Czechoslovak Mathematical Journal

Vlastimil Pták

Characteristics of Hankel matrices

Czechoslovak Mathematical Journal, Vol. 45 (1995), No. 2, 303–313

Persistent URL: http://dml.cz/dmlcz/128518

Czechoslovak Mathematical Journal, 45 (120) 1995, Praha

CHARACTERISTICS OF HANKEL MATRICES

VLASTIMIL PTÁK, Praha

(Received March 31, 1993)

Let H be a Hankel matrix of type (n, n)

$$H = \begin{pmatrix} s_0 & \dots & s_{n-1} \\ \vdots & & \vdots \\ s_{n-1} & \dots & s_{2n-2} \end{pmatrix}.$$

Consider a polynomial f of degree $\leqslant n$

$$f(x) = f_0 + \ldots + f_n x^n.$$

We say [1] that H is compatible with f and write $H \in \mathscr{H}(f)$ if

$$H_{n-1,n+1} f = 0$$

where

$$H_{n-1,n+1} = \begin{pmatrix} s_0 & \dots & s_n \\ \vdots & & \vdots \\ s_{n-2} & \dots & s_{2n-2} \end{pmatrix}$$

and $f = (f_0, \ldots, f_n)^T$. Matrices compatible with f may be characterized by intertwining relations. If the degree of f equals n it is well known that $H \in \mathscr{H}(f)$ if and only if

$$C^T H = HC$$

where C is the companion matrix of the polynomial f. If the degree of f is smaller than the size of the matrix the same characterization remains valid if C is replaced by a matrix R constructed from f in a certain manner: thus $H \in \mathscr{H}(f)$ if and only if

$$R^T H = HR.$$

The matrix R replaces, in this more general situation, the companion matrix of f. These matrices R are called intertwining matrices corresponding to f; the notion of an intertwining matrix for f extends that of the companion of f. The relationship between Hankel matrices and polynomials is less simple and straightfoward though. First of all, a matrix may be compatible with more than one polynomial f. Also, there is more than one possibility of associating a polynomial with a Hankel matrix. G. Heinig and K. Rost [2] consider the whole family of matrices

$$H_{k,2n-k} = \begin{pmatrix} s_0 & \dots & s_{2n-k-1} \\ \vdots & & \vdots \\ s_{k-1} & \dots & s_{2n-2} \end{pmatrix}$$

together with their kernels; the matrices $H_{n-1,n+1}$ and $H = H_{n,n}$ belong to this family. The following facts are known.

If H is nonsingular the set of all polynomials that H is compatible with is a linear manifold of dimension at most 2. In the more important case of a singular H the kernel of H consists of all polynomials of the form $m(x)p(x)$ where m is an arbitrary polynomial whose degree does not exceed a certain bound. The polynomial p, the *gcd* of all polynomials in the kernel of H, and the bound b for the degree of m are two important characteristics of H.

These data contain valuable information about the structure of a Hankel matrix. Characteristics related to these numbers are well known; they are described in detail in the comprehensive monographs of Jochvidov and Heinig-Rost. Thus, for instance, the number $k_\infty = n - (b + \deg p)$ is related to the behaviour of H at infinity: in the terminology of [1] it equals the rank of the degenerate part of H; in the canonical decomposition of H as described in [2] it gives the size of the elementary H-matrix corresponding to ∞.

The present note belongs to a series of papers devoted to applications of the infinite companion matrix: we present a simple approach to characteristic numbers of Hankel matrices. In our opinion this approach has some advantages over the standard one: it puts into evidence the symmetry in the roles of 0 and ∞ and can be related in a simple manner to the behaviour of certain distinguished submatrices of H: the square submatrices obtained by inspecting the four corners of H. All our results are consequences of one simple but important proposition; this proposition is nothing more than a strengthening of a classical result of Frobenius. The techniques used in the proof of the proposition as well as in its applications are based on properties of the infinite companion and yield considerable simplications of the treatment.

An m by n matrix $A = (a_{ij})$ is a mapping which assigns to each pair i, j with $0 \leqslant i \leqslant m-1$ and $0 \leqslant j \leqslant n-1$ a complex number a_{ij}. For $j = 0, 1, \dots, n-1$ we denote by $c(A,j)$ the j-th column of the matrix A

$$c(A,j) = (a_{0j}, a_{1j}, \ldots, a_{m-1,j})^T.$$

If $a_1, \ldots, a_s$ are vectors in a linear space E, we denote by $L(a_1, \ldots, a_s)$ the set of all linear combinations of the vectors $a_1, \ldots, a_s$. If p is a monic polynomial of degree m we denote by $C^\infty(p)$ the (m, ∞) infinite companion matrix of p. The reader is referred to [4] for details. If n is an integer we denote by $C_n(p)$ the (m,n) matrix consisting of the first n columns of $C^\infty(p)$.

Now we consider an n by n Hankel matrix

$$H = \begin{pmatrix} s_0 & \ldots & s_{n-1} \\ \vdots & & \vdots \\ s_{n-1} & \ldots & s_{2n-2} \end{pmatrix}.$$

Its columns will be denoted by h_j:

$$h_j = c(H,j) = (s_j, \ldots, s_{j+n-1})^T$$

so that

$$H = (h_0, \ldots, h_{n-1}).$$

For $k = 1, 2, \ldots, n$ set

$$H_{nk} = (h_0, \ldots, h_{k-1});$$

observe that the second index indicates the number of columns.

The rank of a matrix M will be denoted by $d(M)$.

We shall investigate the behaviour of the function $d(\cdot)$ defined for $1 \leqslant j \leqslant n$ by

$$d(j) = d(H_{nj}) = d(h_0, \ldots, h_{j-1}).$$

We shall assume that H is singular, hence $d(n) < n$. If we exclude the trivial case of a lower singular Hankel, in other words if $h_0 \neq 0$, the behaviour of d can be described as follows:

Since it is obviously nondecreasing and $d(u) < n$ there exists at least one interval where it remains constant. It turns out that there is only one such interval; the indices of its endpoints represent important information about the structure of H.

Our considerations will be based on a combination of methods using the infinite companion and of the following proposition, a strengthening of a classical result of G. Frobenius.

(2,1) Proposition. *Suppose H is a Hankel matrix of type $(n, r+1)$ with columns $h_0, \ldots, h_r$. Suppose that the first r columns $h_0, \ldots, h_{r-1}$ are linearly independent and that $h_r = \alpha_0 h_0 + \ldots + \alpha_{r-1} h_{r-1}$. Denote by p the polynomial*

$$p(z) = -(\alpha_0 + \ldots + \alpha_{r-1} z^{r-1}) + z^r$$

and by H_r the (n, r) matrix $(h_0 \ldots h_{r-1})$. Then:

(1) *the matrix H_{rr} consisting of the first r rows of H_r is nonsingular,*

(2) $H_r = C_n(p)^T H_{rr}$,

(3) *each column of H satisfies the recurrence relation corresponding to p.*

Proof. The last $n - r$ rows of the product $\hat{p}(S_n) C_n(p)^T$ are zero. Let M be the (n, n) matrix defined by the following two requirements: the first r rows of M coincide with the first r rows of the identity matrix, the last $n - r$ rows of M are those of $\hat{p}(S_n)$. It follows that

$$M C_n(p)^T = \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

Denote by P the $(n - r, n)$ last rows of $\hat{p}(S_n)$. For $0 \leqslant j \leqslant n - 2$ let

$$p_j = -\alpha_0 s_j - \alpha_1 s_{j+1} \ldots - \alpha_{r-1} s_{j+r-1} + s_{j+r}.$$

It follows that

$$P H_r = \begin{pmatrix} p_0 & \cdots & p_{r-1} \\ p_1 & \cdots & p_r \\ \vdots & & \vdots \\ p_{n-r-1} & \cdots & p_{n-2} \end{pmatrix}.$$

The assumption $\alpha_0 h_0 + \ldots + \alpha_{r-1} h_{r-1} = h_r$ implies that all $p_j = 0$. Hence

$$M H_r = \begin{pmatrix} H_{rr} \\ 0 \end{pmatrix}.$$

Together with the preceding identity this yields $M H_r - M C_n(p)^T H_{rr} = 0$ and, M being nonsingular, $H_r = C_n(p)^T H_{rr}$. Since rank $H = r$ the conclusion $\det H_{rr} \neq 0$ follows. □

We begin by investigating the behaviour of the function $d(j) = d(h_0, \ldots, h_{j-1})$. It turns out that its graph may contain no more than two bends the location of which we now proceed to determine.

We intend to define four numbers r, t, r', t'.

1° To define r, we distinguish two cases. If $h_0 = 0$, we set $r = 0$. If $h_0 \neq 0$ we have $d(1) = 1$. Since $d(\cdot)$ is nondecreasing and $d(n) \leqslant n - 1$, there exists a $k \geqslant 1$ such that $d(k) = d(k+1)$; in this case let r be the smallest integer of these properties. In a manner of speaking, r is the smallest index such that h_r is a linear combination of the preceding columns.

2° To define t, we observe that the function d is nondecreasing and that $d(n) \leqslant n-1$. It follows that either there exists a $j, 1 \leqslant j \leqslant n-1$ such that $d(j) = d(j+1)$ or all the values of d are distinct and this is only possible if

$$0 = d(1) < d(2) < \ldots < d(n) = n - 1.$$

In the first case we define t as the largest j for which $d(j+1) = d(j)$. In the second case we have $d(s) = s - 1$, for all s, $1 \leqslant s \leqslant n$. In particular $d(1) = 0$; we set $t = 0$. It is easy to see that the following three assertions are equivalent

(1) $d(s) = s - 1$ for all $1 \leqslant s \leqslant n$,

(2) $h_0 = 0$ and the vectors $h_1, \ldots, h_{n-1}$ are linearly independent,

(3) $t = 0$.

Again, the definition of t may be restated in the following form: t is the largest index for which h_t is a linear combination of the preceding columns.

3° To define r' we distinguish two cases. If $h_{n-1} = 0$ set $r' = n - 1$. If $h_{n-1} \neq 0$ then, for every nontrivial linear relation

$$\alpha_0 h_0 + \ldots + \alpha_{n-1} h_{n-1} = 0,$$

at least one $\alpha_j \neq 0$ for some $j < n - 1$. It follows that there exists an $s < n - 1$ such that h_s is a linear combination of the following columns. Let r' be the largest index with these properties.

4° To define t' we distinguish two cases. If there exists a $j < n - 1$ such that $h_j \in L(h_{j+1}, \ldots, h_{n-1})$ we define t' to be the smallest integer of this property. If no such j exists it follows that $h_{n-1} = 0$ and that the vectors $h_0 \ldots, h_{n-2}$ are linearly independent. In this case set $t' = n - 1$.

The four quantities r, t, r', t' satisfy several inequalities two of which we now proceed to prove.

(2,2) Proposition. *The following two inequalities hold*

$$t' \leqslant r, \qquad r' \leqslant t.$$

P r o o f. The inequality $t' \leqslant r$ is inmediate if $r = n - 1$. If $r = 0$ then $h_0 = 0$ so that h_0 is a linear combination of the following columns; hence $t' \leqslant 0$. Hence we may restrict ourselves to the case $0 < r < n-1$. We have then a relation $h_r = \sum_{j<r} \alpha_j h_j$. If all the α_j are zero we have $h_r = 0$ for an $r < n-1$ so that h_r is a linear combination of the following columns and $t' \leqslant r$. Otherwise take the smallest j for which $\alpha_j \neq 0$. Then $h_j \in L(h_{j+1}, \dots, h_r)$ whence $t' \leqslant j < r$.

To prove the inequality $r' \leqslant t$ it suffices to assume $r' > 0$ and $t < n-1$. It follows from the first assumption that the vectors $h_1, \dots, h_{n-1}$ are linearly dependent. Thus we may restrict ourselves to linear dependence relations involving only $h_1 \dots h_{n-1}$. The assumption $t < n-1$ implies that in any relation $\sum_{j>0} \alpha_j h_j = 0$ the coefficient $\alpha_n = 0$. Thus it suffices to consider relations with $\alpha_0 = \alpha_{n-1} = 0$, and $r' \leqslant t$ follows.

□

Consider, again, the function

$$d(j) = d(H_{n,j-1}) = d(h_0, \dots, h_{j-1}).$$

If $d(1) = 0$, in other words, if $h_0 = 0$ we set $p = 1$. If $d(1) = 1$, we have defined r as the smallest integer $r \geqslant 1$ such that $d(r) = d(r+1)$. It is easy to see that there exists a nonzero vector $p = (p_0 \dots p_r)^T$ which annihilates $H_{n,r+1}$. The vector p is uniquely determined by the postulate $p_r = 1$. We identify p with the corresponding polynomial $p(z) = p_0 + p_1 z + \dots + z^r$. The importance of this polynomial was recognized by many authors. Different equivalent characterizations of this polynomial (together with different names) appear in the literature. It coincides with the first characteristic polynomial in the terminology of [4] and with the H-polynomial in [1].

If $d(1) = 0$ it is easy to describe the behaviour of $d(\cdot)$. Indeed $d(1) = 0$ implies $h_0 = 0$ so that H is a lower triangular Hankel matrix. If h_m is the first nonzero column then

$$\begin{aligned} d(j) &= 0 \quad \text{for } 1 \leqslant j < m, \\ d(j) &= j - m + 1 \quad \text{for } j \geqslant m. \end{aligned}$$

If $d(1) = 1$ the graph of d will have two bends in general; to prove this we shall need the following lemma.

(2,3) Proposition. *Suppose $H_{r-1,r+1} p = 0$. Then*

$$H = C^\infty(p)^T H_{rr} C^\infty(p)$$

is a Hankel matrix.

Proof. We observe first that $C(p)^T H_{rr} = H_{rr} C(p)$. If c_j are the columns of $C^\infty(p)$, we have $c_r = C(p)^r c_0$. For the entries of H we obtain $h_{ij} = c_i^T H_{rr} c_j = c_0^T \left(C(p)^T\right)^i H_{rr} C(p)^j c_0 = c_0^T H_{rr} C(p)^{i+j} c_0$. □

Suppose that $(h_0 \dots h_{r-1})$ are linearly independent and that $h_r = \alpha_0 h_0 + \dots + \alpha_{r-1} h_{r-1}$. Denote by p the polynomial $p(z) = -(\alpha_0 + \dots + \alpha_{r-1} z^{r-1}) + z^r$. Applying lemma (2,1) we obtain

$$H_{nr} = C_n(p)^T H_{rr}.$$

Furthermore, $H' = C_n(p)^T H_{rr} C_n(p)$ is Hankel and its first $r+1$ columns coincide with $h_0 \dots h_r$. Since H' is Hankel by (2,3), $H - H'$ is a Hankel matrix whose first $r+1$ columns are zero. Thus $H = H' + R$ where R is a lower triangular Hankel.

For $k > r$ write $C_k(p)$ in the form

$$C_k(p) = (I, Z_k).$$

Also, write R in block form

$$R = \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}$$

the upper left zero being of type r, r.

Denote by M the n by n matrix

$$M = \begin{pmatrix} I & 0 \\ -Z_n^T & I \end{pmatrix}$$

where the unit matrices are of order r and $n - r$. For the product MH and its submatrices MH_{nk} we obtain the following expressions. We only consider $k > n$ and write K_k for the matrix defined by the requirement that the matrix of the first k columns of R be

$$\begin{pmatrix} 0 & 0 \\ 0 & K_k \end{pmatrix},$$

$$MH = \begin{pmatrix} I & 0 \\ -Z_n^T & I \end{pmatrix} \begin{pmatrix} H_{rr} & H_{rr} Z_n \\ Z_n^T H_{rr} & Z_n^T H_{rr} Z + K \end{pmatrix} = \begin{pmatrix} H_{rr} & H_{rr} Z_n \\ 0 & K \end{pmatrix},$$

$$MH_{nk} = \begin{pmatrix} I & 0 \\ -Z_n^T & I \end{pmatrix} \begin{pmatrix} H_{rr} & H_{rr} Z_k \\ Z_n^T H_{rr} & Z_n^T H_{rr} Z_k + K_k \end{pmatrix} = \begin{pmatrix} H_{rr} & H_{rr} Z_k \\ 0 & K_k \end{pmatrix}.$$

If N stands for the (k, k) matrix

$$\begin{pmatrix} 1 & -Z_k \\ 0 & 1 \end{pmatrix}$$

we have

$$d(k) = d(H_{nk}) = d(MH_{nk}N) = d \begin{pmatrix} H_{rr} & 0 \\ 0 & K_k \end{pmatrix}.$$

It follows that $d(k) = r + d(K_k)$ and that $d(r+1) = r$. In view of the form of K_k it is obvious that there exists an $s < n-1$ such that $d(s+1) = r$ and $d(s+1+x) = r+x$

for $x > 0$. Recalling the definition of t, we can identify s with t, since h_{t+1} is the first in the sequence of columns where each column is linearly independent of its predecessors.

Since $d(n) = d$ we have $d = d\big(t+1+n-(t+1)\big) = n-(t+1)$ whence

$$t - r = n - d - 1.$$

Summing up, we have proved the following

(2,4) Proposition. *The function $d(j) = d(h_0, \dots, h_{j-1})$ assumes the following values*

$$\begin{aligned} d(j) &= j \quad \text{for} \quad 1 \leqslant j \leqslant r, \\ &= r \quad \text{for} \quad r \leqslant j \leqslant t+1, \\ &= r + j - t - 1 \quad \text{for} \quad j \geqslant t+1. \end{aligned}$$

Consider the function d_∞ defined for $1 \leqslant x \leqslant n$ by

$$d_\infty(x) = d(h_{x-1}, \dots, h_{n-1}).$$

In a similar manner it is possible to show that

$$\begin{aligned} d_\infty(x) &= d + 1 - x \quad \text{for} \quad 1 \leqslant x \leqslant t' - 1, \\ &= d - t' \quad \text{for} \quad t' - 1 \leqslant x \leqslant r', \\ &= d - t' - (x - r') \quad \text{for} \quad x \geqslant r' \end{aligned}$$

and that

$$n - d - 1 = r' - t'.$$

We identify vectors with the corresponding polynomials: the vector $a = (a_0, \dots, a_{n-1})^T$ will be identified with the polynomial $a(x) = \pi(x)a = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$. As a corollary of the preceding proposition, we obtain the following known description of the kernel of H.

(2,5) Proposition. *The kernel of a singular Hankel matrix consists of all polynomials of the form $m(x)p(x)$ where m is an arbitrary polynomial of degree $\leqslant t - r$.*

P r o o f. This is immediate if $r = 0$. Now suppose $r > 0$. We have defined p by the relation $H_{n,r+1}p = 0$ and proved that $H = C_n^T(p)H_{rr}C_n(p) + R$ with R lower triangular Hankel. Since the first $r+1$ columns of R are zero, we also have $Rp = 0$; the first nonzero column of R being the column with index $t+1$, the matrix R annihilates $p(x), xp(x), \ldots, x^{t-r}p(x)$. The matrix $C_n(p)$ annihilates $p(x)$, $xp(x)$, …, $x^{n-1-r}p(x)$; it follows that $\operatorname{Ker} H$ contains the $t-r+1$ linearly independent polynomials $p(x)$, …, $x^{t-r}p(x)$. The dimension of $\operatorname{Ker} H$ being $n - d(H) = t - r + 1$, these polynomials span $\operatorname{Ker} H$.

Consider the minimal polynomial p. If $p(z) = p_0 + p_1 z + \ldots + p_{r-1} z^{r-1} + z^r$ then the $r+1$ vector $p = (p_0, \ldots, p_{r-1}, 1)$ is characterized by the following two postulates:

1° $H_{n,r+1}p = 0$ where $H_{n,r+1} = (h_0, \ldots, h_r)$

2° $p_r = 1$.

Denote by k the multiplicity of zero in p; thus $p(z) = z^k w(z), w(0) \neq 0$. To exclude the trivial case of upper triangular Hankel matrices we assume $h_{n-1} \neq 0$ so that $k < n-1$. Since

$$p_k h_k + \ldots + p_r h_r = 0$$

and $p_k \neq 0$ we have $h_k \in L(h_{k+1}, \ldots, h_r)$ so that $t' \neq k$. Let us show that $t' = k$. Suppose $t' < k$; then there exists a relation $\sum \alpha_j h_j = 0$ with the following properties

(1) the first index u which $\alpha_u \neq 0$ satisfies $u < k$. Since p represents the only nontrivial relation involving the first $r+1$ columns it follows that the last index v for which $\alpha_v \neq 0$ must be $> r+1$. From the definition of t we easily infer that $v \leqslant t$. Since all columns with indices $r+1 \leqslant j \leqslant t$ belong to $L(h_k, \ldots, h_{r-1})$ this relation would yield another q annihilating $H_{n,r+1}$, a contradiction with the uniqueness of p. It follows that $t' = k$. □

Denote by H' the matrix $(h_k, \ldots, h_{r-1}, h_r)$. The columns $h_k, \ldots, h_{r-1}$ are linearly independent and

$$w_0 h_k + \ldots + w_{m-1} h_{r-1} + h_r = 0$$

where $m = r - k$ and $w_j = p_{j+k}$. Thus the assumptions of Lemma (2,1) are satisfied. It follows that $H' = C_n(w)^T H_{mm}$ where H_{mm} is the matrix consisting of the first m rows of H'. Define $\tilde{H} = C_n(w)^T H_{mm} C(w)^{-k} C_n(w)$. We prove that $\tilde{H}$ is a Hankel matrix. Indeed, $(\tilde{H})_{ij} = c_i^T H_{mm} C(w)^{-k} c_j = c_0^T \left(C(w)^T\right)^i H_{mm} C(w)^{-k} C(w)^j c_0 = c_0^T H_{mm} C(w)^{i-k+j} c_0$.

It is easy to see that

$$c(\tilde{H}, j) = c(H, j)$$

for $k \leqslant j \leqslant r-1$. It follows that

$$H = \tilde{H} + R_0 + R_\infty$$

with upper triangular Hankel R_0 and lower triangular Hankel R_∞. The matrix R_0 has at most k nonzero columns, and the first $r+1$ columns of R_∞ are zero. This makes it possible to give a more precise description of the kernel of H.

(2,6) Proposition. *The kernel of H is generated by all polynomials of the form $x^j w(x)$ with $t' \leqslant j \leqslant t - r$. The polynomial w has $w(0) \neq 0$ and its degree is $r - t' = t - r'$. The indices $k_0 = t', k_\infty = n - t - 1$ and $m = r - t' = t - r'$ satisfy*

$$k_0 + m + k_\infty = d(H).$$

The minimal polynomial p may also be characterized as the *gcd* of all polynomials in the kernel of H; this follows from (2,5). The degree of p equals r; it follows from (2,1) that this important characteristic of H may also be read off from the sequence of north-west subdeterminants of H as the index of the last nonzero determinant in the sequence $\det H_{jj}$.

It is not difficult to obtain a similar characterization for the degree of w.

Denote by $\tilde{H}_p$ the (p,p) matrix consisting of the last p rows of H_p, in other words, the south-west (p,p) corner of H. Thus

$$\tilde{H}_p = \begin{pmatrix} s_{n-2-p} & \cdots & s_{n-1} \\ \vdots & & \vdots \\ s_{n-1} & \cdots & s_{n-2+p} \end{pmatrix}.$$

If $p \geqslant r + 1$ the lower parts of the columns $h_k, \ldots, h_r$ are contained in $\tilde{H}_p$ whence $\det \tilde{H}_p = 0$.

If $r - k < p \leqslant r$, consider the last $r - k$ columns of $\tilde{H}_p$. It is easy to see that this submatrix is also contained in $(h_k, \ldots, h_r)$, a little higher up. Again it follows that $\det \tilde{H}_p = 0$.

In this manner we have shown that $\det \tilde{H}_p = 0$ for all $p > m = r - k$; we shall see, however, that $\tilde{H}_m$ is nonsingular. This is a consequence of the identity $\tilde{H}_m = \left(C(w)^T\right)^{n-m} H_{mm} C(w)^{-k}$.

The number m, the degree of the polynomial w, appears thus as the last index for which the corresponding $\tilde{H}_p$ is nonsingular. All three characteristics k_0, m, k_∞ may be described in a similar manner: let us restate the corresponding facts as follows.

For each $p = 1, 2, \ldots, n$ we define

$$H_p^{(0)} = \begin{pmatrix} s_0 & \cdots & s_{p-1} \\ \vdots & & \vdots \\ s_{p-1} & \cdots & s_{2p-2}, \end{pmatrix},$$

$$H_p^{(\infty)} = \begin{pmatrix} s_{2n-2p} & \dots & s_{2n-p-1} \\ \vdots & & \vdots \\ s_{2n-p-1} & \dots & s_{2n-2}, \end{pmatrix},$$

$$H_p^{(*)} = \begin{pmatrix} s_{n-p-2} & \dots & s_{n-1} \\ \vdots & & \vdots \\ s_{n-1} & \dots & s_{n+p-2} \end{pmatrix},$$

the p by p square submatrices of H in the NW, SE and SW corners respectively. In this manner we obtain three sequences of matrices the last term of each sequence being the matrix H. It follows that the corresponding three sequences of determinants have zero as the last term. Thus we may define the three indices $k_0 + m$, $k_\infty + m$ and m, each the last index in the corresponding sequence for which the determinant is nonzero.

References

[1] *M. Fiedler*: Quasidirect decompositions of Hankel and Toeplitz matrices. Lin. Algebra Appl. *61* (1984), 155–174.

[2] *M. Fiedler*: Polynomials and Hankel matrices. Lin. Algebra Appl. *66* (1985), 235–248.

[3] *M. Fiedler, V. Pták*: Intertwining and testing matrices corresponding to a polynomial. Lin. Algebra Appl. *86* (1978), 53–74.

[4] *G. Heinig, K. Rost*: Algebraic Methods for Toeplitz-like Matrices and Operators. Akademie-Verlag, Berlin, 1984.

[5] *A.S. Iochvidov*: Hankel and Toeplitz Matrices and Forms. Nauka, Moscow, 1974. (In Russian.)

[6] *V. Pták*: The infinite companion matrix. Lin. Algebra Appl. *166* (1992), 65–95.

Author's address: 115 67 Praha 1, Žitná 25, Czech Republic (Mathematical Institute of the Academy of Sciences of the Czech Republic).