Luis A. Cordero; Phillip E. Parker
Examples of sectional curvature with prescribed symmetry on 3-manifolds

EXAMPLES OF SECTIONAL CURVATURE
WITH PRESCRIBED SYMMETRY ON 3-MANIFOLDS

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1. Introduction

Let M be a smooth 3-manifold and g a pseudoriemannian metric tensor on M. Let $G_2(M)$ denote the Grassmannian bundle with fibers $G_2(T_xM)$, the space of (2-dimensional) planes in the tangent space T_xM at a point $x \in M$. Observe that each $G_2(T_xM)$ may be regarded as a (real) algebraic variety, diffeomorphic to the (real) projective plane $\mathbb{P}^2$. As in [3, 5], we shall regard the sectional curvature K_x at each point $x \in M$ as a rational mapping of algebraic varieties $G_2(T_xM) \to \mathbb{R}$, or a rational function for short. The group of all automorphisms of $G_2(T_xM)$ is isomorphic to $PGL_3 \equiv PGL_3(\mathbb{R})$, the group of projective automorphisms of $\mathbb{P}^2$.

In [5] we determined the possible symmetry groups of K at x; *i.e.*, the largest subgroup of PGL_3 which leaves K_x invariant as a rational function. We shall refer to any one of these as a sectional curvature symmetry, or SCS for short. We determined the existence of naturally reductive homogeneous spaces with constant SCS, and gave general descriptions of some examples of them.

In this paper, we exhibit explicit forms of the metric tensors on some of these examples (§4). We also give some inhomogeneous examples utilizing warped products (§3), and begin the study of how the SCS and CF-type can vary on a connected space (§5).

Our Lorentzian metric tensors will have signature $+--$. When necessary, we distinguish among the possible orderings $+--$, $-+-$, $--+$. (To convert to the other signature convention $++-$, see [9, p. 92].) Thus a vector v is *timelike*

if $g(v,v) > 0$, *lightlike* or *null* if $g(v,v) = 0$, *spacelike* if $g(v,v) < 0$, and *causal* if $g(v,v) \geqslant 0$.

When convenient, we regard the Riemann tensor R_{ijkl} as a quadratic form on $\bigwedge^2 TM$; cf. [3, 5]. In local coordinates,

$$R = \begin{bmatrix} R_{1212} & R_{1213} & R_{1223} \\ R_{1213} & R_{1313} & R_{1323} \\ R_{1223} & R_{1323} & R_{2323} \end{bmatrix}.$$

Then the sectional curvature appears as a rational function on $G_2(M)$ in the form of a quotient of two quadratic functions:

$$K = \frac{R}{\bigwedge^2 g}.$$

Also recall that the associated tensor R_{kl}^{ij} represents the curvature operator $\bar{R}$: $\bigwedge^2 TM \to \bigwedge^2 TM$ in local coordinates. Note that if R and $\bar{R}$ are written as matrices with respect to the same local coordinates, then $R = (\bigwedge^2 g)\bar{R}$.

We denote the Lorentz group in $(n = p + q)$ dimensions of signature (p, q) by $O_p^q = O_p^q(\mathbb{R})$, thus the (usual) orthogonal group by $O_n = O_n(\mathbb{R})$. Projectivization of any group of linear transformations is indicated by a prefixed P; for example $PGL_3 = GL_3/\{aI\,;\, 0 \neq a \in \mathbb{R}\} \cong SL_3$.

Some of the results in Section 3 were presented by Parker at the Bolyai Colloquium on Differential Geometry in 1984. Again, Parker thanks Cordero and the Departamento at Santiago for their extraordinary hospitality during his visits.

2. Preliminary recollections

For the convenience of the reader, we state some of the main results of [5]. Recall that $\bigwedge^2 \operatorname{diag}[\,1, -1, -1\,] = \operatorname{diag}[-1, -1, 1\,]$. This is Theorem 2.2 of [5]:

Theorem 2.1. *At each point x of a Lorentzian 3-manifold (M, g), there exists a choice of g-orthonormal coordinates with respect to which the Riemann tensor R_x on $\bigwedge^2 T_x M$ takes on exactly one of these canonical forms:*

CF1 $\operatorname{diag}[B, C, A]$;

$$\text{CF2} \quad \begin{bmatrix} B & 0 & 0 \\ 0 & -A & F \\ 0 & F & A \end{bmatrix}, \quad F \neq 0;$$

$$\text{CF3} \quad \begin{bmatrix} B & 0 & 0 \\ 0 & -\lambda \pm \frac{1}{2} & \pm \frac{1}{2} \\ 0 & \pm \frac{1}{2} & \lambda \pm \frac{1}{2} \end{bmatrix};$$

$$\text{CF4} \quad \begin{bmatrix} -\lambda & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\lambda & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \lambda \end{bmatrix}.$$

We note that these forms can also be characterized in terms of eigenvectors of $\bar{R}$: timelike, spacelike, double null, and triple null, respectively; compare [8, §4.3].

We also give Table 1 of [5]. Recall the group HT of horocyclic translations (called *null rotations* in relativity because there is a fixed null direction). The identity component of this group consists of the matrices

$$\exp \begin{bmatrix} 0 & -t & t \\ t & 0 & 0 \\ t & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -t & t \\ t & 1 - \frac{t^2}{2} & \frac{t^2}{2} \\ t & -\frac{t^2}{2} & 1 + \frac{t^2}{2} \end{bmatrix}, \ t \in \mathbb{R},$$

Each component of $O_1^2(- - +)$ contains one component of HT.

Canonical form of R_x	Symmetry group of $K_x = R_x / \bigwedge^2 g_x$	
CF1 : diag $[B, C, A]$	$A = -B = -C$	PGL_3
	$B = C \neq -A$	PO_2
	$\left. \begin{array}{l} A = -B \neq -C \\ A = -C \neq -B \end{array} \right\}$	PO_1^1
	generic	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
CF2	$\mathbb{Z}_2$	
CF3	$B \neq -\lambda$	$\mathbb{Z}_2$
	$B = -\lambda$	PHT
CF4	1	

Table 1. Lorentzian SCS

Finally, we state Theorem 3.1 of [5].

Theorem 2.2. *If $M = G$ is an irreducible, naturally reductive, Lorentzian homogeneous space of dimension 3, then either M is flat or of constant positive curvature. In the former case, M is Minkowskian 3-space or one of its quotients by a discrete*

group of translations. In the latter, M is $SO_1^{2+}(\mathbb{R})$ or one of its coverings or quotients by a discrete subgroup.

3. Some Warped Products

To begin, we consider Lorentzian warped products of the form $M = \Lambda \times_r \Sigma$. Here, Λ is either the circle S^1 or an open interval $I \subseteq \mathbb{R}$ with coordinate t, r is a positive, smooth function on Λ (the warping function), Σ is a surface with line element $d\sigma^2$, and the line element on M is $ds^2 = dt^2 - r^2 d\sigma^2$. We refer to [9, pp. 204–211] for the geometry of warped products. Using the formulas there, we find $spec\bar{R} = \{\lambda_1, \lambda_2\}$ with λ_i of multiplicity i, $\lambda_2 = \ddot{r}/r$, and $\lambda_1 = (\dot{r}/r)^2 - k/r^2$, where the dot denotes d/dt and k is the intrinsic (Gaussian or sectional) curvature of Σ. Thus, in appropriate local orthonormal coordinates [5], $\bar{R} = \mathrm{diag}[\lambda_2, \lambda_2, \lambda_1]$. Roughly, one may say that $K(\text{plane} \parallel \Sigma) = \lambda_1$ and $K(\text{plane} \perp \Sigma) = \lambda_2$. Consulting Table 1 from [5], we have

Proposition 3.1. *If $\lambda_1 = \lambda_2$, then M has a constant SCS of PGL_3, hence constant curvature. If $\lambda_1 \neq \lambda_2$, then M has a constant SCS of PO_2.*

If we regard λ_1 and k as given, then we find

$$\log r = \pm \int \sqrt{\lambda_1 + k/r^2}\, dt$$

subject to the consistency condition $\lambda_1 \geqslant -k/r^2$. If we regard λ_2 as given, then it follows from ODE theory (*e.g.*, [4]) that a suitable r exists for any continuous λ_2. With a smooth metric, our λ_2 is in fact smooth. Therefore,

Theorem 3.2. *All possible $\bar{R}$ of the form $\mathrm{diag}[\lambda_2, \lambda_2, \lambda_1]$ can be obtained from Lorentzian warped products $\Lambda \times_r \Sigma$.*

It remains to determine which of them have constant curvature. This happens if and only if

$$\frac{\ddot{r}}{r} = \left(\frac{\dot{r}}{r}\right)^2 - \frac{k}{r^2},$$

whence k must be constant. Solving the ODE, we obtain the solution implicitly as

$$\log r = \pm \int \sqrt{c_1 + 2k/r}\, dt + c_2$$

for appropriate constants of integration c_1 and c_2. When $k = 0$, we can write r explicitly as

$$r(t) = r(0)e^{c\,t}$$

for an appropriate constant c. If $c \neq 0$, r cannot be periodic. Thus the only such examples with $\Lambda = S^1$ and a flat Σ are the flat Lorentzian warped products $S^1 \times_r \Sigma$ with constant r.

Reflecting back on [5] and anticipating the next section, we observe that some warped products are homogeneous spaces. For example, if Σ is a closed, orientable surface with a Riemannian metric of constant curvature and the warping function r is constant, then $\Lambda \times_r \Sigma$ is G-homogeneous for $G = \mathbb{R} \times SO_3$, $\mathbb{R} \times \mathbb{R}^2$, or $\mathbb{R} \times SL_2$, respectively, when Σ has genus $g = 0$, 1, or $\geqslant 2$, respectively. In light of the present work, it would be of interest to have effective criteria for determining when a warped product is homogeneous.

In this spirit, we make a simple observation. Assume that a connected, Lorentzian 3-manifold M has a constant SCS of PO_2. Then $\bar{R}$ has a timelike eigenvector at every point. Equivalently, there is a distinguished spacelike plane at every point. We obtain a splitting $TM = L \oplus P$ into a timelike line bundle L and this spacelike plane bundle P. When P is integrable, the constant SCS of PO_2 implies that the leaves are totally umbilic.

Proposition 3.3. *If P is integrable, then M is locally an umbilic product.*

This follows from the obvious Lorentzian version of a theorem of Bishop [2]. (See also [7].) As we shall make no use of this result here, we omit the proof. For the convenience of the reader, however, we recall that an umbilic product is a generalization of a warped product in which the warping function may depend on both factors.

We continue with warped products $M = \Sigma \times_r \Lambda$, where r is now a smooth, positive function on Σ, $ds^2 = r^2\,dt^2 - d\sigma^2$, and we keep the other notations from the first part of this section. Consulting [9] again, we find in suitable local coordinates ($t = x_1, x_2, x_3$)

$$R_{2323} = k,$$
$$R_{1212} = -H^r(\partial_2, \partial_2)/r,$$
$$R_{1313} = -H^r(\partial_3, \partial_3)/r,$$
$$R_{1213} = -H^r(\partial_2, \partial_3)/r,$$
$$R_{1223} = R_{1323} = 0,$$

where H^r denotes the Hessian of r on Σ and $\partial_i = \partial/\partial x_i$. By means of an appropriate change of local coordinates on Σ, we may assume that $R_{1213} = 0$. Then we obtain R in CF1.

We have been able to solve these equations explicitly only when $k = 0$. Even in that case, however, we obtain examples of all SCS in CF1. Indeed, in this case the system becomes

$$\partial_2 \partial_2 r = f,$$
$$\partial_3 \partial_3 r = g,$$
$$\partial_2 \partial_3 r = 0.$$

It follows that f is a function of x_2 only and g is a function of x_3 only. Thus the system can be integrated directly for any such f and g; in particular,

$$\begin{array}{llll} \text{for} & f = g = 0 & \text{we obtain } PGL_3, \\ \text{for} & f = g \neq 0 & \text{we obtain } PO_2, \\ \text{for} & \left\{ \begin{array}{l} f \neq g = 0 \\ f = 0 \neq g \end{array} \right\} & \text{we obtain } PO_1^1, \\ \text{for} & 0 \neq f \neq g \neq 0 & \text{we obtain } \mathbb{Z}_2 \oplus \mathbb{Z}_2. \end{array}$$

Joint work in progress with Dean Allison will explore these warped products further, along with doubly warped products and umbilic products.

4. Some homogeneous examples

In this section we write out explicit details and metric tensors for some of the naturally reductive examples described in [5]. We begin by noting that the case of spaces of constant curvature is well known: *cf.* [10] and references there. Thus we omit explicit examples with constant SCS of PGL_3. In order to simplify the notation, we shall write down our calculations using normalized versions in which the structure constants in [5] are taken as ± 1 whenever possible. We give only examples $M = G/H$ with $\dim G = 4$; among those Lie algebras we have been able to identify and integrate, no new features appear when $\dim G = 5$.

PO_2 For the SCS of PO_2, we take the structure equations of $\mathfrak{g}$ as

$$[e_2, e_3] = e_4,$$
$$[e_3, e_4] = -e_2,$$
$$[e_4, e_2] = -e_3.$$

The left-invariant 1-forms and the dual left-invariant vector fields are

$$\omega_1 = \mathrm{d}x_1,$$
$$\omega_2 = \cos x_4 \, \sinh x_3 \, \mathrm{d}x_2 - \sin x_4 \, \mathrm{d}x_3,$$

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$$\omega_3 = \sin x_4 \, \sinh x_3 \, \mathrm{d}x_2 + \cos x_4 \, \mathrm{d}x_3,$$

$$\omega_4 = \cosh x_3 \, \mathrm{d}x_2 + \mathrm{d}x_4,$$

$$e_1 = \partial_1,$$

$$e_2 = \cos x_4 \, \mathrm{csch}\, x_3 \, \partial_2 - \sin x_4 \, \partial_3 - \cos x_4 \, \coth x_3 \, \partial_4,$$

$$e_3 = \sin x_4 \, \mathrm{csch}\, x_3 \, \partial_2 + \cos x_4 \, \partial_3 - \sin x_4 \, \coth x_3 \, \partial_4,$$

$$e_4 = \partial_4,$$

where $\partial_i = \partial/\partial x_i$. The quotient $M = G/H$ is topologically $\mathbb{R}^3$ and given by $\{e_4 = 0\}$ or $\{x_4 = \text{const.}\}$.

Taking $\varepsilon_4 = -1$ to get the signature $+--$, we find the line element

$$\mathrm{d}s^2 = \mathrm{d}x_1^2 - \cosh 2x_3 \, \mathrm{d}x_2^2 - \mathrm{d}x_3^2$$

with a curvature matrix of $\mathrm{diag}\,[0,0,1]$. Regarding $x_1 \pmod 1$ as a coordinate on S^1, we obtain a G-left invariant metric on $S^1 \times \Sigma$ where Σ is a Riemannian open surface of constant curvature -1. Thus there are compact models $S^1 \times \Sigma_g$ where Σ_g is a closed surface of genus $g \geqslant 2$.

PHT For the SCS of *PHT* in CF3 with $B = -\lambda = 0$, we take the structure equations of $\mathfrak{g}$ as

$$[e_1, e_2] = 0,$$

$$[e_2, e_3] = a\, e_4,$$

$$[e_3, e_1] = 0,$$

$$[e_1, e_4] = 0,$$

$$[e_2, e_4] = -b\, e_3,$$

$$[e_3, e_4] = b\, e_1,$$

where $a = c_{23}^4$ and $ab = \mp 1/2$. We obtain a semidirect product $\mathfrak{g} = \mathfrak{g}_1 \times_\theta \langle e_2 \rangle$ with $\mathfrak{g}_1 = \langle e_3, e_4, e_1 \rangle$ isomorphic to the Heisenberg algebra and

$$\theta(e_2) = \begin{bmatrix} 0 & -b & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, when $ab = -1/2$ we have

$$\exp(-t\theta) = \begin{bmatrix} \cosh \frac{1}{\sqrt 2} t & b\sqrt 2 \sinh \frac{1}{\sqrt 2} t & 0 \\ -a\sqrt 2 \sinh \frac{1}{\sqrt 2} t & \cosh \frac{1}{\sqrt 2} t & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and when $ab = 1/2$ we have

$$\exp(-t\theta) = \begin{bmatrix} \cos\frac{1}{\sqrt{2}}t & b\sqrt{2}\sin\frac{1}{\sqrt{2}}t & 0 \\ -a\sqrt{2}\sin\frac{1}{\sqrt{2}}t & \cos\frac{1}{\sqrt{2}}t & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We consider only the case $ab = 1/2$. Setting $\xi = x_2/\sqrt{2}$ for convenience, the left-invariant 1-forms and vector fields are

$$\omega_1 = dx_1 - bx_3\,dx_4,$$
$$\omega_2 = dx_2,$$
$$\omega_3 = \cos\xi\,dx_3 + b\sqrt{2}\sin\xi\,dx_4,$$
$$\omega_4 = -a\sqrt{2}\sin\xi\,dx_3 + \cos\xi\,dx_4,$$
$$e_1 = \partial_1,$$
$$e_2 = \partial_2,$$
$$e_3 = \frac{x_3}{\sqrt{2}}\sin\xi\,\partial_1 + \cos\xi\,\partial_3 + a\sqrt{2}\sin\xi\,\partial_4,$$
$$e_4 = bx_3\cos\xi\,\partial_1 - b\sqrt{2}\sin\xi\,\partial_3 + \cos\xi\,\partial_4.$$

We perform the change of coordinates

$$y_1 = x_1 + \frac{x_3^2}{2\sqrt{2}}\cot\xi,$$
$$y_2 = x_2,$$
$$y_3 = b\sqrt{2}\,x_4\sin\xi + x_3\cos\xi,$$
$$y_4 = x_4\sec\xi,$$

and obtain (interpreting ξ as $y_2/\sqrt{2}$ now)

$$\omega_1 = dy_1 + \left[\frac{y_3^2\left(\sqrt{2} - 4\sin^2\xi\right)}{4\sqrt{2}\sin^2\xi\cos^2\xi} + \frac{b^2y_4^2}{2}\right]dy_2 - \frac{y_3}{\sqrt{2}}\sec\xi\csc\xi\,dy_3,$$
$$\omega_2 = dy_2,$$
$$\omega_3 = dy_3 + \left(\frac{y_3}{\sqrt{2}}\tan\xi - by_4\right)dy_2,$$
$$\omega_4 = dy_4 - ay_3\tan^2\xi\,dy_2 - a\sqrt{2}\tan\xi\,dy_3,$$
$$e_1 = \partial_1,$$
$$e_2 = -\left(\frac{y_3}{2}\sec\xi\csc\xi - \frac{by_4}{\sqrt{2}}\right)^2\partial_1 + \partial_2 + \left(by_4 - \frac{y_3}{\sqrt{2}}\tan\xi\right)\partial_3 + \frac{y_4}{\sqrt{2}}\tan\xi\,\partial_4,$$
$$e_3 = \frac{y_3}{\sqrt{2}}\sec\xi\csc\xi\,\partial_1 + \partial_3 + a\sqrt{2}\tan\xi\,\partial_4,$$
$$e_4 = \partial_4,$$

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with $\partial_i = \partial/\partial y_i$ now.

On the quotient G/H, we find a family of left-invariant line elements in the parameter y_4. We have not written them out here as we found no reasonable simplifications of the coefficients. Recall that compact models are possible only if $\exp(\theta)$ preserves a suitable lattice in G/H. The case $ab = -1/2$ may be treated similarly.

5. Varying CF and SCS

Here we wish to consider how the canonical form type (CF1–CF4) and the symmetry group (SCS) can vary as one moves about in M. We observe first that with a smooth metric tensor g on M, the curvature tensor R will vary smoothly. Thus, a smooth curve in M lifts to a smooth curve in the bundle of possible curvature tensors. This bundle can be identified either as $\text{End}_{SA}(\bigwedge^2 TM)$, the selfadjoint endomorphisms bundle, or (lowering indices) as $Q(\bigwedge^2 TM)$, the bundle of quadratic forms; *cf.* [5].

To begin, we determine how the CF's and SCS's partition the space of possible curvature tensors at a fixed point $p \in M$. For this purpose, it suffices to fix a normal coordinate chart at p. This identifies $T_pM = \mathbb{R}^3$ and $\bigwedge^2 T_pM = \bigwedge^2 \mathbb{R}^3 \cong \mathbb{R}^3$. Thus we identify the space of possible curvature tensors at p with Sym_3, the symmetric 3×3 matrices regarded as quadratic forms on $\bigwedge^2 \mathbb{R}^3$. We coordinatize $\text{Sym}_3 \cong \mathbb{R}^6$ via

$$\begin{bmatrix} B & D & E \\ D & C & F \\ E & F & A \end{bmatrix} \longmapsto (B, C, A, D, E, F);$$

see [5].

If the metric tensor $g_p = \text{diag}\,[1, -1, -1]$, then $\bigwedge^2 g_p = \text{diag}\,[-1, -1, 1]$. A change of normal coordinates at p acts on $T_pM = \mathbb{R}^3$ by an element of $O_1^2(+--)$, the Lorentz group for g_p. It is an easy exercise in linear algebra to verify that $\bigwedge^2 O_1^2(+--) = SO_1^2(--+)$. Thus we consider the action of $SO_1^2(--+)$ on Sym_3 where $A{\cdot}R = A^t RA$ for $A \in SO_1^2(--+)$ and $R \in \text{Sym}_3$.

Constant curvature k at p is characterized by $R_p = k \bigwedge^2 g_p$. Since $SO_1^2(--+)$ preserves $\bigwedge^2 g_p$, it also preserves R_p in this case: each constant curvature R_p is a fixed point. These comprise a line through the origin in Sym_3. This is the set where the SCS is PGL_3: $\{B = C = -A,\ D = E = F = 0\}$.

For the SCS of PO_2, there is the plane $\{B = C\}$ in BCA-space, containing the line L of constant curvature $\{B = C = -A\}$. The O_2 subgroup of SO_1^2 fixes each point on this plane, and all other elements of SO_1^2 move every point of the two half-planes complementary to L. The two half-planes are not interchanged by the action. We obtain two closed orbits in $\mathbb{R}^6$, PO_2 I and PO_2 II, each homeomorphic to $\mathbb{R}^4$.

Figure 1 represents a plane perpendicular to L in BCA-space. The view is from the third octant, toward the origin along L. In the figure, "dim d" means the full orbit in $\mathbb{R}^6$ has dimension d.

PO_2 I
$-A < C = B$
dim 4

PO_1^1 I
$-A = B < C$
dim 4

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ I
$-A < B < C$
dim 6

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ I
$-A < C < B$
dim 6

PO_1^1 I
$-A = C < B$
dim 4

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ II
$B < -A < C$
dim 6

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ II
$C < -A < B$
dim 6

PO_1^1 II
$B < C = -A$
dim 4

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ III
$B < C < -A$
dim 6

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ III
$C < B < -A$
dim 6

PO_1^1 II
$C < B = -A$
dim 4

PO_2 II
$B = C < -A$
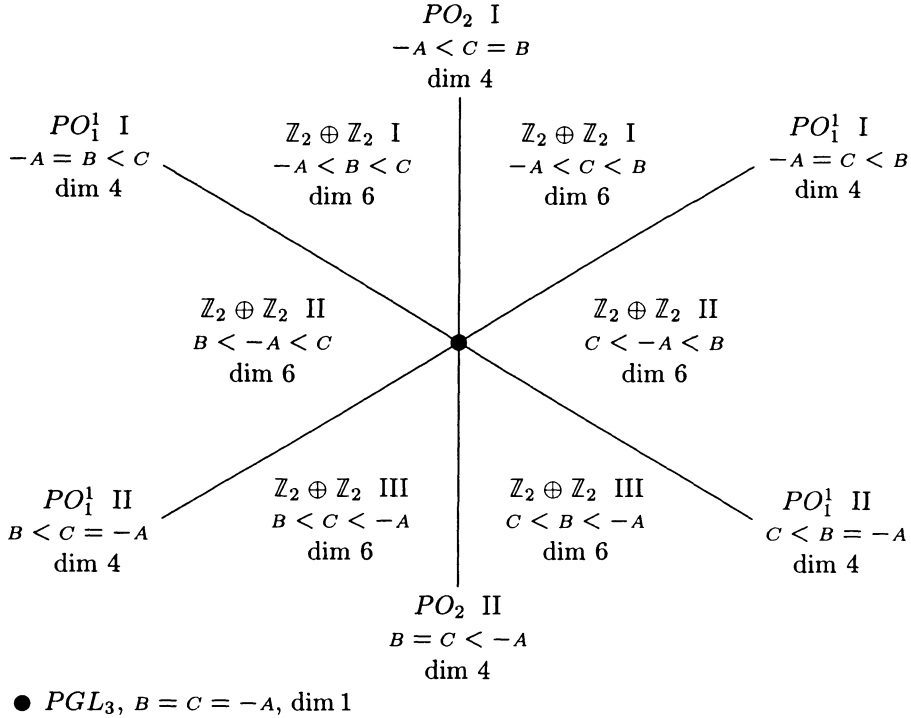dim 4

● PGL_3, $B = C = -A$, dim 1

Figure 1. Contiguity Relations for CF1.

For the SCS of PO_1^1, there are two planes in BCA-space: $\{B = -A\}$ and $\{C = -A\}$. They intersect in L. Each point of each half-plane complementary to L is fixed by the appropriate O_1^1 subgroup of SO_1^2. The two planes are not interchanged by the action of SO_1^2, but the half planes are in pairs across the plane $\{B = C\}$; see Figure 1. We obtain two closed orbits in $\mathbb{R}^6$, PO_1^1 I and PO_1^1 II, each homeomorphic to $S^1 \times \mathbb{R}^3$, which are cones over $\{B = C = -A\}$.

For the SCS of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, we have the complement of the union of the planes $\{B = C\} \cup \{B = -A\} \cup \{C = -A\}$ in BCA-space. The discrete subgroup of SO_1^2 which fixes each point in this set has two elements in each component of SO_1^2. The set consists of six connected components, pairs of which are interchanged across the plane $\{B = C\}$ by the SO_1^2-action. We obtain three open orbits in $\mathbb{R}^6$, each homeomorphic to $S^1 \times \mathbb{R}^5$, which are also cones over $\{B = C = -A\}$. They are pairwise separated in BCA-space by the preceding orbits, as indicated in Figure 1.

CF2 has the SCS of $\mathbb{Z}_2$ and CF3 has the SCS of either $\mathbb{Z}_2$ or PHT. Since $R_{1323} = F \neq 0$ in either, both consist of (affine) planes parallel to one seen previously. No part of the CF2 or CF3 orbits lies in BCA-space.

The planes of CF2 are parallel to $\{C = -A\}$. The discrete subgroup of SO_1^2 which fixes each point of these planes has elements in both components of SO_1^2. The two connected components $\{F > 0\}$ and $\{F < 0\}$ are interchanged by the action. We obtain one open orbit in $\mathbb{R}^6$, homeomorphic to $S^1 \times \mathbb{R}^5$, which is a cone over $\{C = -A\}$. See Figure 2.

CF3 II
$C + A = -2F$
dim 5

CF2
dim 6

CF3 I
$C + A = 2F$
dim 5

CF1
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$C < -A$

CF1
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$-A < C$

CF3 II
$C + A = 2F$
dim 5

CF2
dim 6

CF3 I
$C + A = -2F$
dim 5

● $C = -A$, $F = 0$
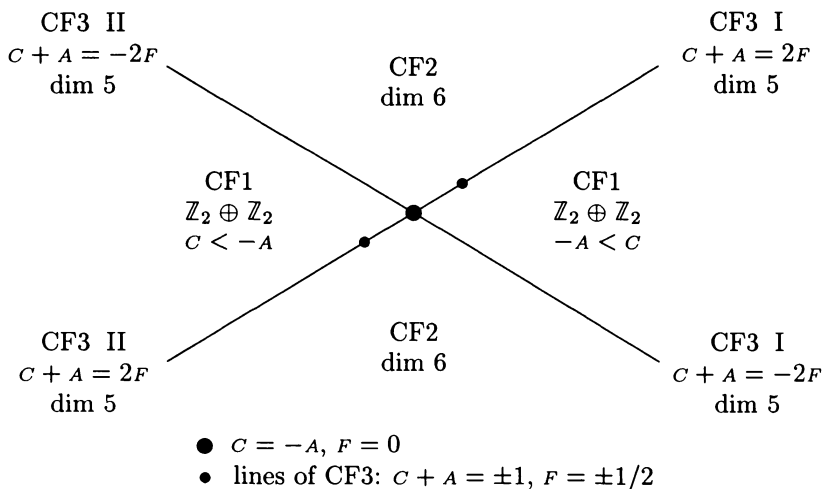●   lines of CF3: $C + A = \pm 1$, $F = \pm 1/2$

Figure 2. Contiguity Relations for CF2 and CF3.

The two planes of CF3 are also parallel to $\{C = -A\}$ and are translates of the two planes of CF2 at $F = \pm 1/2$. The lines $\{B = -\lambda\}$ are the translates of the line of constant curvature L by $(0, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ in $BCAF$-space, and lie inside the small bullets of Figure 2. Each point is fixed by the HT subgroup of SO_1^2. The two lines are not interchanged by the action of SO_1^2. We obtain two closed orbits in $\mathbb{R}^6$, PHT I and PHT II, each homeomorphic to $S^1 \times \mathbb{R}^3$, which are cones over L. Portions of them lie inside the sets CF3 I and CF3 II of Figure 2, respectively.

The complement of these lines in the CF3 planes has the SCS of $\mathbb{Z}_2$. Again, the discrete subgroup of SO_1^2 which fixes each point has elements in both components of SO_1^2, and the translate at $F = \pm 1/2$ interchanges with a correspondent at $F = \mp 1/2$. Two translates at the same F-level, however, do not interchange. We obtain two orbits in $\mathbb{R}^6$, $\mathbb{Z}_2$ I and $\mathbb{Z}_2$ II, each of dimension 5, which are also cones over $\{C = -A\}$.
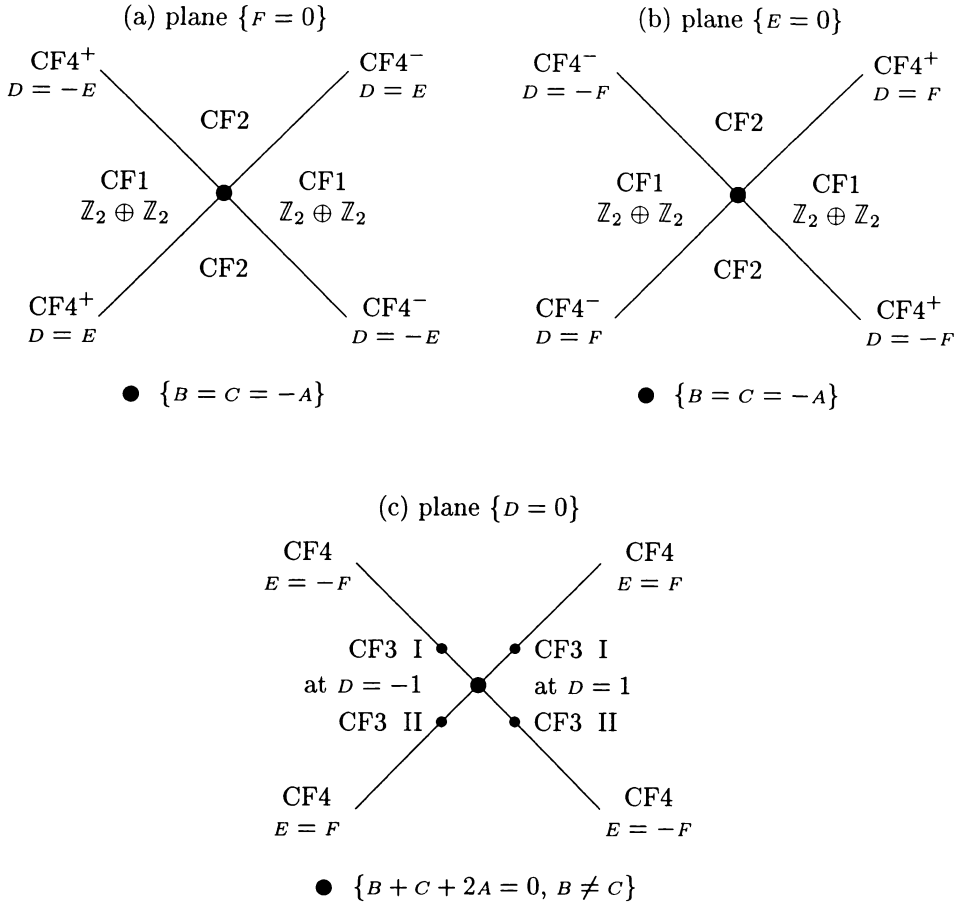
(a) plane $\{F = 0\}$

CF4$^+$
$D = -E$ CF2   CF4$^-$
$D = E$

CF1
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ CF1
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

CF2

CF4$^+$
$D = E$ CF4$^-$
$D = -E$

● $\{B = C = -A\}$

(b) plane $\{E = 0\}$

CF4$^-$
$D = -F$ CF2   CF4$^+$
$D = F$

CF1
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ CF1
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

CF2

CF4$^-$
$D = F$ CF4$^+$
$D = -F$

● $\{B = C = -A\}$

(c) plane $\{D = 0\}$

CF4
$E = -F$ CF4
$E = F$

CF3 I
at $D = -1$ CF3 I
at $D = 1$

CF3 II   CF3 II

CF4
$E = F$ CF4
$E = -F$

● $\{B + C + 2A = 0,\ B \neq C\}$

Figure 3. Contiguity Relations for CF4.

The unions $\mathbb{Z}_2$ I$\cup PHT$ I and $\mathbb{Z}_2$ II$\cup PHT$ II are closed sets in $\mathbb{R}^6$, each homeomorphic to $S^1 \times \mathbb{R}^4$. These are the sets CF3 I and CF3 II of Figure 2, which represents a plane parallel to $\{C = A\}$ in CAF-space. The view is from the fourth quadrant of the CA-plane toward the origin along the line $\{C = -A\}$. The lines have slopes of $\pm 1/\sqrt{2}$, and the two bullets for CF3 have coordinates $\pm(1/\sqrt{2}, 1/2)$. Considering the B-axis as perpendicular to the page, we have a representation of a translate of the 3-space with axes B, $C = A$, and F along the B-axis in $BCAF$-space. The part of the B-axis with $B > C$ is PO_1^1 I and the part with $B < C$ is PO_1^1 II. The SO_1^2-orbits of PO_2 from CF1 are the portions of the elliptic cone $\{(2B - C + A)^2 + 4F^2 = (C + A)^2\}$ above and below the CF1 regions, PO_2 I to the right and PO_2 II to the left.

Finally, CF4 is a line parallel to the line L in BCA-space. Since the SCS here is 1, we obtain one closed orbit of two connected components (CF4$^+$ and CF4$^-$) in $\mathbb{R}^6$, each homeomorphic to $S^1 \times \mathbb{R}^3$. Both components are cones over L. No part of this orbit lies in $BCAF$-space.

In Figure 3, we see the three coordinate planes of DEF-space. Parts (a) and (b) have $B = C = -A$. The orbits of the other canonical forms are contained in the axes; for example, rotations of CF1 are in the D-axis, and those CF2 with $B = -A$ and certain boosts of CF3 are in the F-axis. Part (c) still has $B + C + 2A = 0$, but now we allow $B \neq C$. Note that the four CF3 points lie at $D = \pm 1$, not at $D = 0$; their E- and F-coordinates are $\pm 1/2\sqrt{2}$. The CF4 lines can be either CF4$^+$ or CF4$^-$, depending on the values of B and C. The open regions are parts of the orbit of CF2, the others all being inside the axes. While rotations act naturally in AEF-space, their action in DEF-space is more complicated. Rotations acting here on CF4 sweep out the two branches of the quartic surface $D^2(E^2 + F^2) = (E^2 - F^2)^2$. We recall that a rotation acting through an angle θ clockwise about the A-axis in AEF-space, acts through an angle 2θ counterclockwise about the line $\{B = C, D = 0\}$ in BCD-space.

Now, at each point x of our 3-manifold M, the sectional curvature K has a certain symmetry and the curvature tensor R lies in a certain SO_1^2-orbit in $Q(\bigwedge^2 TM)$. We shall call this latter the *orbit type* of R at x. As a first simple result, we observe

Theorem 5.1. *The SCS is constant if and only if either the orbit type of R is constant or varies between* CF2 *and* CF3 $\mathbb{Z}_2$ *on connected* M.

One now can read information on how the SCS and orbit type can vary from Figures 1–3. As an example, we state

Proposition 5.2. *If a smooth curve connects a point with SCS of PO_2 to a point with SCS of PO_1^1, then there exists either a point of constant curvature or a relatively open set with SCS of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ along the curve.*

On the other hand,

Proposition 5.3. *If a smooth curve connects a point with SCS of PO_1^1 to a point with SCS of $\mathbb{Z}_2$ and R of* CF3, *then there need be no points on the curve with any other orbit type of R. However, there will be a last point with SCS of PO_1^1.*

For warped products, we note

Theorem 5.4. *In a connected warped product $\Lambda \times_r \Sigma$, if there are no points of constant curvature, then $\lambda_2 < \lambda_1$ or $\lambda_1 < \lambda_2$ everywhere.*

Corollary 5.5. *In a connected warped product $\Lambda \times_r \Sigma$, the open sets $\{\lambda_2 < \lambda_1\}$ and $\{\lambda_1 < \lambda_2\}$ are separated by a closed set of points of constant curvature.*

References

[1] *L. Auslander*: An exposition of the structure of solvmanifolds. Part II: G-induced flows. Bull. Amer. Math. Soc. *79* (1973), 262–285.

[2] *R. L. Bishop*: Clairaut submersions. Differential Geometry, in Honor of K. Yano (S. Kobayashi et al., eds.). Tokyo, Kinokuniya, 1972, pp. 21–31.

[3] *J. K. Beem and P. E. Parker*: Values of pseudoriemannian sectional curvature. Comment. Math. Helv. *59* (1984), 319–331.

[4] *E. A. Coddington and N. Levinson*: Theory of Ordinary Differential Equations. New York, McGraw-Hill, 1955.

[5] *L. A. Cordero and P. E. Parker*: Symmetries of sectional curvature on 3-manifolds. To appear in Demonstratio Math. *28* (1995).

[6] *L. A. Cordero and P. E. Parker*: Left-invariant Lorentzian metrics on 3-dimensional Lie groups. To appear in Rend. Mat. Appl.

[7] *R. H. Escobales and P. E. Parker*: Geometric consequences of the normal curvature cohomology class in umbilic foliations. Indiana U. Math. J. *37* (1988), 389–408.

[8] *S. W. Hawking and G. F. R. Ellis*: The Large Scale Structure of Space-time. Cambridge, U.P., 1973.

[9] *B. O'Neill*: Semi-riemannian Geometry. New York, Academic Press, 1983.

[10] *J. A. Wolf*: Spaces of Constant Curvature. 5th ed. Boston, Publish or Perish, 1984.

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