$ seems not to be known.

Our last pair of examples for weighted Sobolev spaces in $\mathbb{R}^N$ are really instances of Corollary 4.1 rather than Theorem 4.1. However since they seem to us both new and interesting we include them. The analysis depends heavily on Lemma 6.1 and resembles the compactness arguments given in Case 2 of Example 6.1.

6.5. Let $\Omega = \mathbb{R}^N \setminus \overline{B(0,1)}$, $v_0(t) = |t|^\gamma$, $v_1(t) = |t|^\alpha$, where $\gamma > -N$ and

$$\frac{\alpha/p - \gamma/q}{m + N/q - N/p} > 1, \tag{6.12}$$

$w(t) = |t|^\beta$, $r(t) = |t|^\delta$, and let $\max\{N/m, 1\} \leqslant p, q < \infty$, the first inequality being sharp if $m < N$. Given $t \in \Omega$ we construct the ball B_t with radius $r(t)$ and center at distance $|t| + r(t)$ from the origin and on the extension of the ray $\overrightarrow{0t}$. Thus if $s \in B_t$, then

$$1 \leqslant \frac{|s|}{|t|} \leqslant 1 + 2\frac{r(t)}{|t|}.$$

Throughout we take $\delta > 1$ and introduce the weights w, v_0 and v_1 into (5.1) in a manner similar to the compactness argument in Case 2 of Example 6.1. Since the reasoning is exactly the same we just summarize the main features.

If γ, $\alpha \geqslant 0$ we obtain the inequality

$$|t|^{\beta/q}|u(t)| \leqslant K\Big\{|t|^{(\beta-\gamma-N\delta)/q}\Big(\int_{B_t}|s|^{\gamma}|u|^{q}\Big)^{1/q} + |t|^{\delta(m-N/p)-\alpha/p+\beta/q}\Big(\int_{B_t}|s|^{\alpha}|\nabla^m u|^{p}\Big)^{1/p}\Big\}.$$

The embedding

$$W^{m,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; v_0) \tag{6.13}$$

holds if

$$\gamma < \begin{cases} \gamma + N(\delta - 1), \\ q(\alpha/p - \delta(m - N/p)) - N. \end{cases} \tag{6.14}$$

Since $\delta > 1$ the first inequality is satisfied. Since γ, α satisfy (6.12) the second inequality in (6.14) can be satisfied for some $\delta > 1$.

If $\gamma < 0, \alpha \geqslant 0$, then proceeding as in Case 2 of Example 6.1 (cf. (6.8)), there exists the compact embedding (6.13) if

$$\gamma < \begin{cases} \delta(\gamma + N) - N, \\ q(\alpha/p - \delta(m - N/p)) - N. \end{cases} \tag{6.15}$$

Since $\delta > 1$, the first inequality holds if $\gamma > -N$. According to (6.12), the second inequality (6.15) is satisfied for some $\delta > 1$.

If $\alpha < 0$ then the second inequality in (6.14) and (6.15) has the form

$$\gamma < \delta q(\alpha/p - (m - N/p)) - N.$$

This yields that $\gamma < -N$ which is inconsistent with the first inequality in (6.15).

Since $\gamma > -N$, $\mathscr{P}_{m-1} \cap W^{m,q,p}(\Omega; |t|^{\gamma}, |t|^{\alpha}) = \{0\}$.

Summarizing, if $\gamma > -N$, $\alpha \geqslant 0$ and if the inequality (6.12) holds, then the Friedrichs inequality

$$\Big(\int_{\Omega}|t|^{\gamma}|u|^{q}\Big)^{1/q} \leqslant K\Big(\int_{\Omega}|t|^{\alpha}|\nabla^m u|^{p}\Big)^{1/p}$$

is true on $W^{m,q,p}(\Omega; |t|^{\gamma}, |t|^{\alpha})$. Furthermore the embedding of $W^{m,q,p}(\Omega; |t|^{\gamma}, |t|^{\alpha})$ into $L^q(\Omega; |t|^{\gamma})$ is compact and so $A = 0$.

6.6. Let Ω be a bounded domain in $\mathbb{R}^N$ having the following integrability property with respect to the distance function:

(I1) There is a number $0 < \mu_0 < N$ such that $\int_\Omega d(t)^{-\mu_0}\,dt < \infty$.

Suppose also that $\max\{N/m, 1\} \leqslant p, q < \infty$, the first inequality being sharp if $m < N$. Set $v_0(t) = d(t)^\gamma$, $v_1(t) = d(t)^\alpha$, $w(t) = d(t)^\beta$, and $r(t) = d(t)^\delta$ where $-\mu_0 < \gamma \leqslant 0$, $\alpha \leqslant 0$, $\beta \in \mathbb{R}$ and $0 < \delta < (\gamma + \mu_0)/(\gamma + N)$. In this case

$$C(\Omega; w^{1/q}) \hookrightarrow\hookrightarrow L^q(\Omega; v_0)$$

if $\gamma > \beta - \mu_0$.

Since $\delta < 1$, the ball $B(t, d(t)^\delta)$ may not be strictly contained in Ω. We overcome this inconvenience by considering $u \in C_0^\infty(\Omega)$. If $s \in B(t, d(t)^\delta) \cap \Omega$ then

$$d(s) < d(t) + d(t)^\delta. \tag{6.17}$$

Thus $d(s) < 2d(t)^\delta$ if $d(t) < 1$. Lemma 6.1, (6.17) and the assumptions on γ, α then yield the inequality

$$d(t)^{\beta/q}|u(t)| \leqslant K\left\{ d(t)^{(-\delta(\gamma+N)+\beta)/q}\left(\int\limits_{B(t,d(t)^\delta)\cap\Omega} d(s)^\gamma |u|^q\right)^{1/q} + d(t)^{\delta(m-N/p-\alpha/p)+\beta/q}\left(\int\limits_{B(t,d(t)^\delta)\cap\Omega} d(s)^\alpha |\nabla^m u|^p\right)^{1/p}\right\}.$$

Pursuing the same arguments as before, we find that $W_0^{m,q,p}(\Omega; d(t)^\gamma, d(t)^\alpha)$ embeds compactly into $L^p(\Omega; d(t)^\gamma)$ if

$$\gamma > \begin{cases} \delta(\gamma + N) - \mu_0, \\ \delta q(\alpha/p - (m - N/p)) - \mu_0. \end{cases}$$

By the conditions on δ and γ both inequalities hold. Because the balls $B(t, d(t)^\delta)$ may overlap $\partial\Omega$, there is no *lower* bound on $d(s)$ in terms of $d(t)$. Unfortunately, this means that in introducing weights into Lemma 6.1 according to the procedure described in Example 6.1 (see the paragraph following (6.8)) we are restricted to non-positive values of α, γ.

It follows that the Poincaré inequality (1.1) holds if Ω is a bounded domain satisfying (I1) and $\alpha \leqslant 0$ and $-\mu_0 < \gamma \leqslant 0$. Since

$$W_0^{m,q,p}(\Omega; d(t)^\gamma, d(t)^\alpha) \hookrightarrow\hookrightarrow L^q(\Omega; d(t)^\gamma)$$

we have $A = 0$.

What kind of domains Ω satisfy (I1)? We show here that if the "Minkowski dimension" of $\partial\Omega$ is less than N then (I1) is true and *vice versa*.

Definition 6.1. Let $0 < \lambda \leqslant N$ and $r > 0$. Then set:

$$M_\Omega^\lambda(\partial\Omega; r) := \frac{|(\partial\Omega + B(0,r)) \cap \Omega|}{r^{N-\lambda}}$$
$$M_\Omega^\lambda(\partial\Omega) := \limsup_{r\to 0+} M_\Omega^\lambda(\partial\Omega; r)$$
$$\dim_{M,\Omega}(\partial\Omega) := \inf\{\lambda\colon M_\Omega^\lambda(\partial\Omega) < \infty\}.$$

The last of these quantities is called the Minkowski dimension of $\partial\Omega$ relative to Ω. It is known and easy to show that $\dim_{M,\Omega}(\partial\Omega) \leqslant N$. However this dimension need not be strictly less than N. There exist Ω such that $M_\Omega^\lambda(\partial\Omega) = \infty$ for all $\lambda \in (0, N)$. See [8, Remark 4.3 and the associated reference].

Recall now that a Whitney covering $\mathfrak{W}$ of Ω is a family of cubes Q each of edge length $L_Q = 2^{-k}$, $k \in \mathbb{N}$ and diameter $D_Q = L_Q\sqrt{N}$ such that the following three properties hold:

(i) $\Omega = \bigcup_{Q\in\mathfrak{W}} Q$;
(ii) the interiors of distinct cubes are disjoint;
(iii) $1 \leqslant \operatorname{dist}(Q, \partial\Omega)/D_Q \leqslant 4$.

It is known (Stein [19]) that such a covering exists for any bounded Ω. Condition (iii) in particular means that there are fixed constants c_1, c_2 such that

$$c_1 L_Q \leqslant d(s) \leqslant c_2 L_Q \tag{6.18}$$

for any $s \in Q$; (we will abbreviate inequalities like (6.18) or $d(s) \leqslant c_2 L_Q$ by the notation $d(s) \approx L_Q$ or $d(s) \lesssim L_Q$). Note that we can take in (6.18) $c_1 = \sqrt{N}$ and $c_2 = 5\sqrt{N}$.

Now let $n(k)$ denote the number of cubes in $\mathfrak{W}_k$ where

$$\mathfrak{W}_k := \{Q \in \mathfrak{W}\colon L_Q = 2^{-k}\}$$

and k is a positive integer. The domain Ω is said to satisfy a Whitney cube #-condition if there is a continuous increasing function $h\colon (0,\infty) \to (0,\infty)$ such that $n(k) \leqslant h(k)$ for all $k \geqslant k_0 \geqslant 1$.

We use the following Lemma about Whitney cube #-conditions which is essentially that found in [15, Theorem 3.11] for the standard Minkowski dimension; however, the proof may be easily modified (also see [8, Lemma 2.2]).

Lemma 6.1. *Let Ω be bounded and $\lambda \in (0, N]$. Then $M_\Omega^\lambda(\partial\Omega) < \infty$ if, and only if, $n(k) \leqslant K2^{\lambda k}$ for all $k \geqslant k_0$ where K and k_0 are finite positive constants.*

Proof. Assume that $\dim_{M,\Omega}(\partial\Omega) < \lambda$. Then there exist $K, r_0 > 0$ such that

$$|(\partial\Omega + B(0,r)) \cap \Omega| \leqslant K r^{N-\lambda} \tag{6.19}$$

for all $r \leqslant r_0$. Take $k \in \mathbb{N}$, $k \geqslant (\log 2)^{-1} \log(12\sqrt{N}/r_0)$ and set $r = 6\sqrt{N}2^{-k}$. Then $2r \leqslant r_0$. According to de Guzman-type covering lemma (cf. for example [7, Theorem XI.5.3]) there exist points $x_1, \dots, x_m \in \partial\Omega$ and a positive constant C dependent only on dimension N such that

$$\partial\Omega \subset \bigcup_{i=1}^{m} B(x_i, r), \tag{6.20}$$

$$\sum_{i=1}^{m} \chi_{B(x_i,2r)} \leqslant C. \tag{6.21}$$

Every cube $Q \in \mathfrak{W}_k$ is contained in some of the balls $B(x_i, 2r)$, $i = 1, \dots, m$. Indeed, given $x \in Q$ take $y \in \partial\Omega$ such that $d(x) = |x - y|$. By (6.20), $y \in B(x_i, r)$ for some $i = 1, \dots, m$. Using (6.18), for every $z \in Q$ we obtain that

$$|z - x_i| \leqslant |z - x| + |x - y| + |y - x_i| \leqslant 12\sqrt{N}2^{-k} = 2r.$$

Denote by $n_i(k)$ the number of cubes $Q \in \mathfrak{W}_k$ which are contained in $B(x_i, 2r)$. Then, we use (6.19) and (6.21) to obtain

$$\begin{aligned} n(k) &\leqslant \sum_{i=1}^{m} n_i(k) \leqslant \sum_{i=1}^{m} |B(x_i, 2r) \cap \Omega| / |Q| \\ &\leqslant C 2^{Nk} |(\partial\Omega + B(0, 2r)) \cap \Omega| \\ &\leqslant CK(12\sqrt{N})^{N-\lambda} 2^{\lambda k}. \end{aligned}$$

The proof of the reversed implication can be made step by step as in the proof of Lemma (3.4) in [15]. □

Proposition 6.1. *Let $\Omega \subset \mathbb{R}^N$. Then the following conditions are equivalent:*

(i) $\dim_{M,\Omega}(\partial\Omega) < N$.

(ii) *There exists $\mu_0 > 0$ such that $\int_\Omega d(s)^{-\mu_0} < \infty$.*

Proof. We show that (i) $\Rightarrow$ (ii): Let $\mathfrak{W}$ be a Whitney covering of Ω and $\lambda = \dim_{M,\Omega}(\partial\Omega)$. Then

$$\begin{aligned}
\int_\Omega d(s)^{-\mu_0} &= \sum_{Q\in\mathfrak{W}} \int_Q d(s)^{-\mu_0} = \sum_{k=0}^{\infty} \Bigg(\sum_{Q\in\mathfrak{W}_k} \int_Q d(s)^{-\mu_0} \Bigg) \\
&\approx \Big\{ \sum_{k=0}^{\infty} n(k) 2^{-kN} (2^{-k})^{-\mu_0} \Big\} \\
&\lesssim K \Big\{ \sum_{k=0}^{\infty} 2^{(\lambda - N + \mu_0)k} \Big\}.
\end{aligned}$$

Since $\lambda < N$ it is evident that the final sum is finite (and therefore (I1) holds) for a suitable $\mu_0 < N - \lambda$.

Next we consider (ii) $\Rightarrow$ (i): Assume to the contrary that $\dim_{M,\Omega}(\partial\Omega) = N$. Then according to the "only if" part of Lemma 6.2, Ω does not satisfy the Whitney cube #-condition with $h(t) = K2^{\lambda t}$ for any $K > 0$ and $\lambda \in (0, N)$. Taking $\lambda = N - \mu_0$, there exists a sequence of natural numbers $k_j = k_j(\lambda)$ such that $n(k_j) > 2^{\lambda k_j}$. Then

$$\begin{aligned}
\int_\Omega d(s)^{-\mu_0} &= \sum_{k=0}^{\infty} \sum_{Q\in\mathfrak{W}_k} \int_Q d(s)^{-\mu_0} \gtrsim \sum_{k=0}^{\infty} n(k) (2^{-k})^{-\mu_0} |Q| \\
&\geqslant \sum_{j=1}^{\infty} n(k_j) (2^{-k_j})^{-\mu_0} 2^{-k_j N} \\
&> \sum_{j=1}^{\infty} 2^{k_j(\lambda + \mu_0 - N)} \\
&= \infty.
\end{aligned}$$

□

If we combine Proposition 6.1 and Example 6.6, we obtain the following result.

Theorem 6.1. *Let* $\max\{N/m, 1\} \leqslant p, q < \infty$, *the first inequality being sharp if* $m < N$, *let* Ω *be bounded,* $\dim_{M,\Omega}(\partial\Omega) = \lambda < N$, $X = L^q(\Omega; d(t)^\gamma)$, $Y = L^p(\Omega; d(t)^\alpha)$ *where* $\alpha \leqslant 0$ *and* $\lambda - N < \gamma \leqslant 0$. *Then if* F *satisfies* (F1), (F2) *on* $W_0^m(X,Y)$ *and* (F3) *on* $\mathscr{P}_{m-1} \cap W_0^m(X,Y)$, *the Poincaré inequality*

$$\left(\int_\Omega d(s)^\gamma |u|^q \right)^{1/q} \leqslant K \left\{ |F(u)| + \left(\int_\Omega d(s)^\alpha |\nabla^m u|^p \right)^{1/p} \right\}$$

holds on $W_0^m(X,Y)$.

Remark 6.3. Theorem 6.1 should be compared with Edmunds and Hurri [8, Corollary 4.4] where it is shown that the Poincaré inequality is true if $p = q > N$, $m = 1$, $F(u) = \int_\Omega u/|\Omega|$, the usual Minkowski dimension $\dim_M(\partial\Omega) = \lambda < N$, $(\lambda - N)/2 < \gamma < 0$, $0 < \alpha < p - N$ and Ω satisfies the φ-quasihyperbolic boundary condition with $\varphi(t) = at^s$, $s = (N - \lambda)/2(p - 1)$. Elsewhere in this paper however the authors show (Theorem 3.1) that their type of Poincaré inequality holds for α, γ in certain ranges dependent on $\dim_M(\partial\Omega) < N$; the inequality is even true for a positive γ and negative α on Ω including those for which $\dim_M(\partial\Omega) = N$. In either case p need not satisfy any condition other than $1 \leqslant p < \infty$.

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