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SOME PROPERTIES OF AN ARCHIMEDEAN ℓ-GROUP

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1. Auxiliary results

We will use the standard notation for ℓ-groups, cf. [5]. Throughout the paper G is an ℓ-group, R is the real group, Q is the rational group and Z is the integer group. If G and H are ℓ-groups, $G \boxplus H$ denotes their cardinal sum. Let $\{G_\alpha \mid \alpha \in A\}$ be a system of ℓ-groups and let $\prod\limits_{\alpha \in A} G_\alpha$ be their product. For an element $g \in \prod\limits_{\alpha \in A} G_\alpha$ we denote the α component of g by g_α. An ℓ-group G is said to be a subdirect sum of ℓ-groups G_α, in symbols $G \subseteq' \prod\limits_{\alpha \in A} G_\alpha$, if G is an ℓ-subgroup of $\prod\limits_{\alpha \in A} G_\alpha$ such that for each $\alpha \in A$ and each $g' \in G_\alpha$ there exists $g \in G$ with the property $g_\alpha = g'$. An ℓ-group G is said to be an ideal subdirect sum of ℓ-groups G_α, in symbols $G \subseteq^* \prod\limits_{\alpha \in A} G_\alpha$, if $G \subseteq' \prod\limits_{\alpha \in A} G_\alpha$ and G is an ℓ-ideal of $\prod\limits_{\alpha \in A} G_\alpha$. We denote the ℓ-subgroup of $\prod\limits_{\alpha \in A} G_\alpha$ consisting of the elements with only finitely many non-zero components by $\sum\limits_{\alpha \in A} G_\alpha$. An ℓ-group G is said to be a completely subdirect sum, if G is an ℓ-subgroup of $\prod\limits_{\alpha \in A} G_\alpha$ and $\sum\limits_{\alpha \in A} G_\alpha \subseteq G$.

A subset $\{0\} \neq D \subset G$ is said to be disjoint, if $g_1 \wedge g_2 = 0$ for any pair of distinct elements $g_1, g_2 \in D$. For any $X \subset G$ we designate $X^\perp = \{g \in G \mid |g| \wedge |x| = 0$ for each $x \in X\}$. For $g \in G$, $[g]$ is the convex ℓ-subgroup of G generated by g, $(g) = g^{\perp\perp}$ is the polar subgroup of G generated by g. We denote the least cardinal α such that $|A| \leqslant \alpha$ for each bounded disjoint subset A of G by vG, where $|A|$ denotes the cardinal of A. G is said to be v-homogeneous if $vH = vG$ for any convex ℓ-subgroup $H \neq \{0\}$ of G. If G is an archimedean v-homogeneous ℓ-group and $vG = \aleph_i$, we call G an archimedean v-homogeneous ℓ-group of $\aleph_i$ type.

In [9] we proved that an ℓ-group G is complete if and only if G is ℓ-isomorphic to an ideal subdirect sum of real groups, integer groups and continuous v-homogeneous complete ℓ-groups. By using this result, we described the structure of an archimedean

ℓ-group in [10]. Suppose that G is a subdirect sum of subgroups of reals and v-homogeneous ℓ-groups, $G \subseteq' \prod_{\delta\in\Delta} T_\delta$. Let $\Delta_1 = \{\delta \in \Delta \mid T_\delta$ is a subgroup of reals$\}$. If $\sum_{\delta\in\Delta_1} T_\delta \subseteq G$, then G is said to be a semicomplete subdirect sum of subgroups of reals and v-homogeneous ℓ-groups of $\aleph_i$ type, in symbols

$$\sum_{\delta\in\Delta_1\subseteq\Delta} T_\delta \subseteq G \subseteq' \prod_{\delta\in\Delta} T_\delta. \tag{1.1}$$

Theorem 1.1 (Theorem 4.7 of [10]). *An ℓ-group G is archimedean if and only if G is ℓ-isomorphic to a semicomplete subdirect sum of subgroups of reals and archimedean v-homogeneous ℓ-groups of $\aleph_i$ type.*

Now let G be an archimedean ℓ-group. Then we have an ℓ-isomorphism ϱ such that

$$\sum_{\delta_1\in\Delta_1\subseteq\Delta} T_{\delta_1} \subseteq \varrho G \subseteq' \prod_{\delta\in\Delta} T_\delta,$$

where T_{δ_1} is a subgroup of reals for each $\delta_1 \in \Delta_1 \subseteq \Delta$ and T_δ is an archimedean v-homogeneous ℓ-group of $\aleph_i$ type for each $\delta \in \Delta \setminus \Delta_1$. For $x \in G$ put $x^1 = (\dots x^1_\delta \dots)$ such that

$$x^1_\delta = \begin{cases} (\varrho x)_\delta & \delta \in \Delta_1, \\ 0 & \delta \in \Delta \setminus \Delta_1. \end{cases}$$

We call x^1 the real part of x. If for any $x \in G$, the real part $x^1 \in \varrho G$, G is said to be real decomposable archimedean ℓ-group. In this case, if we put $x^2 = (\dots x^2_\delta \dots)$ as follows:

$$x^2_\delta = \begin{cases} 0 & \delta \in \Delta_1, \\ (\varrho x)_\delta & \delta \in \Delta \setminus \Delta_1 \end{cases}$$

then

$$\varrho x = x^1 + x^2,$$

and $x^2 = \varrho x - x^1 \in \varrho G$. Put

$$G_1 = \{\varrho x \in \varrho G \mid x \in G, (\varrho x)_\delta = 0 \text{ for } \delta \in \Delta \setminus \Delta_1\},$$
$$G_2 = \{\varrho x \in \varrho G \mid x \in G, (\varrho x)_\delta = 0 \text{ for } \delta \in \Delta_1\}.$$

Then both G_1 and G_2 are ℓ-subgroups of ϱG, moreover,

$$\varrho G = G_1 \boxplus G_2.$$

It is clear that $G_2 = R(\varrho G)$ (the radical of G) and $G_1 = R(\varrho G)^\perp$.

Corollary 1.2. *Let G be a real decomposable archimedean ℓ-group. Then G is ℓ-isomorphic to a cardinal sum $G_1 \boxplus G_2$, where G_1 is a completely subdirect sum of subgroups of reals and G_2 is a subdirect sum of archimedean v-homogeneous ℓ-groups of $\aleph_i$ type.*

So, if G is a real decomposable archimedean ℓ-group, then $G = R(G) \boxplus R(G)^{\perp}$. However, in general, $R(G)$ is not a cardinal summand of G. If G is complete or laterally complete, then $R(G)$ is a cardinal summand.

2. A completely subdirect sum of subgroups of reals

Now we can characterize those ℓ-groups which can be represented as completely subdirect sums of subgroups of reals.

Theorem 2.1. *Let $G \neq \{0\}$ be an archimedean ℓ-group. Then the following properties are equivalent:*
(1) G is ℓ-isomorphic to a completely subdirect sum of subgroups of reals;
(2) G is ℓ-isomorphic to an ideal subdirect sum of real groups and integer groups;
(3) G has a basis.

Proof. (1) $\Rightarrow$ (2): Without loss of generality, assume

$$\sum_{\delta\in\Delta} T_\delta \subseteq G \subseteq' \prod_{\delta\in\Delta} T_\delta,$$

where each T_δ is a subgroup of R for $\delta \in \Delta$. Then

$$G^{\wedge} \subseteq^{*} \prod_{\delta\in\Delta} T_\delta^{\wedge},$$

where $T_\delta^{\wedge} = R$ or Z for $\delta \in \Delta$.

(2) $\Rightarrow$ (1): It is similar to the proof of Theorem 1.1.

(1) $\Rightarrow$ (3): If we have the formula (1.1), then for each $\delta \in \Delta$ we choose a fixed t_δ with $0 < t_\delta \in T_\delta$; the system $\{t_\delta \mid \delta \in \Delta\}$ is a basis for G.

(3) $\Rightarrow$ (2): See Theorem 3.5 in [5]. □

By Theorem 4 and Corollary IV of Chapter 3 in [5] we see that an archimedean ℓ-group G has a finite basis if and only if G is ℓ-isomorphic to a completely subdirect sum of a finite number of subgroups of reals. However, a completely subdirect sum of a finite number of subgroups of reals is a cardinal sum of a finite number of subgroups of reals. So we get

Corollary 2.2. *An archimedean ℓ-group G has a finite basis if and only if G is ℓ-isomorphic to a cardinal sum of a finite number of subgroups of reals.*

3. Hyper-archimedean property

An ℓ-group G is called hyper-archimedean if each ℓ-homomorphic image of G is archimedean.

Proposition 3.1. *An ℓ-group G is hyper-archimedean if and only if G is projectable and $[g] = (g)$ for each $0 < g \in G$.*

Proof. Necessity. Suppose that G is hyper-archimedean. For any $0 < g \in G$ we have $g^{\perp} \boxplus (g) \subseteq G$. But $g^{\perp} \boxplus (g) \supseteq g^{\perp} \boxplus [g] = G$ by Theorem 2.4 in [5]. So $G = g^{\perp} \boxplus (g)$, and G is projectable. From $G = g^{\perp} \boxplus [g] = g^{\perp} \boxplus (g)$ we get $[g] = (g)$.

Sufficiency. If G is projectable and $0 < g \in G$, then $G = g^{\perp} \boxplus (g)$. Since $[g] = (g)$, $G = g^{\perp} \boxplus [g]$. Hence G is hyper-archimedean. □

An ℓ-group G is an a-extension of an ℓ-group H if and only if H is an ℓ-subgroup of G and the map $L \to L \cap H$ is a one-to-one map of the set of all convex ℓ-subgroups of G onto those of H. G is a-closed if it admits no proper a-extension.

Corollary 3.2. *Let G be a hyper-archimedean ℓ-group with a basis. If G is a-closed, then $G/P \simeq R$ for each proper prime P.*

Proof. Let G be an a-closed hyper-archimedean ℓ-group with a basis. By the above Theorem 2.1, without loss of generality, we have

$$\sum_{\delta \in \Delta} T_\delta \subseteq G \subseteq' \prod_{\delta \in \Delta} T_\delta,$$

where each T_δ is a subgroup of reals. Let P be a proper prime. By Theorem 2.4 in [5] P is maximal and $P = \{x \in G \mid x_{\delta_0} = 0 \text{ for some } \delta_0 \in \Delta\}$. So

$$G = T_{\delta_0} \boxplus P$$

and $G/P \simeq T_{\delta_0}$. If G/P_0 fails to be isomorphic to R for some proper prime P_0, then $G' = R \boxplus P_0 \supseteq Q \boxplus P_0$ or $G' = R \boxplus P_0 \supseteq Z \boxplus P_0$ is clearly an a-extension of G, a contradiction. Therefore $G/P = R$ for each proper prime P. □

This corollary partly answers the question of the Corollary 2 in [2].

Next we discuss the hyper-archimedean kernel $\mathrm{Ar}(G)$ of an archimedean ℓ-group G. $\mathrm{Ar}(G)$ is a convex ℓ-subgroup of G which is hyper-archimedean and contains every hyper-archimedean convex ℓ-subgroup of G. An ℓ-group G is continuous if for each $0 < x \in G$ there exist x_1, $x_2 \in G$ such that $x = x_1 + x_2$, $x_1 \wedge x_2 = 0$, $x_1 \neq 0$ and $x_2 \neq 0$.

Lemma 3.3. *Let G be a complete (laterally complete and archimedean) divisible v-homogeneous ℓ-group of $\aleph_i$ type. Then* $\mathrm{Ar}(G) = 0$.

Proof. First we can show that a projectable v-homogeneous ℓ-group of $\aleph_i$ type is continuous. In fact, $v[x] = vG = \aleph_i$ for any $0 < x \in G$. So there exist $0 < a_1 < x$ and $0 < a_2 < x$ such that $a_1 \wedge a_2 = 0$. Then $G = a_1^{\perp} \boxplus a_2^{\perp\perp}$ and so $x = x_1 + x_2$ with $x_1 \in a_1^{\perp}$ and $x_2 \in a_1^{\perp\perp}$. It is clear that $x \notin a_1^{\perp}$ and $x \notin a_1^{\perp\perp}$. Hence $x_1 \neq 0$, $x_2 \neq 0$.

Now let G be a complete (laterally complete and archimedean) divisible v-homogeneous ℓ-group of $\aleph_i$ type. Then G is projectable (see [5], [4]) and continuous. Consider the Bernau representation ([3])

$$\varrho \colon G \to \widehat{G} \subseteq D(X_G).$$

For any $0 < x \in G$ there exists a maximal disjoint subset X in G such that $x \in X$. By Theorem 3.3 in [6] we can choose ϱ such that $\widehat{x}$ is the characteristic function of a clopen subset S in X_G. Since G is continuous, $\widehat{G}$ is also continuous. So $\widehat{x} = x_1^1 + x_1^2$ with $x_1^1 \wedge x_1^2 = 0$ and $x_1^1 \neq 0$, $x_1^2 \neq 0$. For $0 < x_1^2 \in \widehat{G}$ we also have $x_1^2 = x_2^1 + x_2^2$ with $x_2^1 \wedge x_2^2 = 0$ and $x_2^1 \neq 0$, $x_2^2 \neq 0$. We continue to get a sequence $\{x_n^1 \mid n = 1, 2, \ldots\}$ in $\widehat{G}$ such that

$$x_n^1 = \chi_{S(x_n^1)}, \quad x_n^1 \wedge x_m^1 = 0 \ (n \neq m)$$

and

$$S(x_n^1) \subseteq S(x), \quad S(x_n^1) \cap S(x_m^1) = \emptyset \ (n \neq m),$$

where $S(x_n^1)$ is the support of x_n^1 and $\chi_{S(x_n^1)}$ is the characteristic function on $S(x_n^1)$. Put

$$x_n = \frac{1}{n} x_n^1 \quad \text{and} \quad \overline{x} = \bigvee_{n=1}^{\infty}{}^{(\widehat{G})} x_n.$$

Then x_n, $\overline{x} \in G$. Now $(\widehat{x} \wedge n\overline{x})(t) = \frac{n}{n+1}$ for $t \in S(x_{n+1}^1)$. On the other hand $[\widehat{x} \wedge (n+1)\overline{x}](t) = 0$. Therefore

$$\widehat{x} \wedge n\overline{x} = \widehat{x} \wedge (n+1)\overline{x}.$$

This proves that $\mathrm{Ar}(G) = 0$ by Lemma 2.1 in [8]. □

Proposition 3.4. *Let G be a complete (laterally complete and archimedean) v-homogeneous ℓ-group of $\aleph_i$ type. Then* $\mathrm{Ar}(G) = 0$.

Proof. By Lemma 3.3, $\mathrm{Ar}(G^d) = 0$ where G^d is the divisible hull of G. For any $0 < x \in G$ and any $n \in N$ we have

$$[x]^G = \left[\frac{x}{n}\right]^{G^d} \quad \text{and} \quad x_G^{\perp} = \left(\frac{x}{n}\right)_{G^d}^{\perp},$$

where $[x]^G$ is the convex ℓ-subgroup of G generated by x and $\left[\frac{x}{n}\right]^{G^d}$ is the convex ℓ-subgroup of G^d generated by $\frac{x}{n}$, $x_G^{\perp}$ and $\left(\frac{x}{n}\right)_{G^d}^{\perp}$ are polars in G and in G^d, respectively. Hence

$$[x]^G \boxplus x_G^{\perp} \subseteq \left[\frac{x}{n}\right]^{G^D} \boxplus \left(\frac{x}{n}\right)_{G^d}^{\perp}.$$

By Corollary 2.1.1 in [8] we get

$$\mathrm{Ar}(G) = \bigcap_{0<x\in G} \left([x]^G \boxplus x_G^{\perp}\right) = \bigcap_{\substack{0<x\in G\\ n\in N}} \left(\left[\frac{x}{n}\right]^{G^d} \boxplus \left(\frac{x}{n}\right)_{G^d}^{\perp}\right) = \mathrm{Ar}(G^d) = 0.$$

□

Theorem 3.5. *Let G be a complete ℓ-group. Then $\mathrm{Ar}(G)$ is an ideal subdirect sum of real groups and integer groups.*

Proof. By Proposition 2.2 in [9], without loss of generality, we have

$$\sum_{\delta\in\Delta} T_\delta \subseteq G \subseteq^* \prod_{\delta\in\Delta} T_\delta,$$

where each T_δ ($\delta \in \Delta$) is R or Z or a complete v-homogeneous ℓ-group of $\aleph_i$ type. Put $\Delta_1 = \{\delta \in \Delta \mid T_\delta = R \text{ or } Z\}$, $\Delta_2 = \Delta \setminus \Delta_1$. Assume $x \in \mathrm{Ar}(G)$. For any $\delta_0 \in \Delta_2$ and any $a_{\delta_0} \in T_{\delta_0}$ we have $\overline{a}_{\delta_0} = (\dots 0 \dots a_{\delta_0} \dots 0 \dots) \in G$. So there exists $n \in N$ such that

$$x \wedge n\overline{a}_{\delta_0} = x \wedge (n+1)\overline{a}_{\delta_0}.$$

Hence

$$x_{\delta_0} \wedge n a_{\delta_0} = x_{\delta_0} \wedge (n+1) a_{\delta_0}.$$

By Lemma 2.1 in [8] this means that $x_{\delta_0} \in \mathrm{Ar}(T_{\delta_0})$. However, by Proposition 3.4, $\mathrm{Ar}(T_{\delta_0}) = 0$. So $x_{\delta_0} = 0$. Therefore

$$\mathrm{Ar}(G) \subseteq' \prod_{\delta\in\Delta_1} T_\delta.$$

By Lemma 2.1. in [8] it is clear that $\sum_{\delta\in\Delta_1} T_\delta \subseteq \mathrm{Ar}(G)$. Since $\mathrm{Ar}(G)$ is convex in G and G is convex in $\prod_{\delta\in\Delta_1} T_\delta$, $\mathrm{Ar}(G)$ is convex in $\prod_{\delta\in\Delta_1} T_\delta$. So we have

$$\sum_{\delta\in\Delta_1} T_\delta \subseteq \mathrm{Ar}(G) \subseteq^* \prod_{\delta\in\Delta_1} T_\delta.$$

□

Corollary 3.6. *If a complete ℓ-group G is hyper-archimedean, then G is an ideal subdirect sum of real groups and integer groups.*

Theorem 3.7. *Let G be a complete ℓ-group. Then $\mathrm{Ar}(G)$ is dense in G if and only if G is an ideal subdirect sum of real groups and integer groups.*

Proof. Necessity. By the proof of Theorem 3.5 we have

$$\sum_{\delta\in\Delta_1} T_\delta \subseteq \mathrm{Ar}(G) \subseteq G \subseteq^* \prod_{\delta\in\Delta} T_\delta$$

and

$$\sum_{\delta\in\Delta_1} T_\delta \subseteq \mathrm{Ar}(G) \subseteq' \prod_{\delta\in\Delta_1} T_\delta, \tag{3.1}$$

where $T_\delta = R$ or Z $(\delta \in \Delta_1)$. Since G is complete, $\mathrm{Ar}(G)^{\perp\perp}_G \subseteq \prod_{\delta\in\Delta_1} T_\delta$ by (3.1). Since $\mathrm{Ar}(G)$ is dense in G, $G = \mathrm{Ar}(G)^{\perp\perp}_G$. Hence

$$\sum_{\delta\in\Delta_1} T_\delta \subseteq \mathrm{Ar}(G) \subseteq \mathrm{Ar}(G)^{\perp\perp}_G = G \subseteq^* \prod_{\delta\in\Delta_1} T_\delta.$$

Sufficiency. Let

$$\sum_{\delta\in\Delta_1} T_\delta \subseteq G \subseteq^* \prod_{\delta\in\Delta_1} T_\delta,$$

where each $T_\delta = R$ or Z $(\delta \in \Delta_1)$. Since

$$\sum_{\delta\in\Delta_1} T_\delta \subseteq \mathrm{Ar}(G) \subseteq G,$$

$\mathrm{Ar}(G)$ is dense in G. □

Corollary 3.8. *Let G be a complete ℓ-group. Then $\mathrm{Ar}(G)$ is dense in G if and only if G has a basis.*

Theorem 3.7 and Corollary 3.8 partly answer the question and conjecture in [8].

4. Projectability

It is well known that a complete (σ-complete) ℓ-group is projectable. M. Anderson defined some weak concepts of projectability in [1]. An ℓ-group G is called subprojectable if for each $0 < x \in G$ and each non-zero polar $P \subseteq x^{\perp\perp}$ there exists a non-zero polar Q such that $Q \subseteq P$ and $x = Q \boxplus Q^{\perp}$. G is called densely projectable if it has a family $\mathcal{F}$ of non-trivial cardinal summands such that if $\{0\} \neq P \in P(G)$ then there exists a $Q \in \mathcal{F}$ such that $Q \subseteq P$, where $P(G)$ is the Boolean algebra of all polars in G.

Suppose that H is an ℓ-subgroup of an ℓ-group G. H is called a signature for G if $P \to P \cap H$ is a Boolean isomorphism from $P(G)$ onto $P(H)$. An ℓ-group G is a specker group if it is generated as a group by its singular elements. Assume $0 < x \in G$. If $x = x_1 + x_2$, $x_1 \wedge x_2 = 0$ in G, we call x_1 (and x_2) a component of x. We call $0 \leqslant x \in G$ a specker sign if for each $0 < y \leqslant x$ there exists a non-zero component x_1 of x in $y^{\perp\perp}$. We will say that G has a specker signature if it has a signature if it has a signature which happens to be a specker ℓ-subgroup.

Let G be an archimedean ℓ-group. We denote by G^e the essential closure of G (see [6]). An element $0 < x \in G$ is said to be saturated if, whenever there exist x_1, $x_2 \in G^e$ with $x_1 \wedge x_2 = 0$ in G^e such that $x = x_1 + x_2$, then $x_1 \in G$. An archimedean ℓ-group G is said to be saturated if each $0 < x \in G$ is saturated. For example, a divisible complete ℓ-group is saturated.

Proposition 4.1. *A subprojectable v-homogeneous ℓ-group G of $\aleph_i$ type is continuous.*

Proof. By Theorem 6 in [7] each $0 \leqslant x \in G$ is a specker sign. $v[x] = vG = \aleph_i$ implies that x is not basic. It follows from Lemma 9 of [7] that G is continuous. □

Proposition 4.2. *A saturated archimedean ℓ-group is subprojectable.*

Proof. Let G be a saturated archimedean ℓ-group. Consider the Bernau representation

$$\pi\colon G \to \widehat{G} \subseteq D(X_G),$$
$$x \to \hat{x} \in \widehat{G}.$$

Let $0 < y \leqslant x \in G$. By Theorem 3.3 in [6] the ℓ-isomorphism π can be chosen so that $\hat{y}$ is the characteristic function of a clopen subset S of the Stone space X_G. Put $S' = S(\hat{x}) \setminus S$ where $S(\hat{x})$ is the support of $\hat{x}$. Then S' is also a clopen subset of X_G and

$$S(\hat{x}) = S \cup S'.$$

So we have

$$D\big(S(\hat{x})\big) = D(S) \boxplus D(S'),$$
$$\hat{x} = \hat{x}_1 + \hat{x}_2,$$

where $\hat{x}_1 \in D(S)$, $\hat{x}_2 \in D(S')$ and $D(S)\big(D(S')\big) = \{f\colon S(S') \to (R, \pm\infty) \mid f$ is continuous and f is real on a dense open subset of $S(S')\}$. Since G is saturated, so is $\widehat{G}$. Hence $\hat{x}_1 \in \widehat{G}$. It is clear that

$$y_{\widehat{G}}^{\perp\perp} = \{\hat{g} \in \widehat{G} \mid S(\hat{g}) \subseteq S\}$$

(see [3]). So we have $\hat{x}_1 \in y_{\widehat{G}}^{\perp\perp}$. This proves that $\hat{x}$ is a specker sing. Hence each $0 \leqslant x \in G$ is a specker sign. By Theorem 6 in [7], G is subprojectable. □

Corollary 4.3. *A saturated archimedean v-homogeneous ℓ-group of $\aleph_i$ type is continuous.*

From Theorem 7 in [7] we have

Corollary 4.4. *A saturated archimedean ℓ-group has a specker signature.*

In [1] M. Anderson proved that G is subprojectable if and only if each $[x]$ is densely projectable. so from Proposition 4.2 we have

Corollary 4.5. *Let G be a saturated archimedean ℓ-group. Then each $[x]$ $(x \in G)$ is densely projectable.*

Proposition 4.6. *Let G be an archimedean ℓ-group with a basis. Then G is subprojectable.*

Proof. By Theorem 1.1 we have

$$\sum_{\delta \in \Delta} T_\delta \subseteq G \subseteq' \prod_{\delta \in \Delta} T_\delta,$$

where each T_δ $(\delta \in \Delta)$ is a subgroup of reals. Then for each $\delta \in \Delta$ we choose a fixed t_δ with $0 < t_\delta \in T_\delta$; the system $\{t_\delta \in T_\delta \mid \delta \in \Delta\}$ is a maximal disjoint subset and each t_δ is a specker sign (each basic is a specker sign). By 4(b) in [7], G has a specker signature. It follows from Theorem 7 in [7] that G is subprojectable. □

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