Ethiraju Thandapani

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OSCILLATION THEOREMS FOR SECOND ORDER DAMPED NONLINEAR DIFFERENCE EQUATIONS

E. THANDAPANI, Tamilnadu

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1. INTRODUCTION

In this paper we are concerned with the nonlinear damped difference equation of the type

$$\Delta(a_n\Delta y_n) + p_n\Delta y_n + q_{n+1}f(y_{n+1}) = 0, \quad n = 0, 1, 2, \dots \tag{1}$$

where the forward difference operator Δ is defined by $\Delta y_n = y_{n+1} - y_n$ and the real sequences $\{a_n\}$, $\{p_n\}$ and $\{q_n\}$ and the function f satisfy the following conditions:

(c_1) $a_n > 0$, $p_n \geqslant 0$ and $q_n > 0$ for all $n \geqslant n_0 \geqslant 0$;

(c_2) $f\colon \mathbb{R} \to \mathbb{R} = (-\infty, \infty)$ is a nondecreasing function such that

$$uf(u) > 0 \quad \text{for } u \neq 0.$$

By a solution of (1) we mean a real sequence $\{y_n\}$, $n = 0, 1, 2, \dots$ satisfying (1). We consider only such solutions which are nontrivial for all large n. A solution of (1) is said to be oscillatory if for every $N \geqslant 0$ there exists $n \geqslant N$ such that $y_n y_{n+1} \leqslant 0$. Otherwise it is called nonoscillatory.

In recent years there has been an increasing interest in the study of the qualitative behavior of solutions of difference equations of the type (1) and/or related equations; see, for example, [1, 3, 5, 7, 8, 9] and the references cited therein.

Our purpose in this paper is to establish some new oscillation criteria (sufficient conditions) for oscillation of all solutions of (1).

2. Main results

We begin with the following lemma which is a discrete analogue of Lemma 1 of Baker [2].

Lemma 1. *Assume that*

$$a_n - p_n > 0 \quad \text{for } n \geqslant n_0 \geqslant 0 \tag{2}$$

and

$$\sum^{\infty} \frac{1}{a_n}\left[\prod_{s=0}^{n-1}\left(1 - \frac{p_s}{a_s}\right)\right] = \infty. \tag{3}$$

If $\{y_n\}$ is a nonoscillatory solution of Eq. (1), then there is $N \geqslant 0$ such that $y_n \Delta y_n > 0$ for all $n \geqslant N$.

Proof. Let $\{y_n\}$ be a nonoscillatory solution of Eq. (1) and assume $y_n > 0$ for $n \geqslant n_0 \geqslant 0$. Suppose $\{\Delta y_n\}$ is oscillatory. Then there exists an integer $n_1 \geqslant n_0 \geqslant 0$ such that

$$\Delta y_{n_1} < 0 \quad \text{or} \quad \Delta y_{n_1} = 0.$$

First we consider $\Delta y_{n_1} < 0$. Now Eq. (1) implies

$$\begin{aligned}\Delta(a_{n_1}\Delta y_{n_1})\Delta y_{n_1} &= -p_{n_1}(\Delta y_{n_1})^2 - q_{n_1+1} f(y_{n_1+1})\Delta y_{n_1}\\ &> -p_{n_1}(\Delta y_{n_1})^2\end{aligned}$$

since $-q_{n_1+1} f(y_{n_1+1})\Delta y_{n+1} > 0$. Hence

$$\Delta y_{n_1}[a_{n_1+1}\Delta y_{n_1+1} - a_{n_1}\Delta y_{n_1}] > -p_{n_1}(\Delta y_{n_1})^2$$

or

$$a_{n_1+1}\Delta y_{n_1+1}\Delta y_{n_1} > (a_{n_1} - p_{n_1})(\Delta y_{n_1})^2 > 0.$$

Thus, by dividing the above by a negative term $a_{n_1+1}\Delta y_{n_1}$ we obtain

$$\Delta y_{n_1+1} < 0.$$

By induction, we obtain $\Delta y_n < 0$ for all $n \geqslant n_1$.

Next, consider $\Delta y_{n_1} = 0$. Then Eq. (1) implies

$$\Delta y_{n_1+1} < 0$$

and we obtain as above $\Delta y_n < 0$ for all $n \geqslant n_1$. Hence in both cases we obtain $\Delta y_n < 0$ for all $n \geqslant n_1$ which, however, contradicts the assumption that $\{\Delta y_n\}$ oscillates. Thus $\{\Delta y_n\}$ is eventually of fixed sign.

Let $\Delta y_n < 0$ for $n \geqslant N \geqslant 0$, then

$$\Delta z_n + \frac{p_n}{a_n} z_n \geqslant 0 \quad \text{for } n \geqslant N \tag{4}$$

where

$$z_n = -a_n \Delta y_n.$$

From (4) we obtain

$$z_n \geqslant z_N \prod_{s=N}^{n-1} \left(1 - \frac{p_s}{a_s}\right)$$

or

$$a_n \Delta y_n \leqslant -z_N \prod_{s=N}^{n-1} \left(1 - \frac{p_s}{a_s}\right), \quad n \geqslant N. \tag{5}$$

Now summing (5) and using (3) we obtain a contradiction. The proof for the case of $\{y_n\}$ eventually negative is similar and hence omitted. □

Remark. If $p_n = 0$, then the condition (3) assumes the form

$$\sum^{\infty} \frac{1}{a_n} = \infty$$

which is used in [5, 8].

Lemma 1 is false if we omit the assumption (3). This is illustrated in the following example.

Consider the difference equation

$$\Delta\big(n(n+1)\Delta y_n\big) + n\Delta y_n + (n+1)^2 y_{n+1}^3 = 0, \quad n \geqslant 1. \tag{E$_1$}$$

Let $f(x) = x^3$, $a_n = n(n+1)$, $p_n = n$, $q_n = (n+1)^2$. Eq. (E$_1$) has a nonoscillatory solution $y_n = 1/n$, a contradiction to the conclusion of Lemma 1 since the condition (3) does not hold.

In the following theorem we study the oscillatory behavior of Eq. (1) subject to the conditions

$$\int^{+\infty} \frac{du}{f(u)} < \infty \quad \text{and} \quad \int^{-\infty} \frac{du}{f(u)} < \infty. \tag{6}$$

Theorem 2. *Suppose that the conditions (2), (3) and (6) hold. Assume that there exists a positive sequence $\{h_n\}$ such that*

$$\Delta h_n \geqslant 0 \text{ and } \Delta(a_n \Delta h_n) \leqslant 0 \quad \text{for } n \geqslant n_0 \geqslant 0. \tag{7}$$

If

$$\sum^{\infty} h_n q_{n+1} = \infty \tag{8}$$

then every solution of Eq. (1) is oscillatory.

Proof. Let $\{y_n\}$ be a nonoscillatory solution of Eq. (1) which must then be eventually of constant sign. In view of Lemma 1, there is no loss in generality in assuming that there is an integer $N \geqslant 0$ such that $y_n > 0$ and $\Delta y_n > 0$ for all $n \geqslant N$. Define

$$z_n = \frac{h_n v_n}{f(y_n)}$$

where $v_n = a_n \Delta y_n$. Note that $z_n > 0$.

Then for $n \geqslant N$,

$$\Delta z_n = -h_n q_{n+1} - \frac{p_n h_n \Delta y_n}{f(y_{n+1})} + \frac{\Delta h_n v_{n+1}}{f(y_{n+1})} - \frac{h_n v_n \Delta f(y_n)}{f(y_{n+1}) f(y_n)}. \tag{9}$$

Now using the condition (7) and $v_{n+1} \leqslant v_n$ in (9), we obtain

$$\Delta z_n \leqslant -h_n q_{n+1} + \frac{\Delta h_n v_n}{f(y_{n+1})} \quad \text{for } n \geqslant N.$$

Since $(a_n \Delta h_n)$ is nonincreasing for $n \geqslant N$, we have

$$\Delta z_n < -h_n q_{n+1} + a_N \Delta h_N \frac{\Delta y_n}{f(y_{n+1})}, \qquad n \geqslant N. \tag{10}$$

Now for $y_n \leqslant x \leqslant y_{n+1}$ we have $\frac{1}{f(x)} \geqslant \frac{1}{f(y_{n+1})}$, and it follows that

$$\int_{y_n}^{y_{n+1}} \frac{\mathrm{d}x}{f(x)} \geqslant \frac{\Delta y_n}{f(y_{n+1})}.$$

Using the above inequality in (10) and summing the resulting inequality from N to n leads to

$$\sum_{s=N}^{n} h_s q_{s+1} \leqslant z_N - z_{n+1} + a_N \Delta h_N \int_{y_N}^{y_{n+1}} \frac{\mathrm{d}x}{f(x)}.$$

In view of (6) and $z_n > 0$, $n \geqslant N$, the above inequality gives

$$\sum_{s=N}^{n} h_s q_{s+1} < \infty,$$

which contradicts (8). □

Remark. In Theorem 2, let $p_n = 0$, $a_n = 1$, $f(u) = u^\alpha$, $\alpha > 1$ ratio of odd positive integers and $h_n = n$. Then it reduces to Theorem 4.1 of Hooker and Patula [3]. Also Theorem 2 reduces to Theorem 4.2 of Kulenovic and Budincevic [6] if $p_n = 0$ and $h_n = \sum\limits_{s=0}^{n-1} \frac{1}{a_n}$.

All solutions of the difference equation

$$(\mathrm{E}_2) \qquad \Delta\big((n+1)\Delta y_n\big) + \frac{1}{n+1}\Delta y_n + (n+1)(4n^2+10n+5)y_{n+1}^3 = 0, \quad n \geqslant 1$$

are oscillatory by Theorem 2. One such solution of (E_2) is $y_n = (1)^n/n$.

We now state a lemma which will be used in the proof of our next theorem. The proof is similar to Lemma 4.1 of [3].

Lemma 3. *If $y_N \geqslant 0$, $\Delta(a_n\Delta y_n) \leqslant 0$ and $\Delta y_n > 0$ for $n \geqslant N \geqslant 1$, then*

$$y_{n+1} \geqslant R(n)a_n\Delta y_n$$

where

$$R(n) = \sum_{s=N}^{n} \frac{1}{a_s}.$$

Theorem 4. *Suppose that the conditions* (2) *and* (3) *are satisfied. Assume that*

$$(11) \qquad \int^{\pm c} \frac{du}{f(u)} < \infty \quad \text{for every positive constant } c > 0$$

and f satisfies

$$(12) \qquad f(xy) \geqslant Kf(x)f(y) \quad \text{and} \quad -f(-xy) \geqslant Kf(x)f(y)$$

on $(0,\infty) \cup (-\infty,0)$ where K is a positive constant. If

$$(13) \qquad \sum^{\infty} f\big(R(n)\big)q_{n+1} = \infty$$

then every solution of Eq. (1) *is oscillatory.*

Proof. Let $\{y_n\}$ be a nonoscillatory solution of (1). As before, there exists an integer $N \geqslant 0$ such that

$$y_n > 0 \text{ and } \Delta y_n > 0 \quad \text{for all } n \geqslant N.$$

Since $p_n \geqslant 0$, we have from (1)

$$\Delta(a_n \Delta y_n) + q_{n+1} f(y_{n+1}) \leqslant 0. \tag{14}$$

Since $\Delta(a_n \Delta y_n) \leqslant 0$ for $n \geqslant N$, we can use Lemma 3 in (14) and then using (12) we obtain

$$\Delta(a_n \Delta y_n) + K q_{n+1} f\big(R(n)\big) f(a_n \Delta y_n) \leqslant 0 \quad \text{for } n \geqslant N,$$

or

$$\frac{\Delta(a_n \Delta y_n)}{f(a_n \Delta y_n)} + K q_{n+1} f\big(R(n)\big) \leqslant 0. \tag{15}$$

Observe that for $a_n \Delta y_n \geqslant x \geqslant a_{n+1} \Delta y_{n+1}$ we have $\frac{1}{f(x)} \geqslant \frac{1}{f(a_n \Delta y_n)}$ and it follows that

$$-\int_{a_{n+1}\Delta y_{n+1}}^{a_n \Delta y_n} \frac{\mathrm{d}x}{f(x)} \leqslant \frac{\Delta(a_n \Delta y_n)}{f(a_n \Delta y_n)}.$$

Using the last inequality in (15) and summing the resulting inequality from N to n leads to

$$K \sum_{s=N}^{n} f\big(R(s)\big) q_{s+1} \leqslant \int_{a_{n+1}\Delta y_{n+1}}^{a_N \Delta y_N} \frac{\mathrm{d}x}{f(x)},$$

which is by (11) an immediate contradiction. □

Remark. Let $p_n = 0$ in Theorem 4, then it reduces to Theorem 4.1 of Kulenovic and Budincevic [5]. If $p_n = 0$, $a_n = 1$ and $f(u) = u^\alpha$, $0 < \alpha < 1$, then Theorem 4 reduces to Theorem 4.3 of Hooker and Patula [3].

Consider the difference equation

$$\Delta\big((n+1)\Delta y_n\big) + \frac{1}{n+1}\Delta y_n + \frac{4n^2 + 10n + 5}{(n+1)^{5/3}} y_{n+1}^{1/3} = 0, \quad n \geqslant 1. \tag{E$_3$}$$

All conditions of Theorem 4 are satisfied and hence all solutions of (E$_3$) are oscillatory. One such solution is $y_n = (-1)^n/n$.

Finally, we discuss the oscillatory behavior of Eq. (1) subject to the condition

$$f(u) - f(v) = g(u,v)(u - v), \quad g(u,v) \geqslant M > 0 \text{ for } u, v \neq 0. \tag{16}$$

Theorem 5. *Let the conditions* (2), (3) *and* (16) *be satisfied. Assume there exists a positive non-decreasing sequence* $\{h_n\}$ *such that*

$$\limsup_{n\to\infty} \frac{1}{(n)^{(\alpha)}} \sum_{s=N}^{n-1} (n-s)^{(\alpha)} h_s \left[q_{s+1} - \frac{a_s}{4M}\left(\frac{p_s}{a_s} - \frac{\Delta h_s}{h_s} + \frac{\alpha}{n-s+\alpha-1}\right)^2\right] = \infty \tag{17}$$

for some positive integer $\alpha \geqslant 1$, where $(n)^{(\alpha)} = n(n-1)\dots(n-\alpha+1)$ is the usual factorial notation. Then every solution of Eq. (1) *is oscillatory.*

Proof. Let $\{y_n\}$ be a nonoscillatory solution Eq. (1). As before, there exists an integer $N \geqslant 0$ such that

$$y_n > 0 \text{ and } \Delta y_n > 0 \quad \text{for all } n \geqslant N.$$

Consider the function z_n defined in the proof of Theorem 2. We obtain (9) and using the condition (16), we get

$$\Delta z_n \leqslant -h_n q_{n+1} - \frac{p_n h_n \Delta y_n}{f(y_{n+1})} + \frac{\Delta h_n v_{n+1}}{f(y_{n+1})} - \frac{M h_n v_n \Delta y_n}{f(y_n) f(y_{n+1})} \quad \text{for } n \geqslant N. \tag{18}$$

Using the inequalities $v_{n+1} \leqslant v_n$ and $f(y_n) \leqslant f(y_{n+1})$, we obtain from (18)

$$\Delta z_n \leqslant -h_n q_{n+1} - \frac{p_n h_n}{a_n h_{n+1}} z_{n+1} + \frac{\Delta h_n}{h_{n+1}} z_{n+1} - \frac{M h_n}{a_n h_{n+1}^2} z_{n+1}^2.$$

Since

$$\sum_{s=N}^{n-1} (n-s)^{(\alpha)} \Delta z_s = -(n-N)^{(\alpha)} z_N + \alpha \sum_{s=N}^{n-1} (n-s)^{(\alpha-1)} z_{s+1},$$

we get

$$\begin{aligned}
&\frac{1}{(n)^{(\alpha)}} \sum_{s=N}^{n-1} (n-s)^{(\alpha)} h_s q_{s+1} \\
&\leqslant \frac{(n-N)^{(\alpha)}}{(n)^{(\alpha)}} z_N - \frac{1}{(n)^{(\alpha)}} \sum_{s=N}^{n-1} \frac{(n-s)^{(\alpha)} M h_s}{a_s h_{s+1}^2} \\
&\times \left\{ z_{s+1}^2 + \frac{a_s h_{s+1}}{M} \left(\frac{p_s}{a_s} - \frac{\Delta h_s}{h_s} + \frac{\alpha}{n-s+\alpha-1} \right) \right\} \\
&\frac{(n-N)^{(\alpha)}}{(n)^{(\alpha)}} z_N + \frac{1}{(n)^{(\alpha)}} \sum_{s=N}^{n-1} \frac{(n-s)^{(\alpha)} a_s h_s}{4M} \times \left\{ \left(\frac{p_s}{a_s} - \frac{\Delta h_s}{h_s} \right) + \frac{\alpha}{n-s+\alpha-1} \right\}^2
\end{aligned}$$

or

$$\begin{aligned}
\frac{1}{(n)^{(\alpha)}} \sum_{s=N}^{n-1} (n-s)^{(\alpha)} h_s \left[q_{s+1} - \frac{a_s}{4M} \left(\frac{p_s}{a_s} - \frac{\Delta h_s}{h_s} + \frac{\alpha}{n-s+\alpha-1} \right)^2 \right] \\
\leqslant \frac{(n-N)^{(\alpha)}}{(n)^{(\alpha)}} z_N \to z_N \qquad \text{as } n \to \infty,
\end{aligned}$$

which contradicts (17). This completes the proof of the theorem. □

Corollary 6. *If the condition* (17) *is replaced by*

$$\limsup_{n\to\infty} \frac{1}{(n)^{(\alpha)}} \sum_{s=N}^{n-1} (n-s)^{(\alpha)} h_s q_{s+1} = \infty, \tag{19}$$

$$\lim_{n\to\infty} \frac{1}{(n)^{(\alpha)}} \sum_{s=N}^{n-1} \frac{(n-s)^{(\alpha)} h_s a_s}{(n-s+\alpha-1)^2} \Big[(n-s+\alpha-1)\Big(\frac{p_s}{a_s} - \frac{\Delta h_s}{h_s}\Big) + \alpha\Big]^2 < \infty \tag{20}$$

for a positive integer $\alpha \geqslant 1$, then every solution of Eq. (1) *is oscillatory.*

Remark. Corollary 6 is a discrete analogue of Theorem 1 when $a_n = 1$ and $h_n = n$ and of Theorem 2 when $a_n = 1$ and $h_n = 1$ of C. C. Yeh [10]. If $f(u) = u$, $a_n = 1$, $h_n = 1$ and $p_n = 0$, then the condition (20) holds for $\alpha = 1$ and in this case Corollary 6 reduces to the discrete analogue of Kamanev's result [4].

It follows from (20) that $p_n \not\equiv 0$ and $a_n = 1$ and $h_n = n$ in Corollary 6, in which $\{p_n\}$ can be thought of as a small perturbation of $\frac{1}{n}$. If $a_n = 1$ and $h_n = 1$, it follows from (20) that $\{p_n\}$ may be equal to zero in Corollary 6, in which $\{p_n\}$ can be thought of as a small perturbation of 0.

Consider the difference equation

$$\Delta^2 y_n + \frac{1}{n+2}\,\Delta y_n + \frac{4n^2+4n+1}{n(n+2)}\, y_{n+1} = 0, \quad n \geqslant 1. \tag{E$_4$}$$

All conditions of Corollary 6 are verified for $\alpha = 1$. Hence every solution of (E_4) is oscillatory. One such solution is $y_n = (-1)^n/n$.

Theorem 7. *In addition to* (2), (3), (6) *and* (16) *assume that there is a constant $K > 0$ and a positive nondecreasing sequence $\{h_n\}$ such that*

$$\Delta(a_{n+1}\Delta h_n) \leqslant 0 \quad \textit{and} \quad p_n \geqslant -\frac{K}{h_n} \quad \textit{for all } n \geqslant N > 0. \tag{21}$$

If

$$\sum^{\infty} h_n q_{n+1} = \infty \tag{22}$$

then all solutions of (1) *are oscillatory.*

Proof. Let $\{y_n\}$ be a nonoscillatory solution of (1) such that $y_n > 0$ and $\Delta y_n > 0$ for $n \geqslant N \geqslant 0$. We multiply (1) by $h_n/f(y_{n+1})$, summing from N to $n-1$, and use (16) and (21) to obtain

$$\begin{aligned}\frac{h_n a_n \Delta y_n}{f(y_{n+1})} &+ M\sum_{s=N}^{n-1}\frac{h_{s+1}a_{s+1}(\Delta y_{s+1})^2}{f(y_{s+1})f(y_{s+2})} + \sum_{s=N}^{n-1} h_s q_{s+1}\\ &\leqslant c + K\sum_{s=N}^{n-1}\frac{\Delta y_s}{f(y_{s+1})} + a_{N+1}\Delta h_N \sum_{s=N}^{n-1}\frac{\Delta y_{s+1}}{f(y_{s+2})}\\ &< c + K\int_{y_N}^{y_n}\frac{\mathrm{d}x}{f(x)} + a_{N+1}\Delta h_N \int_{y_{N+1}}^{y_{n+1}}\frac{\mathrm{d}x}{f(x)}\end{aligned}$$

where c is a constant. Taking the limit as $n \to \infty$ and using (6) and (22) we arrive at a contradiction that $\Delta y_n < 0$ for all $n \geqslant N$. This completes the proof. □

Remark. Let $a_n = 1$ and $h_n = n$, then Theorem 7 is a discrete analogue of Theorem 4 of Naito [6].

The equation

$$(\mathrm{E}_5) \qquad \Delta^2 y_n - \frac{1}{n}\Delta y_n + \frac{4n^2+6n+1}{n(n+1)^3}\, y_{n+1}^3 = 0, \quad n \geqslant 1$$

has an oscillatory solution $y_n = (-1)^n n$. All conditions of Theorem 7 are satisfied.

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Author's address: Department of Mathematics, Madras University P. G. Centre, Salem-636 011, Tamilnadu, India.