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UNIQUE IRREDUNDANT DECOMPOSITIONS IN UPPER CONTINUOUS LATTICES

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1. INTRODUCTION

Let L be a complete lattice. Lattice join, meet, inclusion and proper inclusion are denoted respectively by the symbols $\vee$, $\wedge$, $\leqslant$ and $<$. Let 0 be the least element of L, and 1 the greatest element of L.

An element $m \in L$ is called irreducible if and only if, for all $x, y \in L$, $m = x \wedge y$ implies $m = x$ or $m = y$. $M(L)$ is defined to be the set of all irreducible elements of L.

If a is an element of the lattice L, then a representation $a = \bigwedge T$ with $T \subseteq M(L)$ is called a (meet) decomposition of a. A decomposition $a = \bigwedge T$ is irredundant if $\bigwedge(T - \{t\}) \neq a$ for all $t \in T$.

If every element of L has exactly one irredundant decomposition, then we say that L has unique irredundant decompositions.

A complete lattice L is called upper continuous iff, for every $a \in L$ and for every chain $C \subseteq L$, $a \wedge \bigvee C = \bigvee(a \wedge c\colon c \in C)$.

For two elements $a, b \in L$ $(a < b)$ we define

$$b/a := \{x\colon a \leqslant x \leqslant b\}.$$

If $b/a = \{a, b\}$, then we say that b covers a, notation $a \prec b$. A lattice L is said to be weakly atomic if, for every pair of elements $a, b \in L$ with $a < b$, there exist elements $u, v \in L$ such that $a \leqslant u \prec v \leqslant b$.

L is strongly atomic if, whenever $a < b$, there is an element $p \in L$ with $a \prec p \leqslant b$. In a complete strongly atomic lattice L, for each $a \in L$ let P_a denote the set of all elements covering a, and we set $u_a = \bigvee P_a$.

If for every $a \in L$ the sublattice u_a/a is distributive, then we say that L is locally distributive.

We know that each element of an upper continuous weakly atomic lattice has a decomposition ([3], p. 338). Furthermore, if an upper continuous lattice L is strongly atomic, then every element of L has an irredundant decomposition ([4], Theorem 10).

In this paper we prove the following

Theorem. *An upper continuous strongly atomic lattice L has unique irredundant decompositions if and only if L is locally distributive.*

2. Some lemmas

We start this section with the following

Lemma 1. *If a, b are elements of an upper continuous strongly atomic lattice L and $a \ngeqslant b$, then there exists an element $m \in M(L)$ such that $m \geqslant a$ and $m \ngeqslant b$.*

Proof. Since L is strongly atomic and $a < a \vee b$, there exists an element $p \in L$ such that $a \prec p \leqslant a \vee b$. Let

$$T := \{x \in L\colon x \geqslant a,\ x \ngeqslant p\}.$$

T is nonempty, since $a \in T$. Let C be a chain in T. Then upper continuity yields

$$p \wedge \bigvee C = \bigvee (p \wedge c\colon c \in C) = a.$$

Thus $\bigvee C \in T$, and by Zorn's lemma T contains a maximal element m. Clearly, $m \in M(L)$, $m \geqslant a$ and $m \ngeqslant b$. $\square$

In the proofs of Theorems 3.7 and 7.3 from [1] it was not used that L is an algebraic lattice, only that L is upper continuous. Therefore, Lemmas 2 and 3 below can be proved analogously, and their proofs will be omitted.

Lemma 2. *If an upper continuous, strongly atomic lattice L has the property that, for all $a, b \in L$, $a \wedge b \prec a, b$ implies $a, b \prec a \vee b$, then L is semimodular.*

In view of this lemma, every locally distributive, upper continuous, strongly atomic lattice is semimodular.

Lemma 3. *If a, b, p_1, p_2 are elements of a locally distributive, upper continuous, strongly atomic lattice L, and if $p_1, p_2 \in P_a$, $b \wedge (p_1 \vee p_2) = a$ and $p_1 \vee b = p_2 \vee b$, then $p_1 = p_2$.*

By Zorn's lemma we get

Lemma 4. *Let L be an upper continuous lattice and let $a, b, c \in L$. If $a = b \wedge c$, then the set $\{x \in L\colon x \geqslant c,\ a = b \wedge x\}$ has a maximal element.*

The next lemma is a generalization of Lemma 6.2 from [2].

Lemma 5. *If an upper continuous strongly atomic lattice L has unique irredundant decompositions, then L is semimodular.*

Proof. By the proof of Lemma 6.2 ([2], p. 17) we conclude that our lemma follows from Lemmas 1 and 4. $\square$

For our investigations we need the following concept. A subset A of a complete lattice L is said to be independent if $a \wedge \bigvee(A - \{a\}) = 0$ for all $a \in A$.

Lemma 6. *Let L be an upper continuous strongly atomic lattice. If an element $a \in L$ has a unique irredundant decomposition, then P_a is an independent subset of $1/a$.*

Proof. Let p be an arbitrary element of the set P_a. Now we prove that

$$\text{(1)} \qquad \text{for every finite subset } X \text{ of } P_a - \{p\}, \qquad p \nleqslant \bigvee X.$$

Suppose that there is a finite subset Q of $P_a - \{p\}$ which contains a minimal number of elements such that $p \leqslant \bigvee Q$. Let q be an element of Q and set $s := \bigvee(Q - \{q\})$. Obviously $p \nleqslant s$. By Lemma 1 there exist irreducible elements m_1 and m_2 such that $m_1 \geqslant q$, $m_2 \geqslant s$, $m_1 \ngeqslant p$ and $m_2 \ngeqslant p$. Consequently $m_1 \wedge p = m_2 \wedge p = a$. Lemma 4 implies that there are maximal elements $w_1, w_2 \geqslant p$ such that $m_1 \wedge w_1 = m_2 \wedge w_2 = a$.

Since L is an upper continuous strongly atomic lattice, every element of L has an irredundant decomposition. Let $w_1 = \bigwedge T_1$ and $w_2 = \bigwedge T_2$ be irredundant decompositions of w_1 and w_2, respectively. Then

$$a = m_1 \wedge \bigwedge T_1 = m_2 \wedge \bigwedge T_2.$$

Moreover, these decompositions are irredundant, since $\bigwedge T_1, \bigwedge T_2 \geqslant p > a$, and the maximality of w_1 and w_2 implies that $m_1 \wedge \bigwedge(T_1 - \{t_1\}), m_2 \wedge \bigwedge(T_2 - \{t_2\}) > a$ for every $t_1 \in T_1$ and $t_2 \in T_2$.

Note that $m_1 \notin T_2$, otherwise $m_1 \geqslant w_2 \geqslant p$, contradicting $m_1 \ngeqslant p$. Since a has a unique irredundant decomposition we have $m_1 = m_2$. Therefore $m_1 \geqslant q$ and $m_1 \geqslant s$. Hence $m_1 \geqslant q \vee s = \bigvee Q \geqslant p$, a contradiction. Thus we obtain (1). Therefore, by 2.4 [1] we have $p \nleqslant \bigvee(P_a - \{p\})$. Thus P_a is an independent subset of $1/a$. $\square$

Now we will prove

Lemma 7. *Let L be an upper continuous semimodular lattice and let P be the set of all atoms of L. If P is an independent subset of L and every element of L is a join of elements of P, then L is distributive.*

Proof. By Theorem 4.1 from [1], L is both atomic and complemented. Observe that

$$\text{if } 1 = \bigvee T, \text{ where } T \subseteq P, \text{ then } T = P. \tag{2}$$

Indeed, if $T \neq P$, then there exists an element $p \in P - T$, and hence

$$0 < p = p \wedge \bigvee T \leqslant p \wedge \bigvee (P - \{p\}),$$

contrary to the independence of P.

Now we prove that L is a uniquely complemented lattice. Let $x \in L$. Suppose $u_1, u_2 \in L$ are such that

$$x \vee u_1 = x \vee u_2 = 1 \tag{3}$$

and

$$x \wedge u_1 = x \wedge u_2 = 0. \tag{4}$$

Since every element of L is a join of atoms, there are subsets X, U_1, U_2 of P such that $x = \bigvee X$, $u_1 = \bigvee U_1$ and $u_2 = \bigvee U_2$. By (3), $1 = \bigvee (X \cup U_1) = \bigvee (X \cup U_2)$ and from (2) it follows that $X \cup U_1 = X \cup U_2 = P$. By (4) we have $X \cap U_1 = X \cap U_2 = \emptyset$. Consequently, $U_1 = U_2$ and hence $u_1 = u_2$. Thus L is a uniquely complemented lattice. Then, by Theorem 4.5 [1], L is distributive. □

Lemma 8. *Let L be a locally distributive upper continuous strongly atomic lattice and let $a \in L$. Then $m \geqslant a$ and $m \ngeqslant u_a$ imply $m \wedge u_a \prec u_a$ for each $m \in M(L)$.*

Proof. Lemma 2 implies that L is semimodular. Let

$$b := m \wedge u_a.$$

Since $m \ngeqslant u_a$ we have $b < u_a$. Then there is an element $p_1 \in P_a$ such that $p_1 \nleqslant b$. By semimodularity, $b \prec b \vee p_1$. Suppose that u_a does not cover b. Then $b \vee p_1 < u_a$, and therefore there exists an element $p_2 \in P_a$ such that

$$p_2 \nleqslant b \vee p_1. \tag{5}$$

Since $b, p_1, p_2 \in u_a/a$ and u_a/a is distributive by hypothesis, we obtain

$$b \wedge (p_1 \vee p_2) = (b \wedge p_1) \vee (b \wedge p_2).$$

We have $p_1 \not\leqslant b$ and $p_2 \not\leqslant b$, and hence $b \wedge p_1 = b \wedge p_2 = a$. Therefore, $b \wedge (p_1 \vee p_2) = a$. Then

$$m \wedge (p_1 \vee p_2) = m \wedge u_a \wedge (p_1 \vee p_2) = b \wedge (p_1 \vee p_2) = a.$$

Since $b = m \wedge u_a \not\geqslant p_1$ we conclude that $m \not\geqslant p_1$. Also $m \not\geqslant p_2$, since otherwise $b \vee p_1 \geqslant b = u_a \wedge m \geqslant p_2$, contrary to (5). Now, by semimodularity, $m \prec m \vee p_1$ and $m \prec m \vee p_2$, and as $m \in M(L)$ and hence is covered by a unique element, we conclude that

$$m \vee p_1 = m \vee p_2.$$

By Lemma 3 we obtain $p_1 = p_2$, contrary to (5). Thus $m \wedge u_a \prec u_a$, and proof of Lemma 8 is completed. □

Finally, we prove

Lemma 9. *Let L be a locally distributive upper continuous strongly atomic lattice, and let a be an element of L. If $p \in P_a$, $x, y \geqslant a$ and $x \in M(L)$, then*

$$p \wedge (x \vee y) = (p \wedge x) \vee (p \wedge y). \tag{6}$$

P r o o f. Suppose the assumptions of Lemma 9 are fulfilled but $p \wedge (x \vee y) > (p \wedge x) \vee (p \wedge y)$. Consequently, $p \wedge (x \vee y) = p$ and $p \wedge x = p \wedge y = a$. Then $p \leqslant x \vee y$, $p \not\leqslant x$ and $p \not\leqslant y$. Set

$$b := x \wedge y.$$

We have $b < y$, since otherwise $y = x \wedge y \leqslant x$ and hence $p \leqslant x \vee y = x$, a contradiction. Since L is strongly atomic, there is an element $q \in L$ such that $b \prec q \leqslant y$. By Lemma 2, L is semimodular. The semimodularity of L and the fact that $p \not\leqslant b$ imply that $b \prec p \vee b$. We denote $w := x \wedge u_b$. By the assumption, $x \in M(L)$. Note that $x \not\geqslant u_b$, otherwise $x \geqslant u_b \geqslant p \vee b \geqslant p$, contradicting $p \not\leqslant x$. It follows from Lemma 8 that

$$w \prec u_b. \tag{7}$$

We shall prove that $w \not\geqslant q$. Suppose on the contrary that $w \geqslant q$. Then $x \geqslant x \wedge u_b = w \geqslant q$. But also $y \geqslant q$, and hence $b = x \wedge y \geqslant q$, a contradiction. Therefore, $w \not\geqslant q$. From this and (7) we obtain

$$w \vee q = u_b. \tag{8}$$

Since $x \not\geqslant p$ and $y \not\geqslant p$ we have $w \not\geqslant p \vee b$ and $p \vee b \neq q$. This together with the fact that $b \prec p \vee b$ and $b \prec q$ yields that

$$(p \vee b) \wedge w = b \quad \text{and} \quad (p \vee b) \wedge q = b.$$

Since $p \vee b$, q, $w \in u_b/b$, by the distributivity of u_b/b we infer

$$(p \vee b) \wedge (w \vee q) = \big[(p \vee b) \wedge w\big] \vee \big[(p \vee b) \wedge q\big] = b.$$

On the other hand, by (8),

$$(p \vee b) \wedge (w \vee q) = (p \vee b) \wedge u_b = p \vee b > b.$$

This contradiction shows that (6) holds. □

3. Proof of Theorem

Let L be an upper continuous strongly atomic lattice. Suppose that L has unique irredundant decompositions. Consider a particular element $a \in L$.

By Lemma 5, L is semimodular. Lemma 6 implies that P_a is an independent subset of u_a/a. Therefore, in view of Lemma 7, to show that u_a/a is distributive we need only to show that each element of u_a/a is a join of elements covering a.

Let x be an arbitrary element of u_a/a, and let b be the join in the sublattice u_a/a of all elements $p \in P_a$ for which $p \leqslant x$. Suppose that $b < x$. Since L is strongly atomic, there exists an element $q \in L$ such that $b \prec q \leqslant x$. By semimodularity, if $p \in P_a$ and $p \not\leqslant b$, then $b \prec p \vee b$.

Observe that $q \neq p \vee b$ for every $p \in P_a$. Indeed, if $q = p_0 \vee b$ for some element $p_0 \in P_a$, then $p_0 \leqslant p_0 \vee b = q \leqslant x$ and hence $p_0 \leqslant b$. Consequently $q = b$, a contradiction. Therefore

$$\{p \vee b \colon p \in P_a,\ p \not\leqslant b\} \subseteq P_b - \{q\}.$$

Then

$$q \leqslant x \leqslant u_a = \bigvee (p \vee b \colon p \in P_a,\ p \not\leqslant b) \subseteq \bigvee (P_b - \{q\}),$$

contrary to the fact that the set P_b is an independent subset of $1/b$. Thus $x = b$, and every element of u_a/a is a join of elements covering a.

Now, suppose that L is locally distributive. Since L is an upper continuous strongly atomic lattice, every element of L has an irredundant decomposition.

Let a be an arbitrary element of L, and let $a = \bigwedge S = \bigwedge T$ be two irredundant decompositions of a. Pick any element $s \in S$ and set $w := \bigwedge(S - \{s\})$. Obviously $w > a$. Then, as L is strongly atomic. there exists $p \in L$ such that $a \prec p \leqslant w$. Clearly, there must be an element $t \in T$ such that $t \not\geqslant p$. Consequently, $p \wedge s = p \wedge t = a$. By Lemma 9,

$$p \wedge (s \vee t) = (p \wedge s) \vee (p \wedge t) = a.$$

Hence $p \not\leqslant s \vee t$. From Lemma 2 it follows that L is semimodular. Therefore

$$s \prec p \vee s \quad \text{and} \quad t \prec p \vee t.$$

Suppose that $s \neq t$. Then either $s \vee t > s$ or $s \vee t > t$. If $s \vee t > s$, then there exists $v \in s \vee t/s$ such that $s \prec v$. Since $s \in M(L)$ and hence is covered by a unique element, $p \vee s = v \leqslant s \vee t$. But this is impossible since $p \not\leqslant s \vee t$. Similarly, if $s \vee t > t$, then $p \vee t \leqslant s \vee t$. Hence $p \leqslant s \vee t$, a contradiction. Therefore $s = t$, and we infer that $S = T$. Consequently, L has unique irredundant decompositions, and the proof of our theorem is complete.

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