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# ON A RESULT OF J. JOHNSON

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J. Johnson proved in [4] that if $Y$ is a Banach space having the bounded approximation property then the anulator $K(X,Y)$ in $L(X,Y)^*$ is the kernel of a projection $P$ in $L(X,Y)^*$. Here $X$ is an arbitrary Banach space and $K(X,Y) = K$, $L(X,Y) = L$, denote respectively the space of all compact or bounded operators $f\colon X \to Y$. Moreover, the range space of the projection $P$ is isomorphic to $K^*$. In [3] the same statement was shown to be true for the spaces $X = P$ and $Y = P^*$ where $P$ is any separable Pisier space. Notice that here Johnson's result cannot be applied since $P^*$ (and $P$) do not even have the approximation property. The proof in [3] was based on the fact that every $f\colon P \to P^*$ is factorable through a Hilbert space. In this note we observe (see Proposition 2 and Remarks 1 and 2) that Johnson's result holds for any couples of Banach spaces $X$, $Y$ such that any $f\colon X \to Y$ is factorable through a Banach space $Z$, $Z^*$ having the bounded approximation property and $Z^*$ being separable. In fact much weaker assumptions are shown to be sufficient for J. Johnson's result (Proposition 1 and Remark 5).

Following N. Kalton [6] we denote on by $w'$ the topology $L(X,Y) = L$ (projectively) generated by all $x^{**} \otimes y^*$ where $x^{**} \in X^{**}$ and $y^* \in Y^*$. Thus we write $f_n \overset{w'}{\to} f$ to denote that for any $x^{**}$, and $y^*$ we have $x^{**}\big(f_n^*(y^*)\big) \to x^{**}\big(f^*(y^*)\big)$. We will make crucial use of the following result of Kalton:

(K) If $\{f_n\} \subset K$ is a sequence of compact operators such that $f_n \overset{w'}{\to} f$ and if $f\colon x \to y$ is compact then $f_n \to f$ in the weak topology of $L(X,Y)$.

We say that the operator $f\colon X \to Y$ is factorable through a Banach space $Z$ if $f = f_1 f_2$ where $f_2\colon X \to Z$ and $f_1\colon Z \to Y$ are operators. All operators in the paper are bounded linear operators.

**Proposition 1.** *Let $X$, $Y$ be Banach spaces such that for every $f \in L(X,Y) = L$ there is a sequence $\{f_n\} \subset K(X,Y) = K$ such that $f_n \xrightarrow{w'} f$. Then there exists a continuous bilinear form $J\colon K^* \times L \to R$ (scalars) and a number $c > 0$ such that*

a) *if* $f \in K$ *and* $\Phi \in K^*$ *then* $J(\Phi, f) = \Phi(f)$;

b) $|J(\Phi, f)| \leqslant c\|\Phi\| \cdot \|f\|$ *for all* $f \in L$ *and* $\Phi \in K^*$;

c) $J(\Phi, f) = \lim \Phi(f_n)$ *where* $\{f_n\}$ *is any sequence of compact operators* $f_n \in K$ *tending* $w'$ *to* $f$.

Proof. First we observe that if $f_n \xrightarrow{w'} f$, $f \in L$ and $f_n \in K$ then $\lim \Phi(f_n)$ exists for all $\Phi \in K^*$. Indeed, $\{\Phi(f_n)\}$ is bounded by the uniform boundedness principle and thus $\limsup_n \Phi(f_n) = \lim_k \Phi(f_{n_k})$ and $\liminf_n \Phi(f_n) = \lim_k \Phi(f_{m_k})$ for suitable subsequences $\{n_k\}$ and $\{m_k\}$ of natural numbers. Thus $\limsup \Phi(f_n) - \liminf \Phi(f_n) = \lim_k \Phi(f_{n_k} - f_{m_k}) = 0$, because $f_{n_k} - f_{m_k} \to 0$ weakly by (K). Similarly we show that if $f_n \xrightarrow{w'} f$ and $g_n \xrightarrow{w'} f$ with $\{f_n\} \subset K$ and $\{g_n\} \subset K$ then $\lim \Phi(f_n) = \lim \Phi(g_n)$ for any $\Phi \in K^*$. Thus we may define $J(\Phi, f)$ by c). $J$ is evidently bilinear and if $f \in K$ then $J(\Phi, f) = \lim \Phi(f_n) = \Phi(f)$ because $f_n = f \xrightarrow{w'} f$. To show b) let us assume

(i) there is $c > 0$ such that for any $f \in L$ there is $\{f_n\} \subset K$ with $f_n \xrightarrow{w'} f$ and $\|f_n\| \leqslant c\|f\|$.

If (i) is satisfied and $\Phi \in K^*$ then

$$|J(\Phi, f)| = |\lim \Phi(f_n)| \leqslant \|\Phi\| \sup \|f_n\| \leqslant c\|\Phi\| \cdot \|f\|.$$

To complete the proof it is sufficient to show (i). □

**Lemma.** *Let* $X, Y$ *be such that for every* $f \in L(X, Y)$ *there is a sequence* $\{f_n\} \subset K(X, Y)$ *such that* $f_n \xrightarrow{w'} f$. *Then the condition (i) is satisfied. In deed, the norm* $|\!|\!| \cdot |\!|\!|$

$$|\!|\!| f |\!|\!| = \inf\{\sup_n \|f_n\|;\ f_n \subset K,\ f_n \xrightarrow{w'} f\} \quad \text{for } f \in L(X, Y)$$

*is an equivalent norm on* $L(X, Y)$.

Proof. The uniform boundedness theorem yields that if $f_n \xrightarrow{w'} f$ then $\{f_n\}$ is bounded in the norm so that $|\!|\!| f |\!|\!|$ is finite. We observe that $\|\cdot\| \leqslant |\!|\!| \cdot |\!|\!|$ on $L$. In fact for any $\varepsilon > 0$ let $\|x\| \leqslant 1$ and $\|y^*\| \leqslant 1$ be such that

$$\|f\| - \varepsilon \leqslant \left|y^*\big(f(x)\big)\right| = \lim \left|y^*\big(f_n(x)\big)\right| \leqslant \sup \|f_n\|.$$

Passing to the infimum gives the claim. Evidently $|\!|\!| \cdot |\!|\!|$ is a norm on $L$. Now we observe that $(L, |\!|\!| \cdot |\!|\!|)$ is complete. To prove this it is sufficient to show that if $f_p \in L$, $\sum_{p=1}^{\infty} |\!|\!| f_p |\!|\!| < \infty$ then $\sum_{p=1}^{\infty} f_p \in L$ exists in $L$ and $|\!|\!| \sum f_p |\!|\!| \leqslant \sum |\!|\!| f_p |\!|\!|$ (cf. Theorem 6.2.3

[7]). To see this let $f_{np} \in K$ be such that $f_{np} \overset{w'}{\underset{n}{\to}} f_p$, $\sup_n \|f_{np}\| \leqslant |||f_p||| + \frac{\varepsilon}{2^p}$. If $\|x^{**}\| \leqslant 1$, $\|y^*\| \leqslant 1$ then we have

$$\left|x^{**}\left(f^*_{np}(y^*)\right)\right| \leqslant |||f_p||| + \frac{\varepsilon}{2^p} \quad \text{for all } n.$$

Thus $\sum_p x^{**}\left(f^*_{np}(y^*)\right)$ converges uniformly in $n$ and

$$(1) \qquad \lim_n \sum_{p=1}^{\infty} x^{**}\left(f^*_{np}(y^*)\right) = \sum_{p=1}^{\infty} \lim x^{**}\left(f^*_{np}(y^*)\right) = \sum_{p=1}^{\infty} x^{**}\left(f^*_p(y^*)\right).$$

Observe now that $\sum_p f_p \in L$ exists because $\|f_p\| \leqslant |||f_p|||$ and similarly also $\sum_{p=1}^{\infty} f_{np} \in K$ exists because $K$ is $\|\ \|$-complete. Thus (1) implies that

$$\sum_p f_{np} \overset{w'}{\underset{n}{\to}} \sum f_p.$$

Then $|||\sum f_p||| \leqslant \sup_n \|\sum_p f_{np}\| \leqslant \sup_n \sum_p \|f_{np}\| \leqslant \varepsilon + \sum_p |||f_p|||$, showing that $|||\sum_p f_p||| \leqslant \sum_p |||f_p|||$. Finally, the open mapping theorem gives that $|||\cdot||| \leqslant \frac{c}{2}\|\cdot\|$ which implies (i). □

**Proposition 2.** *Suppose that every $f \in L(X,Y)$ is factorable through a Banach space $Z$, $f = f_1 f_2$ ($Z$ depending on $f$) such that $Z^*$ is separable and has the approximation property. Then for every $f \in L$ there is a sequence $\{f_n\} \subset K$ with $f_n \xrightarrow{w'} f$, i.e. the assumptions of Proposition 1 are satisfied.*

Proof. Under the assumptions $Z^*$ has the metric approximation property. Let $f = f_1 f_2$ be any factorization of $f \in L$ through the Banach space $Z$, let $p_n(z^*) \to z^*$ for every $z^* \in Z^*$. We may suppose that $p_n = P^*_n$ where $P_n \in K(Z)$, $\|P_n\| \leqslant 1$ are finite-dimensional operators [5]. Let us define $f_n = f_1 P_n f_2 \in K$. Then $f_n \xrightarrow{w'} f$. □

**Remark 1.** $J$ gives rise to two isomorphic imbeddings:

$$J_K\colon K^* \to L^* \quad J_K\Phi(f) = J(\Phi, f)$$

and

$$J_L\colon L \to K^{**} \quad J_L f(\Phi) = J(\Phi, f), \quad J_L = J^*_K/L.$$

Evidently $J_L f = f$ if $f \in K$.

Moreover,

$$\|\Phi\| \leqslant \|J_K \Phi\| \leqslant c\|\Phi\| \quad \text{for all } \Phi \in K^*$$

and

$$\|f\| \leqslant \|J_L f\| \leqslant c\|f\| \quad \text{for all } f \in L.$$

Thus $J_K$ and $J_L$ are $c$ isomorphisms and $\|J_K\| \leqslant c$, $\|J_L\| \leqslant c$.

Proof. Indeed, given $f \in L$ and $\varepsilon > 0$ we have for suitable $\|x\| = 1$, $\|y^*\| = 1$

$$\begin{aligned} \|f\| - \varepsilon \leqslant \left|y^*\big(f(x)\big)\right| &= \left|\lim x\big(f_n^*(y^*)\big)\right| \\ &= |J(x \otimes y^*, f)| = |J_L f(x \otimes y^*)| \\ &\leqslant \sup\left\{|J_L f(\Phi)|;\ \|\Phi\| \leqslant 1\right\} = \|J_L f\|. \end{aligned}$$

Similarly $\|\Phi\| = \sup\left\{|\Phi(f)|;\ f \in K;\ \|f\| \leqslant 1\right\}$. But $\Phi(f) = J(\Phi, f) = J_K \Phi(f)$. Thus

$$\|\Phi\| \leqslant \sup\left\{|J_K \Phi(f)|;\ f \in L;\ \|f\| \leqslant 1\right\} = \|J_K \Phi\|.$$

□

**Remark 2.** If $\mathrm{Re} : L^* \to K^*$ is the restriction operator then $P = J_K \mathrm{Re}$ is a projection in $L^*$ whose range is $c$-isomorphic to $K^*$ and $\operatorname{Ker} P = K^\circ$.

This is J. Johnson's type of statement and it follows immediately from Remark 1.

**Remark 3.** Let every $f \in L(X, Y)$ be factorable as indicated in the assumption of Proposition 2. Let us put

$$p(f) = \inf \|f_1\| \cdot \|f_2\|$$

where the infimum is taken over all factorizations of $f$ through any $Z$ such that $Z^*$ has the bounded approximation property and is separable. Then

a) $p$ is an equivalent norm on $L(X, Y)$;

b) for every $\varepsilon > 0$ there are $f_n \in K$ such that

$$f_n \xrightarrow{w'} f \quad \text{and} \quad p(f_n) \leqslant (1 + \varepsilon) p(f).$$

Thus

$$b_1) \qquad |J(\Phi, f)| \leqslant p^*(\Phi) p(f) \quad \text{for } f \in L \text{ and } \Phi \in K^*.$$

Easy observations similar as in Remark 1 give that $J_K$ and $J_L$ are $p$-isometries and $p(P) = 1$. Thus $K$ is an ideal in $(L, p)$ in the terminology of [2]. The question when e.g. $(L, p)$ is a $u$-ideal or an $M$-ideal will be treated in a subsequent paper.

P r o o f. We show e.g. a). As in the proof of Proposition 1 we have $\|\cdot\| \leqslant p(\cdot)$ on $L$. Evidently $p$ is subadditive. In fact, let $f_i = B_i A_i$ be factorizations of $f_i$ through suitable $Z_i$ so that $\sum_i \|A_i\| \cdot \|B_i\| \leqslant \varepsilon + \sum_i p(f_i)$, $\|A_i\| = \|B_i\|$. Let us put

$$Z = (Z_i)_{\ell_2} \quad \text{and} \quad A = (A_i)\colon X \to Z,$$

$B\colon Z \to Y$, $B(\{z_i\}) = \sum B_i(z_i)$. Then $\|A\|^2 \leqslant \sum \|A_i\|^2$ and $\|B\|^2 = \|B^*\|^2 \leqslant \sum \|B_i\|^2$. Thus

$$p\Big(\sum f_i\Big) \leqslant \|A\| \cdot \|B\| \leqslant \sum \|A_i\| \cdot \|B_i\| \leqslant \varepsilon + \sum p(f_i).$$

To see that $(L, p)$ is complete it suffices as in the proof of the Lemma to show the following: Let $f_i \in L$ be such that $\sum p(f_i) < \infty$. Then $\sum f_i \in L$ and $p(\sum f_i) \leqslant \sum p(f_i)$. But this is exactly the above proof of the subadditivity of $p$. □

**Remark 4.** The isomorphism $J_L\colon L \to K^{**}$ together with the local reflexivity of $K$ gives: Under the assumptions of Proposition 1 the Banach space $L$ is $(c+\varepsilon)$-finitely representable in $K$ so that the representations are the identity on $K$.

**Remark 5.** It is not necessary to assume in Proposition 2 that $Z^*$ is separable. In fact, the following is sufficient for the statement of Proposition 2 (and for Remark 3): Every $f \in L$ is factorable through a Banach space $Z$, $f = f_1 f_2$ ($Z$ depending on $f$) such that $Z^*$ has the bounded approximation property and $f_1^*(Y^*) \subset Z^*$ is separable.

**Remark 6.** Another modification of Proposition 2 is the following:

Suppose that every $f \in L(X, Y)$ is factorable through a Banach space $Z$, ($Z$ depending on $f$) such that there is a sequence $\{P_n\}$ in the unit ball of $K(Z)$ such that $P_n \to Id_Z$ in the weak operator topology and such that $Z$ has the property (**) defined in [1, p. 678]. Then the assumptions of Proposition 1 are satisfied.

In order to have (**) it is sufficient that $Z$ has the unique extension property in the sense of [1].

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