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ON THE NEUMANN PROBLEM OF ONE-DIMENSIONAL NONLINEAR THERMOELASTICITY WITH TIME-INDEPENDENT EXTERNAL FORCES

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1. INTRODUCTION

This paper is concerned with one-dimensional nonlinear thermoelastic motion, which is described by the deformation function $X = X(t,x) \in \mathbb{R}$ and the absolute temperature $T = T(t,x) > 0$, where $t \in \mathbb{R}$ denotes a time and $x \in \mathbb{R}$ a point in the unit interval $\Omega = (0,1)$ which is identified with the reference body having the natural temperature $\tau_0 > 0$. The equation of motion and the balance of energy are given by the following formulae:

$$\varrho X_{tt} - \tilde{S}_x = f, \tag{1.1}$$

$$\Big(\tilde{\varepsilon} + \frac{\varrho}{2}X_t^2\Big)_t - (\tilde{S}X_t)_x = q_x + fX_t + g, \tag{1.2}$$

for $x \in \Omega$ and $t > 0$, where subscripts indicate partial differentiations, $\tilde{S}$ denotes the stress, $\tilde{\varepsilon}$ denotes the internal energy, q stands for the heat flux, $\varrho = \varrho(x)$ is a positive smooth function defined on $\bar{\Omega} = [0,1]$ describing the mass density of Ω, f is an external force and g is an external heat supply. One of the essential assumptions of this paper is that $f = f(x)$, that is, f depends only on x. Here and hereafter, all the functions are assumed to be real valued. As a boundary condition, we consider the following Neumann type condition:

$$\tilde{S} = q = 0 \tag{1.3}$$

for $x \in \partial\Omega$ and $t > 0$, which describes the traction free and thermally insulated condition. Note that the boundary $\partial\Omega$ of Ω consists of only two points 0 and 1. And also, the following initial condition is considered:

$$X(0,x) = X_0(x), \ \ X_t(0,x) = X_1(x), \ \ T(0,x) = T_0(x) \tag{1.4}$$

for $x \in \Omega$.

Now, we shall discuss the constitutive relations of $\tilde{S}$ and $\tilde{\varepsilon}$. Let $\tilde{\psi}$ and $\tilde{\eta}$ be the Helmholtz free energy function and the specific entropy function, respectively, and let F be a variable corresponding to X_x. Assume that $\tilde{\psi}$, $\tilde{\eta}$, $\tilde{\varepsilon}$ and $\tilde{S}$ are functions with respect to X_x and T only, that is,

$$\tilde{\psi} = \psi(X_x, T),\ \tilde{\eta} = \eta(X_x, T),\ \tilde{\varepsilon} = \varepsilon(X_x, T),\ \tilde{S} = S(X_x, T), \tag{1.5}$$

and that $\psi(F,T)$, $\eta(F,T)$, $\varepsilon(F,T)$ and $S(F,T)$ are in $C^\infty(G(\delta))$, where

$$G(\delta) = \{(F,T) \in \mathbb{R}^2 \mid |(F,T) - (1, \tau_0)| \leqslant \delta\}$$

and δ is a positive constant. Then, the 2nd Law of Thermodynamics implies that the following two formulae are equivalent:

$$\mathrm{d}\varepsilon = S\,\mathrm{d}F + T\,\mathrm{d}\eta \iff \mathrm{d}\psi = S\,\mathrm{d}F - \eta\,\mathrm{d}T,$$

from which it follows that

$$S = \frac{\partial \psi}{\partial F},\ \ \varepsilon = \psi - T\frac{\partial \psi}{\partial T},\ \ \eta = -\frac{\partial \psi}{\partial T}. \tag{1.6}$$

And then, we have the equation:

$$T\tilde{\eta}_t = q_x + g \tag{1.7}$$

for $x \in \Omega$ and $t > 0$, which is equivalent to (1.2) under (1.1). In fact, using the constitutive relation (1.6) and the assumption (1.5), we have

$$\begin{aligned}\tilde{\varepsilon}_t &= \varepsilon(X_x, T)_t \\ &= \frac{\partial \psi}{\partial T}T_t + \frac{\partial \psi}{\partial F}X_{tx} - T_t\frac{\partial \psi}{\partial T} - T\Big(\frac{\partial \psi}{\partial T}\Big)_t \\ &= T\tilde{\eta}_t + \tilde{S}X_{tx}.\end{aligned}$$

On the other hand, multiplication of (1.1) with X_t implies that

$$\Big(\frac{\varrho}{2}X_t^2\Big)_t = \tilde{S}_x X_t + f X_t.$$

Combining these two formulae, we have (1.7) from (1.2).

For the simplicity, we assume that

$$g = 0, \tag{1.8}$$

$$q = Q(X_x, T)T_x, \tag{1.9}$$

$$Q(F,T) \in C^\infty(G(\delta)),\ \ Q(F,T) > 0 \text{ for all } (F,T) \in G(\delta). \tag{1.10}$$

Moreover, let us assume that

$$\frac{\partial^2\psi}{\partial F^2} > 0, \ \frac{\partial^2\psi}{\partial F \partial T} \neq 0, \ \frac{\partial^2\psi}{\partial T^2} < 0 \quad \text{in } G(\delta), \tag{1.11}$$

$$S(1, \tau_0) = 0. \tag{1.12}$$

The assumption (1.12) means that $(1, \tau_0)$ is an equilibrium state with $f = g = 0$.

The purpose of this paper is to show a unique existence theorem globally in time of smooth solutions X and T to the problem (1.1)–(1.4). Moreover, we investigate that (X_t, X_x, T) converges to $(0, X'_\infty, T_\infty)$ exponentially as $t \longrightarrow \infty$, where (X'_∞, T_∞) is another equilibrium state which is different from $(1, \tau_0)$ in general. When f depends on t essentially, it seems to be difficult to prove a global in time existence of smooth solutions even if f is bounded in t. This is different from the case of the constant temperature boundary condition: $T = \tau_0$ (cf. [14], [12]). Such difference comes about in applying Poincaré's inequality.

In concluding this section, let us state recent results concerning global in time existence theorems of smooth solutions to one-dimensional nonlinear thermoelasticity for small and smooth initial data. From now to the end of this section, we consider the case of Ω being unbounded (a half line) as well as the case of Ω being bounded (a unit interval). And, as boundary conditions, one of the following is also considered:

(D.D)	$X = x$	and	$T = \tau_0$	for $x \in \partial\Omega$ and $t > 0$,
(D.N)	$X = x$	and	$q = 0$	for $x \in \partial\Omega$ and $t > 0$,
(N.D)	$\tilde{S} = 0$	and	$T = \tau_0$	for $x \in \partial\Omega$ and $t > 0$.

Here, (D.D), (D.N) and (N.D) mean the rigidly clamped and constant temperature condition, the rigidly clamped and thermally insulated condition and the traction free and constant temperature condition, respectively.

M.Slemrod [14] solved the problem (1.1), (1.2) and (1.4) in cases of (N.D) and (D.N), where Ω was assumed to be bounded. When Ω is unbounded, the same problem as in [14] was solved by Jiang Song [5]. These authors used the usual L^2-energy method and thanks to the special form of the boundary condition, the essential difficulty was not created by the boundary term. The Cauchy problem to (1.1) and (1.2) was solved by Kawashima [8], Kawashima and Okada [9], Zheng and Shen [15], and Hrusa and Tarabek [4], using also the L^2-energy method. In cases of (D.D) and (1.3), in using the L^2-energy method, the essential difficulty arose from the boundary. Racke and Shibata [11] overcame this difficulty by showing the polynomial decay rate of solutions to the corresponding linear problem, which was obtained by use of a spectral analysis, where the boundary condition was (D.D) and Ω was bounded. Subsequently, Shibata [13] also solved the problem in the case that the boundary condition was (1.3) and Ω was bounded, by reducing the problem to the (D.D) case and by modifying the method developed in [11]. Afterwards, Muñoz Rivera [10] obtained an exponential decay result for one-dimensional linear thermoelasticity with (D.D) boundary condition, where Ω was bounded, by using the L^2-energy method and by choosing some multipliers wisely to control the boundary terms. Extending Rivera's method to the nonlinear case, Jiang Song [6] solved the problem in case of (D.D), where he treated the case of Ω being unbounded as well as the case of Ω being bounded. And also, Racke, Shibata and Zheng [12] proved the exponential stability and the existence of periodic solutions in case of (D.D), where Ω was bounded. Being inspired by Rivera's work [10], Jiang Song [7] solved the problem (1.1)–(1.4) in the case that $f = g = 0$, where he treated the case of Ω being unbounded as well as the case of Ω being bounded. But, the asymptotic behaviour of solutions was obtained in the linear case only, so that one knows the asymptotic behaviour of solutions to the problem (1.1)–(1.4) from the present paper. Anyhow, our proof will proceed in the spirit of the Jiang Song and Muñoz Rivera method.

Finally, we note that a globally in time defined smooth solutions should not be expected for large data in general. Indeed, Dafermos and Hsiao [1] and Hrusa and Messaoudi [3] showed that for specialized constitutive equations, the smooth solutions to the Cauchy problem blow up in finite time provided that the initial data are large. To the authors, it seems that one-dimensional nonlinear thermoelasticity was almost settled, except for the existence of periodic solutions to the problem (1.1)–(1.3) and the global in time existence of smooth solutions to the problem (1.1)–(1.4) with external force f depending on t essentially. These problems seem to be open.

2. Statement of Main Results

Throughout the paper, we use the following notation. For differentiation, we put

$$v_s = \partial_s v = \frac{\partial v}{\partial s}, \ \partial_s^k v = \frac{\partial^k v}{\partial s^k} \qquad (s = t \text{ and } x,\ v = v(t,x)),$$
$$w^{(k)} = \frac{\mathrm{d}^k w}{\mathrm{d}x^k}, \ w' = w^{(1)}, \ w'' = w^{(2)}, \ w''' = w^{(3)} \qquad (w = w(x)),$$
$$\bar{\partial}_s^k v = (v, \partial_s v, \dots, \partial_s^k v), \ D^k v = \{\partial_t^i \partial_x^j v \mid i + j = k\}, \ \bar{D}^k v = \{\partial_t^i \partial_x^j v \mid i + j \leqslant k\}.$$

We denote the usual L^2 space on Ω, its norm and its innerproduct by L^2, $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. Put

$$H^j = \left\{ w(x) \in L^2 \ \middle| \ \|w\|_j = \left(\sum_{k=0}^{j} \|w^{(k)}\|^2 \right)^{\frac{1}{2}} < \infty \right\},$$

$$H^0 = L^2, \ \|\cdot\|_0 = \|\cdot\|, \ \|w\|_\infty = \sup_{x\in(0,1)} |w(x)|,$$

$$\langle u, v\rangle = u(1)v(1) - u(0)v(0), \ \langle u\rangle = \left\{u(1)^2 + u(0)^2\right\}^{\frac{1}{2}}.$$

For a Banach space X and an interval $I \subset \mathbb{R}$, $C^j(I,X)$ denotes the set of all X-valued continuous functions which are j-times continuously differentiable on I and $L^2(I,X)$ denotes the set of all X-valued strongly measurable functions on I which are square integrable on I. As a class of solutions to the problem (1.1)–(1.4), let us introduce the following functional space:

$$Z^N(t_0) = \Big\{(X(t,x), T(t,x)) \mid$$

$$X(t,x) \in \bigcap_{j=0}^{N} C^j([0,t_0], H^{N-j}), \tag{2.1}$$

$$T(t,x) \in C^{N-1}([0,t_0], L^2) \cap \bigcap_{j=0}^{N-2} C^j([0,t_0], H^{N-j}), \tag{2.2}$$

$$\partial_t^{N-1} T(t,x) \in L^2((0,t_0), H^1), \tag{2.3}$$

$$(X_x(t,x), T(t,x)) \in G(\delta) \text{ and } T(t,x) > 0 \quad \text{for } (t,x) \in [0,t_0] \times \bar{\Omega}\Big\}. \tag{2.4}$$

Let us begin with stating a local in time unique existence theorem, which was obtained by W. Dan [2]. The problem treated in [2] is more general than the problem (1.1)–(1.4) of the present paper. Before stating the theorem, let us discuss the conditions on the initial data and the right members. Of course, it is not necessary to assume that $f = f(x)$ to obtain a local in time existence theorem, so that for a moment we shall consider the case where $f = f(t,x)$ and $g = g(t,x)$. Let (X,T) be a solution in $Z^N(t_0)$ to the problem (1.1)–(1.4). Put

$$X_j(x) = \partial_t^j X(0,x), \ T_j(x) = \partial_t^j T(0,x), \tag{2.5}$$

and then $X_j(x)$ and $T_j(x)$ are successively determined through the equations (1.1) and (1.7). For example, for $f = f(x)$ and $g = 0$, we get

$$X_2(x) = S(X_0'(x), T_0(x))' + f(x),$$
$$T_1(x) = \Big(T_0(x)\frac{\partial\eta}{\partial T}(X_0'(x), T_0(x))\Big)^{-1}$$
$$\times \Big\{(Q(X_0'(x), T_0(x))T_0'(x))' - T_0(x)\frac{\partial\eta}{\partial F}(X_0'(x), T_0(x))X_1'(x)\Big\},$$

and so on. The conditions (2.1) and (2.2) imply that

$$X_j(x) \in H^{N-j} \ (0 \leqslant j \leqslant N), \tag{2.6}$$

$$T_j(x) \in H^{N-j} \ (0 \leqslant j \leqslant N-2), \quad T_{N-1}(x) \in L^2. \tag{2.7}$$

Assuming that $N \geqslant 3$, we see that

$$T_x(t,x), S(X_x(t,x), T(t,x)) \in \bigcap_{j=0}^{N-2} C^j\left([0,t_0], H^{N-1-j}\right),$$

for $(X,T) \in Z^N(t_0)$. Since (1.3) is satisfied for all $t \in [0,t_0]$, we meet the following requirement from (1.3):

$$\partial_t^j S(X_x(t,x), T(t,x))|_{t=0} = 0, \tag{2.8}$$

$$T_j'(x) = 0 \tag{2.9}$$

for $x \in \partial\Omega$ and for $0 \leqslant j \leqslant N-2$. Moreover, the condition (2.8) is written in terms of X_j and T_j and their derivatives $(0 \leqslant j \leqslant N-2)$. Indeed, for $N = 3$, we have

$$S(X_0'(x), T_0(x)) = 0, \tag{2.10}$$

$$\frac{\partial S}{\partial F}(X_0'(x), T_0(x))X_1'(x) + \frac{\partial S}{\partial T}(X_0'(x), T_0(x))T_1(x) = 0, \tag{2.11}$$

$$T_0'(x) = T_1'(x) = 0 \tag{2.12}$$

for $x \in \partial\Omega$. And then, we know the following local in time existence theorem (cf. W. Dan [2]).

Theorem 2.1. *Suppose that* (1.5), (1.6), (1.9), (1.10) *and* (1.11) *hold and that* N *is an integer* $\geqslant 3$. *Suppose that* (2.6), (2.7), (2.8) *and* (2.9) *hold and that*

$$f, g \in \bigcap_{j=0}^{N-2} C^j([0,t_0], H^{N-2-j}) \text{ and } \partial_t^{N-1} f, \partial_t^{N-1} g \in L^2((0,t_0), L^2). \tag{2.13}$$

Let $B > 0$ *be a number such that*

$$\sum_{j=0}^{3} \|X_j\|_{3-j} + \sum_{j=0}^{1} \|T_j\|_{3-j} + \|T_2\| + \sum_{j=0}^{1} \sup_{0 \leqslant s \leqslant t_0} \|(\partial_t^j f, \partial_t^j g)(s,\cdot)\|_{1-j} + \left\{ \int_0^{t_0} \|(\partial_t^2 f, \partial_t^2 g)(s,\cdot)\|^2 \, \mathrm{d}s \right\}^{\frac{1}{2}} \leqslant B. \tag{2.14}$$

Suppose that

$$(X_0'(x), T_0(x)) \in G(\delta/2) \text{ and } T_0(x) > 0 \text{ for } x \in \bar{\Omega}. \tag{2.15}$$

Then, there exists a time $t_1 \in (0, t_0)$ depending on B and δ essentially such that the problem (1.1)–(1.4) admits a unique solution $(X, T) \in Z^N(t_1)$ satisfying the condition:

$$(2.16) \qquad (X_x(t,x), T(t,x)) \in G(2\delta/3) \quad \text{and} \quad T(t,x) > 0$$

for $(t, x) \in [0, t_1] \times \bar{\Omega}$.

Remark 2.2. The essential point in Theorem 2.1 is that the existence time t_1 depends only on B, so that it is enough to get an *a priori* bound for $\|\bar{D}^3 X(t,\cdot)\|$, $\|\bar{D}^1 T(t,\cdot)\|_2$ and $\|\partial_t^2 T(t,\cdot)\|$ to prove the global in time existence theorem, in view of (2.14).

Now, we are going to state our global in time existence theorem and the estimation of solutions. From now on, we assume that

$$(2.17) \qquad f = f(x) \in H^{N-2} \ (N \geqslant 3) \quad \text{and} \quad g = 0.$$

Without loss of generality, we may assume that

$$(2.18) \qquad \int_0^1 f(x)\,\mathrm{d}x = \int_0^1 \varrho(x) X_0(x)\,\mathrm{d}x = \int_0^1 \varrho(x) X_1(x)\,\mathrm{d}x = 0.$$

In fact, let us define a compensating function $r(t)$ by the formula:

$$r(t) = \left\{ \int_0^1 \varrho(x) X_0(x)\,\mathrm{d}x + t \int_0^1 \varrho(x) X_1(x)\,\mathrm{d}x + \frac{t^2}{2} \int_0^1 f(x)\,\mathrm{d}x \right\} \left(\int_0^1 \varrho(x)\,\mathrm{d}x \right)^{-1}.$$

Put $\tilde{X}(t,x) = X(t,x) - r(t)$ and put $\tilde{X}_j(x) = \partial_t^j \tilde{X}(0,x)$. Then, from the definition of $r(t)$ it follows immediately that

$$\int_0^1 \varrho(x) \tilde{X}_0(x)\,\mathrm{d}x = \int_0^1 \varrho(x) \tilde{X}_1(x)\,\mathrm{d}x = 0.$$

Moreover, what $X_k(x) \in H^{N-k}$, $0 \leqslant k \leqslant N$, is equivalent to what $\tilde{X}_k(x) \in H^{N-k}$, $0 \leqslant k \leqslant N$. Since $\tilde{X}_k'(x) = X_k'(x)$ for all $k \geqslant 0$, the following two conditions for our local in time existence theorem also hold:

$$\partial_t^j S(\tilde{X}_x(t,x), T(t,x))|_{t=0} = 0 \quad \text{for } x \in \partial\Omega \text{ and } 0 \leqslant j \leqslant N-2,$$

$$(\tilde{X}_0'(x), T_0(x)) \in G(\delta/2) \quad \text{for } x \in \bar{\Omega}.$$

And also, we see that

$$\begin{aligned}
&\varrho(x)\tilde{X}_{tt} - S(\tilde{X}_x, T)_x \\
&\qquad = \varrho(x)X_{tt} - S(X_x, T)_x - \varrho(x)r''(t) \\
&\qquad = f(x) - \varrho(x)\int_0^1 f(x)\,\mathrm{d}x \left(\int_0^1 \varrho(x)\,\mathrm{d}x\right)^{-1},
\end{aligned}$$

and that

$$f(x) - \varrho(x)\int_0^1 f(x)\,\mathrm{d}x \left(\int_0^1 \varrho(x)\,\mathrm{d}x\right)^{-1} \in H^{N-2},$$

$$\int_0^1 \left[f(x) - \varrho(x)\int_0^1 f(x)\,\mathrm{d}x \left(\int_0^1 \varrho(x)\,\mathrm{d}x\right)^{-1}\right] \mathrm{d}x = 0.$$

From this observation, we see that the assumption (2.18) gives us no restriction.

Now, let us define an equilibrium state $(X'_\infty(x), T_\infty) \in H^{N-1} \times \mathbb{R}$ which is different from $(1, \tau_0)$ in general and to which the solution converges exponentially as $t \to \infty$. Put

$$F(x) = \int_0^x f(y)\,\mathrm{d}y, \tag{2.19}$$

and then (2.18) implies that

$$F(0) = F(1) = 0. \tag{2.20}$$

Integrating (1.2) over Ω and using (1.3), we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\tilde{\varepsilon} + \frac{\varrho}{2}X_t^2\right) \mathrm{d}x = \int_0^1 X_t f\,\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 X_x F\,\mathrm{d}x, \tag{2.21}$$

where we have used (2.19) and (2.20) in the second equality. Integrating (2.21) over $(0,t)$ and using (1.5), we have the following conservative quantity:

$$\int_0^1 \left\{\varepsilon(X_x(t,x), T(t,x)) + \frac{\varrho(x)}{2}X_t(t,x)^2 + X_x(t,x)F(x)\right\} \mathrm{d}x = e_0, \tag{2.22}$$

where

$$e_0 = \int_0^1 \left\{\varepsilon(X'_0(x), T_0(x)) + \frac{\varrho(x)}{2}X_1(x)^2 + X'_0(x)F(x)\right\} \mathrm{d}x. \tag{2.23}$$

Another equilibrium state $(X_\infty(x), T_\infty)$ is given in the following lemma which will be proved in the Appendix below.

Lemma 2.3. *Suppose that (1.6), (1.11) and (1.12) hold and that* $X_0(x) \in H^2$, $T_0(x) \in H^1$, $X_1(x) \in L^2$ *and* $f(x) \in H^{N-2}$, *where* N *is an integer* $\geqslant 3$. *Then, for any* $\sigma > 0$ *there exists a* $\kappa > 0$ *such that if*

$$\|(X_0', T_0) - (1, \tau_0)\|_\infty + \|X_1\| + \|f\|_1 < \kappa \tag{2.24}$$

then there exist a $X_\infty(x) \in H^N$ *and a constant* $T_\infty > 0$ *such that*

$$(X_\infty'(x), T_\infty) \in G(\delta/2) \text{ for all } x \in \bar{\Omega}, \quad \|X_\infty' - 1\|_2 + |T_\infty - \tau_0| < \sigma \tag{2.25}$$

and the following two equalities hold:

$$S(X_\infty'(x), T_\infty) = -F(x) \quad \text{for } x \in \Omega, \tag{2.26}$$

$$\int_0^1 \{\varepsilon(X_\infty'(x), T_\infty) + X_\infty'(x)F(x)\} \, \mathrm{d}x = e_0. \tag{2.27}$$

Remark 2.4. By (2.26) and (2.20), we see that

$$S(X_\infty'(x), T_\infty) = 0 \quad \text{for } x \in \partial\Omega. \tag{2.28}$$

To state our main result exactly, we introduce an additional notation. Put

$$u(t,x) = X(t,x) - X_\infty(x), \quad \theta(t,x) = T(t,x) - T_\infty, \tag{2.29}$$

$$N(t) = \sup_{0<s<t} \|\bar{D}^2(u_x, u_t, \theta)(s, \cdot)\|, \tag{2.30}$$

$$E_0 = \|X_0' - X_\infty'\|_2 + \|T_0 - T_\infty\|_3 + \sum_{j=1}^{3} \|X_j\|_{3-j} + \|T_1\|_2 + \|T_2\|. \tag{2.31}$$

Moreover, for $\alpha > 0$ let us put

$$N_\alpha(t) = \sup_{0<s<t} \mathrm{e}^{\alpha s} \left\{ \|\bar{D}^2(u_x, u_t, \theta)(s, \cdot)\| + \|(\theta_{xxt}, \theta_{xxx})(s, \cdot)\| \right\}, \tag{2.32}$$

$$M_\alpha(t) = \left\{ \int_0^t \mathrm{e}^{2\alpha s} \|(D^2 u, D^3 u, D^1\theta, D^2\theta, \theta_{xtt}, \theta_{xxt}, \theta_{xxx})(s, \cdot)\|^2 \, \mathrm{d}s \right\}^{\frac{1}{2}}. \tag{2.33}$$

Note that

$$E_0 = \|u_x(0, \cdot)\|_2 + \sum_{j=1}^{3} \|\partial_t^j u(0, \cdot)\|_{3-j} + \|\bar{D}^1\theta(0, \cdot)\|_2 + \|\partial_t^2\theta(0, \cdot)\|. \tag{2.34}$$

Under these preparations, we can state our main result in the following way.

Theorem 2.5. *Suppose that* (1.5), (1.6), (1.8), (1.9), (1.10), (1.11) *and* (1.12) *hold, that N is an integer* $\geqslant 3$, *that* (2.6), (2.7), (2.8), (2.9) *and* (2.15) *hold and that* $f = f(x) \in H^{N-2}$. *In addition, suppose that* (2.18) *holds. Then, there exists a* $\sigma > 0$ *such that if*

$$\|(X_0', T_0) - (1, \tau_0)\|_\infty + E_0 + \|f\|_1 < \sigma \tag{2.35}$$

then the problem (1.1)–(1.4) *admits a unique solution* $(X(t,x), T(t,x)) \in Z^N(\infty)$ *which satisfy the following estimate:*

$$N_\alpha(t)^2 + M_\alpha(t)^2 \leqslant CE_0^2 \tag{2.36}$$

for suitable positive constants α and C.

Remark 2.6. (1) In view of Lemma 2.3, (2.35) guarantees the existence of X_∞ and T_∞ as well as the existence of global in time solutions. Moreover, $\|(X_\infty', T_\infty) - (1, \tau_0)\|_\infty$, $\|X_\infty''\|_\infty$ and $\|X_\infty''\|_1$ become smaller according to the choice of σ. Put

$$E_1 = \|X_0' - 1\|_2 + \|T_0 - \tau_0\|_3 + \sum_{j=1}^{3} \|X_j\|_{3-j} + \|T_1\|_2 + \|T_2\| + \|f\|_1.$$

Then, by Lemma 2.3 we see that (2.35) can be replaced by the condition: $E_1 < \sigma$.

(2) In particular, (2.36) implies the following asymptotic behaviour:

$$\|\bar{D}^2(X_t, X_x - X_\infty', T - T_\infty)(t, \cdot)\| \leqslant Ce^{-\alpha t} E_0^2, \tag{2.37}$$

that is, (X_t, X_x, T) converges to $(0, X_\infty', T_\infty)$ exponentially as $t \to \infty$. In general, (X_∞', T_∞) may be different from $(1, \tau_0)$.

3. A Proof of Theorem 2.5

Let $(X, T) \in Z^N(t_0)$, $t_0 > 0$, be a solution to the problem (1.1)–(1.4) and we shall use the notation defined by the formulae (2.29)–(2.33), below. In a manner like that of §5 of the reference [11], we can establish Theorem 2.5 if we show the following assertion: There exist positive constants C, α and σ such that the following estimate holds:

$$N_\alpha(t)^2 + M_\alpha(t)^2 \leqslant CE_0^2 \tag{3.1}$$

provided that

$$N(t) \leqslant \sigma \qquad \text{for } 0 \leqslant t \leqslant t_0, \tag{3.2}$$

$$\|(X'_\infty, T_\infty) - (1, \tau_0)\|_\infty + \|X''_\infty\|_\infty + \|X''_\infty\|_1 + \|f\| \leqslant \sigma. \tag{3.3}$$

Therefore, we shall derive (3.1) under the assumptions (3.2) and (3.3). Below, we assume that $0 < \sigma, \alpha < 1$ and the derivation will be divided into ten steps. In view of (1.10), (1.11) and the fact that $\varrho(x) > 0$ for $x \in \bar{\Omega}$, we can choose positive constants β_0 and β_1 in such a way that

$$\beta_0 \leqslant \varrho(x),\ \frac{\partial^2\psi}{\partial F^2}(F,T),\ \left|\frac{\partial^2\psi}{\partial F \partial T}(F,T)\right|,\ -\frac{\partial^2\psi}{\partial T^2}(F,T),\ Q(F,T) \leqslant \beta_1 \tag{3.4}$$

for all $x \in \bar{\Omega}$ and $(F,T) \in G(\delta)$. Put

$$M_g(F,T) = \frac{\partial S}{\partial F}(F,T)\frac{\partial g}{\partial T}(F,T) - \frac{\partial S}{\partial T}(F,T)\frac{\partial g}{\partial F}(F,T) \tag{3.5}$$

for $g = \varepsilon$ and η. Choosing β_0 and δ small enough if necessary, we may also assume that

$$T \geqslant \beta_0, \qquad M_g(F,T) \geqslant \beta_0 \ (g = \varepsilon \text{ and } \eta) \tag{3.6}$$

for all $(F,T) \in G(\delta)$, because

$$\begin{aligned}
&\frac{\partial \varepsilon}{\partial T}(1,\tau_0) = \frac{\partial \eta}{\partial T}(1.\tau_0) = -\tau_0\frac{\partial^2\psi}{\partial T^2}(1,\tau_0), \qquad \frac{\partial S}{\partial F}(1,\tau_0) = \frac{\partial^2\psi}{\partial F^2}(1,\tau_0),\\
&\frac{\partial \varepsilon}{\partial F}(1,\tau_0) = \frac{\partial \eta}{\partial F}(1,\tau_0) = -\tau_0\frac{\partial^2\psi}{\partial F \partial T}(1,\tau_0), \quad \frac{\partial S}{\partial T}(1,\tau_0) = \frac{\partial^2\psi}{\partial F \partial T}(1,\tau_0)
\end{aligned}$$

where we have used (1.6) and (1.12). Thus, (1.11) implies that

$$M_g(1,\tau_0) = \tau_0\left\{\frac{\partial^2\psi}{\partial F^2}(1,\tau_0)\left(-\frac{\partial^2\psi}{\partial T^2}(1,\tau_0)\right) + \frac{\partial^2\psi}{\partial F \partial T}(1,\tau_0)^2\right\} > 0 \tag{3.7}$$

for $g = \varepsilon$ and η. Since σ will be chosen very small later on, we may also assume that

$$(X'_\infty(x), T_\infty) + \ell(u_x(t,x), \theta(t,x)) \in G(\delta) \quad \text{for all } \ell \in [0,1] \text{ and } (t,x) \in [0,t_0] \times \bar{\Omega}.$$

For $K = S$, ε, η and Q and for $L = F$ and T, we put

$$\begin{aligned}
K_L &= K_L(t,x) = \frac{\partial K}{\partial L}(X'_\infty(x) + u_x(t,x), T_\infty + \theta(t,x)),\\
K_L^0 &= K_L^0(t,x) = \int_0^1 \frac{\partial K}{\partial L}((X'_\infty(x), T_\infty) + \ell(u_x(t,x), \theta(t,x)))\, d\ell.
\end{aligned}$$

Note that

$$S_T = -\eta_F \quad \text{and} \quad S_T^0 = -\eta_F^0, \tag{3.8}$$

which follows from (1.6). Below, to denote various constants independent of α and σ, we shall use the same letter C. In each step of our derivation of (3.1), we shall frequently use the following relations:

$$\|\bar{D}^1(u_t, u_x, \theta)(t,\cdot)\|_\infty \leqslant CN(t) \leqslant C\sigma \leqslant C, \tag{3.9}$$

$$\|D^1(Q, K_L, K_L^0)(t,x)\|_\infty \leqslant C\,(N(t) + \|X''_\infty\|_\infty) \leqslant C\sigma \leqslant C, \tag{3.10}$$

$$\|\partial_t^2 P(X'_\infty + \ell u_x, T_\infty + \ell\theta)(t,\cdot)\| \leqslant C\|(u_{xtt}, \theta_{tt}, u_{xt}, \theta_t)(t,\cdot)\| \leqslant C\sigma, \tag{3.11}$$

$$\|\partial_t\partial_x P(X'_\infty + \ell u_x, T_\infty + \ell\theta)(t,\cdot)\| \leqslant C\|(u_{xxt}, \theta_{xt}, u_{xt}, \theta_t)(t,\cdot)\| \leqslant C\sigma, \tag{3.12}$$

(3.13)

$$\|\partial_x^2 P(X'_\infty + \ell u_x, T_\infty + \ell\theta)(t,\cdot)\| \leqslant C\,\{\|(u_{xxx}, \theta_{xx}, u_{xx}, \theta_x)(t,\cdot)\| + \|X''_\infty\|_1\} \leqslant C\sigma$$

where $0 \leqslant \ell \leqslant 1$, $Q = Q(X'_\infty + u_x, T_\infty + \theta)$, $K = S$, ε, η and Q, $L = F$ and T and $P(F,T) \in C^\infty(G(\delta))$. In fact, relations (3.9) and (3.10) follows from Sobolev's inequality and the fact that $\sigma < 1$. Relations (3.11)–(3.13) can be obtained easily by direct calculation and by use of (3.2), (3.3) and (3.9).

Step 1. *We verify the relations*

$$\|u_t(t,\cdot)\| \leqslant C\|u_{tx}(t,\cdot)\|, \tag{3.14}$$

$$\|(u_x, \theta)(t,\cdot)\| \leqslant C\|(u_t, u_x, \theta)_x(t,\cdot)\| \tag{3.15}$$

provided that σ is small enough.

Integrating (1.1) over $\Omega \times (0,t)$ and using (1.3) and (2.18), we have

$$\int_0^1 \varrho(x) u_t(t,x)\,\mathrm{d}x = 0,$$

which combined with Poincaré's inequality:

$$\|v\| \leqslant C\left\{\left|\int_0^1 p(x)v(x)\,\mathrm{d}x\right| + \|v'\|\right\} \tag{3.16}$$

for $v \in H^1$, where $p(x) \in L^2$ such that $\int_0^1 p(x)\,\mathrm{d}x \neq 0$, implies (3.14). Put $S_\infty = S(X'_\infty(x), T_\infty)$. Since $S - S_\infty = 0$ on $\partial\Omega$ (cf. (2.28),) by another Poincaré's inequality:

$$\|v\| \leqslant C\,\{\langle v\rangle + \|v'\|\} \tag{3.17}$$

for $v \in H^1$, we have

$$\|S - S_\infty\| \leqslant C\|(S - S_\infty)_x\|. \tag{3.18}$$

Since $S - S_\infty = S_F^0 u_x + S_T^0 \theta$ as follows from the Taylor expansion, it follows from (3.18) and (3.10) that

$$\|(S_F^0 u_x + S_T^0 \theta)(t,\cdot)\| \leqslant C\left\{\|(u_x,\theta)_x(t,\cdot)\| + \sigma\|(u_x,\theta)(t,\cdot)\|\right\}. \tag{3.19}$$

Since $\varepsilon(X_x, T) = \varepsilon(X_\infty'(x), T_\infty) + \varepsilon_F^0 u_x + \varepsilon_T^0 \theta$ as follows from the Taylor expansion, combining (2.22) and (2.27) implies that

$$\int_0^1 \left\{\varepsilon_F^0 u_x + \varepsilon_T^0 \theta + \frac{\varrho}{2} u_t^2 + u_x F\right\} \mathrm{d}x = 0, \tag{3.20}$$

where we have used the fact that $X_t = u_t$ and $X_x F = X_\infty' F + u_x F$. On the other hand, by Poincaré's inequality (3.16) with $p(x) = 1$, we know that

$$\|\varepsilon_F^0 u_x + \varepsilon_T^0 \theta\| \leqslant C\left\{\left|\int_0^1 (\varepsilon_F^0 u_x + \varepsilon_T^0 \theta)\,\mathrm{d}x\right| + \|(\varepsilon_F^0 u_x + \varepsilon_T^0 \theta)_x\|\right\},$$

which combined with (3.20) implies that

$$\begin{aligned} &\|(\varepsilon_F^0 u_x + \varepsilon_T^0 \theta)(t,\cdot)\| \\ &\quad\leqslant C\Big\{\|u_t(t,\cdot)\|_\infty \|u_t(t,\cdot)\| + \|F\|\|u_x(t,\cdot)\| + \|(u_x,\theta)_x(t,\cdot)\| \\ &\qquad + \|(\varepsilon_F^0, \varepsilon_T^0)_x(t,\cdot)\|_\infty \|(u_x,\theta)(t,\cdot)\|\Big\} \\ &\quad\leqslant C\left\{\sigma\|(u_x,\theta)(t,\cdot)\| + \|(u_x,u_t,\theta)_x(t,\cdot)\|\right\}, \end{aligned} \tag{3.21}$$

where we have used (3.9), (3.10) with $K = \varepsilon$, (3.14) and the fact that $\|F\| \leqslant \|f\| \leqslant \sigma$ (cf. (3.3)). Let us define the matrix U by

$$U = \begin{pmatrix} S_F^0 & \varepsilon_F^0 \\ S_T^0 & \varepsilon_T^0 \end{pmatrix},$$

and then $(S_F^0 u_x + S_T^0 \theta, \varepsilon_F^0 u_x + \varepsilon_T^0 \theta) = (u_x, \theta)U$. Since

$$\left|K_L^0 - \frac{\partial K}{\partial L}(X_\infty'(x), T_\infty)\right| \leqslant C\sigma$$

for $K = S$ and ε and $L = F$ and T as follows from Taylor expansion and (3.9), by (3.6) we have

$$\det\ U = M_\varepsilon(X_\infty'(x), T_\infty) - C\sigma \geqslant \beta_0 - C\sigma$$

because $(X'_\infty(x), T_\infty) \in G(\delta)$ for all $x \in \bar{\Omega}$ (cf. Lemma 2.3). Choose σ so small that $\beta_0 - C\sigma \geqslant \beta_0/2 > 0$ implies that $(u_x, \theta) = (S_F^0 u_x + S_T^0 \theta, \varepsilon_F^0 u_x + \varepsilon_T^0 \theta)U^{-1}$, and then combining (3.19) and (3.21) implies (3.15).

Step 2. *We verify the relation*

$$\text{(3.22)}\quad e^{2\alpha t}\|(u_x, u_t, \theta)(t,\cdot)\|^2 + c_1 \int_0^t e^{2\alpha s}\|\theta_x(s,\cdot)\|^2\,ds \leqslant C\left\{E_0^2 + (\sigma+\alpha)M_\alpha(t)^2\right\}$$

for suitable $c_1 > 0$.

Multiplying (1.1) by u_t and integrating the resulting formula over Ω, we have

$$(f, u_t) = \frac{1}{2}\frac{d}{dt}(\varrho u_t, u_t) + (S, u_{tx}).$$

Since

$$\begin{aligned}(S, u_{tx}) &= (-F, u_{tx}) + (S_F^0 u_x + S_T^0\theta, u_{tx}) \\ &= (f, u_t) + \frac{1}{2}\frac{d}{dt}(S_F^0 u_x, u_x) - \frac{1}{2}((S_F^0)_t u_x, u_x) + (S_T^0\theta, u_{tx})\end{aligned}$$

where we have used (2.26), (2.20) and (2.19), finally we have

$$\text{(3.23)}\qquad \frac{1}{2}\frac{d}{dt}\left\{(\varrho u_t, u_t) + (S_F^0 u_x, u_x)\right\} + (S_T^0\theta, u_{tx}) - \frac{1}{2}((S_F^0)_t u_x, u_x) = 0.$$

By (1.5) and Taylor expansion, we can write the equivalent equation (1.7) to (1.2) in the following way:

$$\text{(3.24)}\qquad (T_\infty + \theta)(\eta_T^0\theta + \eta_F^0 u_x)_t = (Q\theta_x)_x.$$

Since $\|T_\infty + \theta(t,\cdot) - \tau_0\|_\infty \leqslant \delta$ for $t \in [0, t_0]$ and $T_\infty + \theta(t,x) \geqslant \beta_0$ as follows from (2.4), (2.29) and (3.6), we have

$$\text{(3.25)}\qquad (\delta + \tau_0)^{-1} \leqslant (T_\infty + \theta(t,x))^{-1} \leqslant \beta_0^{-1} \qquad \text{for all } (t,x) \in [0,t_0]\times\bar{\Omega}.$$

Since $T_x = \theta_x = 0$ on $\partial\Omega$, multiplying (3.24) by $(T_\infty + \theta)^{-1}\theta$ implies that

$$\text{(3.26)}\qquad \begin{aligned}&\frac{1}{2}\frac{d}{dt}(\eta_T^0\theta, \theta) + (\eta_F^0 u_{tx}, \theta) + \frac{1}{2}((\eta_T^0)_t\theta, \theta) \\ &\qquad + \frac{1}{2}((\eta_F^0)_t u_x, \theta) + ((T_\infty + \theta)^{-2} T_\infty Q\theta_x, \theta_x) = 0.\end{aligned}$$

Combining (3.23) and (3.26) and using (3.10), (3.8) and (3.25), we have

$$(3.27)\quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{(\varrho u_t,u_t)+(S_F^0u_x,u_x)+(\eta_T^0\theta,\theta)\}\leqslant -c_0\|\theta_x(t,\cdot)\|^2+C\sigma\|(u_x,\theta)(t,\cdot)\|^2$$

for a suitable positive constant c_0. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{2\alpha t}f(t)\right)=\mathrm{e}^{2\alpha t}\frac{\mathrm{d}}{\mathrm{d}t}f(t)+2\alpha\mathrm{e}^{2\alpha t}f(t),$$

multiplying (3.27) by $\mathrm{e}^{2\alpha t}$ and using the relation:

$$(3.28)\qquad \beta_0\|(p,q,r)\|^2\leqslant(\varrho p,p)+(S_F^0q,q)+(\eta_T^0r,r)\leqslant\beta_1\|(p,q,r)\|^2$$

which follows from (3.4) and (1.6), we have

$$(3.29)\qquad \begin{aligned}&\beta_0\mathrm{e}^{2\alpha t}\|(u_t,u_x,\theta)(t,\cdot)\|^2+c_0\int_0^t\mathrm{e}^{2\alpha s}\|\theta_x(s,\cdot)\|^2\,\mathrm{d}s\\&\quad\leqslant\beta_1\|(u_t,u_x,\theta)(0,\cdot)\|^2+C\sigma\int_0^t\mathrm{e}^{2\alpha s}\|(u_x,\theta)(s,\cdot)\|^2\,\mathrm{d}s\\&\quad+2\beta_1\alpha\int_0^t\mathrm{e}^{2\alpha s}\|(u_t,u_x,\theta)(s,\cdot)\|^2\,\mathrm{d}s.\end{aligned}$$

Inserting (3.14) and (3.15) of Step 1 into the right-hand side of (3.29) and using the definitions of $M_\alpha(t)$ and E_0 (cf. (2.33) and (2.34)), we have (3.22).

Step 3. *We verify the relation*

$$(3.30)\quad \mathrm{e}^{2\alpha t}\|(u_x,u_t,\theta)_t(t,\cdot)\|^2+c_1\int_0^t\mathrm{e}^{2\alpha s}\|\theta_{xt}(s,\cdot)\|^2\,\mathrm{d}s\leqslant C\left\{E_0^2+(\alpha+\sigma)M_\alpha(t)^2\right\}$$

for suitable $c_1>0$.

Differentiating (1.1), (1.2) and (1.3) once in t and using the relation (1.5) imply that

$$(3.31)\qquad \varrho(x)u_{ttt}-(S_Fu_{xt}+S_T\theta_t)_x=0\qquad\text{for } x\in\Omega,$$

$$(3.32)\qquad S_Fu_{xt}+S_T\theta_t=0\qquad\text{for } x\in\partial\Omega,$$

$$(3.33)\qquad (T_\infty+\theta)(\eta_T\theta_{tt}+\eta_Fu_{xtt})=(Q\theta_{xt})_x-g_1+g_{2x}\qquad\text{for } x\in\Omega,$$

$$(3.34)\qquad \theta_x=\theta_{xt}=0\qquad\text{for } x\in\partial\Omega,$$

where

$$(3.35)\qquad g_1=(T_\infty+\theta)((\eta_T)_t\theta_t+(\eta_F)_tu_{xt})+\theta_t(\eta_T\theta_t+\eta_Fu_{xt})\ \text{ and }\ g_2=Q_t\theta_x.$$

Multiplying (3.31) and (3.33) by u_{tt} and $(T_\infty+\theta)^{-1}\theta_t$, respectively, integrating the resulting formulae over Ω and using (3.32) and (3.34), we have

$$\tag{3.36} \begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{(\varrho u_{tt},u_{tt})+(S_F^0 u_{xt},u_{xt})+(S_T^0\theta_t,\theta_t)\}+c_0\|\theta_{xt}(t,\cdot)\|^2 \\ &\leqslant C\sigma\|(u_x,\theta)_t(t,\cdot)\|^2+\|g_1(t,\cdot)\|\,\|(T_\infty+\theta)^{-1}\theta_t(t,\cdot)\| \\ &\quad+\|g_2(t,\cdot)\|\,\|((T_\infty+\theta)^{-1}\theta_t)_x(t,\cdot)\|, \end{aligned}$$

where we have used the fact that

$$(g_{2x},(T_\infty+\theta)^{-1}\theta_t)=-(g_2,((T_\infty+\theta)^{-1}\theta_t)_x).$$

Applying (3.9) and (3.10) to estimate g_1, g_2 and $\|((T_\infty+\theta)^{-1}\theta_t)_x(t,\cdot)\|$ and multiplying (3.36) by $\mathrm{e}^{2\alpha t}$, we have (3.30) immediately.

Step 4. *We verify the relation*

$$\tag{3.37} \begin{aligned} &\mathrm{e}^{2\alpha t}\|(u_x,u_t,\theta)_{tt}(t,\cdot)\|^2+c_1\int_0^t \mathrm{e}^{2\alpha s}\|\theta_{xtt}(s,\cdot)\|^2\,\mathrm{d}s \\ &\qquad\leqslant C\left\{E_0^2+(\alpha+\sigma)M_\alpha(t)^2+\sigma N_\alpha(t)^2\right\}. \end{aligned}$$

Differentiating (3.31)–(3.34) once in t implies that

$$\tag{3.38} \varrho(x)u_{tttt}-(S_F u_{xtt}+S_T\theta_{tt}+g_3)_x=0 \qquad \text{for } x\in\Omega,$$
$$\tag{3.39} S_F u_{xtt}+S_T\theta_{tt}+g_3=0 \qquad \text{for } x\in\partial\Omega,$$
$$\tag{3.40} (T_\infty+\theta)(\eta_T\theta_{ttt}+\eta_F u_{xttt})=(Q\theta_{xtt})_x-g_4+g_{5x} \qquad \text{for } x\in\Omega,$$
$$\tag{3.41} \theta_x=\theta_{xt}=\theta_{xtt}=0 \qquad \text{for } x\in\partial\Omega,$$

where

$$\begin{aligned} g_3 &= (S_F)_t u_{xt}+(S_T)_t\theta_t, \\ g_4 &= 2(T_\infty+\theta)((\eta_T)_t\theta_{tt}+(\eta_F)_t u_{xtt})+(T_\infty+\theta)((\eta_T)_{tt}\theta_t+(\eta_F)_{tt}u_{xt}) \\ &\quad+2\theta_t(\eta_T\theta_{tt}+\eta_F u_{xtt}+(\eta_T)_t\theta_t+(\eta_F)_t u_{xt})+\theta_{tt}(\eta_T\theta_t+\eta_F u_{xt}), \\ g_5 &= 2Q_t\theta_{xt}+Q_{tt}\theta_x. \end{aligned}$$

Multiplying (3.38) and (3.40) by u_{ttt} and $(T_\infty+\theta)^{-1}\theta_{tt}$, respectively, and using (3.39) and (3.41), we have

$$\tag{3.42} \begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{(\varrho(x)u_{ttt},u_{ttt})+(S_F u_{xtt},u_{xtt})+(\eta_T\theta_{tt},\theta_{tt})+2(g_3,u_{xtt})\}+c_0\|\theta_{xtt}(t,\cdot)\|^2 \\ &\leqslant C\sigma\|(u_{xtt},\theta_{tt})(t,\cdot)\|^2+\|g_{3t}(t,\cdot)\|\|u_{xtt}(t,\cdot)\| \\ &\quad+\|g_4(t,\cdot)\|\|((T_\infty+\theta)^{-1}\theta_{tt})(t,\cdot)\|+\|g_5(t,\cdot)\|\|((T_\infty+\theta)^{-1}\theta_{tt})_x(t,\cdot)\|. \end{aligned}$$

To estimate the terms: $\|g_{3t}(t,\cdot)\|$, $\|g_4(t,\cdot)\|$ and $\|g_5(t,\cdot)\|$, we use (3.9), (3.10), (3.11) and Sobolev's inequality, and then

$$\|g_{3t}(t,\cdot)\|,\ \ \|g_4(t,\cdot)\|,\ \ \|g_5(t,\cdot)\| \leqslant C\sigma\|(u_{xtt}, u_{xxt}, u_{xt}, \theta_{tt}, \theta_{xt}, \theta_t, \theta_x)\|.$$

In fact, for example, we have

$$\begin{aligned}\|g_{3t}(t,\cdot)\| &\leqslant \|(S_F, S_T)_t\|_\infty \|(u_x,\theta)_{tt}\| + \|(S_F, S_T)_{tt}\| \|(u_x,\theta)_t\|_\infty \\ &\leqslant C\sigma\|(u_{xtt}, u_{xxt}, u_{xt}, \theta_{tt}, \theta_{xt}, \theta_t)\|,\end{aligned}$$

where we have used (3.10), (3.11) and Sobolev's inequality in the second inequality. Multiplying (3.42) by $e^{2\alpha t}$, integrating the resulting inequality with respect to t and using the relation

$$|(g_3, u_{xtt})| \leqslant C\sigma\|(u_{xtt}, u_{xt}, \theta_t)(t,\cdot)\|^2$$

which follows also from (3.10), we have (3.37).

Combining Step 2, Step 3 and Step 4, we have

$$e^{2\alpha t}\|\bar{\partial}_t^2(u_t, u_x, \theta)(t,\cdot)\|^2 + c_1 \int_0^t e^{2\alpha s}\|\bar{\partial}_t^2\theta_x(s,\cdot)\|^2\,ds \leqslant CR_\alpha(t) \tag{3.43}$$

where

$$R_\alpha(t) = E_0^2 + (\sigma+\alpha)M_\alpha(t)^2 + \sigma N_\alpha(t)^2$$

The relation (3.43) was derived by the usual L^2-energy estimate.

Step 5. *For σ small enough, we verify the relation*

$$N_\alpha(t)^2 \leqslant CR_\alpha(t). \tag{3.44}$$

In view of (3.43) and the definition of $N_\alpha(t)$ (cf. (2.32)), to get (3.44) we have to estimate the terms: θ_x, θ_{xt}, u_{xx}, θ_{xx}, u_{xxx}, u_{xxt}, θ_{xxx} and θ_{xxt}. From (1.7) it follows that

$$(Q\partial_t^k\theta_x)_x = -k(Q_t\theta_x)_x + \partial_t^k((T_\infty+\theta)\eta_t) \quad \text{for } k=0 \text{ and } 1. \tag{3.45}$$

Multiplying (3.45) by $\partial_t^k\theta$ and integrating the resulting equation over Ω, by integration by parts and by (3.9), (3.10) and (3.11) we have

$$\begin{aligned}&|(Q\theta_x,\theta_x)| + |(Q\theta_{xt},\theta_{xt})| \\ &\qquad \leqslant |((T_\infty+\theta)\eta_t,\theta)| + |(Q_t\theta_x,\theta_{xt})| + |(((T_\infty+\theta)\eta_t)_t,\theta_t)| \\ &\qquad \leqslant C\sigma\|(\theta_x,\theta_{xt})(t,\cdot)\|^2 + C\|(\theta,\theta_t,u_{xt},u_{xtt},\theta_{tt})(t,\cdot)\|^2.\end{aligned}$$

Since $Q \geqslant \beta_0$, choosing σ so small that $C\sigma \leqslant \beta_0/2$ and using (3.43), we see that

(3.46) $$e^{2\alpha t}\|(\theta_x, \theta_{xt})(t,\cdot)\|^2 \leqslant CR_\alpha(t).$$

Since $f = F_x = -S_{\infty x}$, (1.1) can be rewritten as follows:

(3.47) $$S_F^0 u_{xx} = \varrho u_{tt} - \{S_T^0 \theta_x + (S_F^0)_x u_x + (S_T^0)_x \theta\}.$$

Applying (3.10), (3.43) and (3.46), we have

(3.48) $$e^{2\alpha t}\|u_{xx}(t,\cdot)\|^2 \leqslant CR_\alpha(t).$$

Since (1.7) can be rewritten as follows:

(3.49) $$Q\theta_{xx} = -Q_x\theta_x + (T_\infty + \theta)(\eta_T\theta_t + \eta_F u_{xt}),$$

applying (3.46), (3.10) and (3.43) to estimate the terms: $\|(Q_x\theta_x)(t,\cdot)\|$, $\|u_{xt}(t,\cdot)\|$, $\|\theta_t(t,\cdot)\|$, we have also

(3.50) $$e^{2\alpha t}\|\theta_{xx}(t,\cdot)\|^2 \leqslant CR_\alpha(t).$$

Differentiating (3.47) once in ℓ $\;(\ell = t$ and $x)$, we have

(3.51) $$S_F^0 u_{xx\ell} = \varrho u_{tt\ell} - S_T^0\theta_{x\ell} + \varrho_\ell u_{tt} - g_6 - g_7,$$

where

$$g_6 = (S_F^0)_\ell u_{xx} + (S_F^0)_x u_{x\ell} + (S_T^0)_\ell \theta_x + (S_T^0)_x \theta_\ell,$$
$$g_7 = (S_F^0)_{x\ell} u_x + (S_T^0)_{x\ell}\theta.$$

Since $\|(u_x,\theta)\|_\infty \leqslant C\|(u_x,u_t,\theta)_x\|$ as follows from Sobolev's inequality and (3.15), by (3.10), (3.12) and (3.13) we have

$$\|g_6\| \leqslant C\sigma\|(u_{xx}, u_{x\ell}, \theta_x, \theta_\ell)\|,$$
$$\|g_7\| \leqslant \|(S_F^0, S_T^0)_{x\ell}\|\|(u_x,\theta)\|_\infty \leqslant C\sigma\|(u_x,u_t,\theta)_x\|.$$

Therefore, by (3.43), (3.46) and (3.50), we have

(3.52) $$e^{2\alpha t}\|(u_{xxx}, u_{xxt})(t,\cdot)\|^2 \leqslant CR_\alpha(t).$$

For $\ell = t$ and x, differentiation of (3.49) once in ℓ implies that

(3.53) $$Q\theta_{xx\ell} - (T_\infty + \theta)\eta_T\theta_{t\ell} - (T_\infty + \theta)\eta_F u_{xt\ell} + g_8 + g_9 = 0$$

where

$$g_8 = Q_\ell \theta_{xx} + Q_x \theta_{x\ell} - ((T_\infty + \theta)\eta_T)_\ell \theta_t - ((T_\infty + \theta)\eta_F)_\ell u_{xt},$$
$$g_9 = Q_{x\ell}\theta_x.$$

Employing the same arguments as above, we have

$$\|g_8\| \leqslant C\sigma \|(\theta_{xx}, \theta_{x\ell}, \theta_t, u_{xt})\| \text{ and } \|g_9\| \leqslant C\sigma \|(\theta_x, \theta_{xx})\|.$$

Therefore, applying (3.43), (3.46) and (3.52) implies that

$$e^{2\alpha t} \|(\theta_{xxx}, \theta_{xxt})(t, \cdot)\|^2 \leqslant C R_\alpha(t),$$

which combined with (3.46), (3.48), (3.50), (3.52) and (3.43) implies (3.44).

Now, we are going to estimate $M_\alpha(t)$. By (3.43), we know the estimation corresponding to the terms: $\bar{\partial}_t^2 \theta_x$, and then we shall estimate the terms: $D^2 u$, $D^3 u$, θ_t, θ_{tt}, θ_{xx}, θ_{xxt} and θ_{xxx}.

Step 6. *For σ small enough, we verify the relations*

$$\text{(3.54)} \qquad \int_0^t e^{2\alpha s} \|\theta_{xxx}(s, \cdot)\|^2 \, ds \leqslant C \left\{ \int_0^t e^{2\alpha s} \|u_{xxt}(s, \cdot)\|^2 \, ds + R_\alpha(t) \right\},$$

$$\text{(3.55)} \quad \int_0^t e^{2\alpha s} \|(\theta_{xx}, \theta_{xxt})(s, \cdot)\|^2 \, ds \leqslant \delta \int_0^t e^{2\alpha s} \|u_{xxt}(s, \cdot)\|^2 \, ds + C\delta^{-1} R_\alpha(t)$$

for $\delta \in (0, 1)$.

Since it follows from the formula (3.53) with $\ell = x$ that

$$\text{(3.56)} \quad \|\theta_{xxx}(t, \cdot)\|^2 \leqslant C\{\|\theta_{xt}(t, \cdot)\|^2 + \|u_{xxt}(t, \cdot)\|^2) + \sigma \left\{ \|(u_{xxx}, \theta_{xx}, u_{xx}, \theta_x, \theta_t, u_{xt})(t, \cdot)\|^2 \right\},$$

multiplying (3.56) by $e^{2\alpha t}$, integrating the resulting inequality, using (3.43) and recalling the definitions of $R_\alpha(t)$ and $M_\alpha(t)^2$, we have (3.54).

Multiplying (3.49) by θ_{xx} and integrating the resulting equation over Ω implies that

$$(Q\theta_{xx}, \theta_{xx}) = -(Q_x \theta_x, \theta_{xx}) - (((T_\infty + \theta)(\eta_T \theta_t + \eta_F u_{xt}))_x, \theta_x)$$

where we have used integration by parts and the fact that $\theta_x = 0$ for $x \in \partial\Omega$ to get the second term of the right hand side. Since

$$|(\eta_F u_{xxt}, \theta_x)| \leqslant \delta \|u_{xxt}\|^2 + C\delta^{-1} \|\theta_x\|^2,$$

as follows from Schwarz's inequality, by (3.43), (3.9), (3.10) and (3.4) we have

$$\int_0^t e^{2\alpha s}\|\theta_{xx}(s,\cdot)\|^2 \leqslant \delta \int_0^t e^{2\alpha s}\|u_{xxt}(s,\cdot)\|^2\,ds + C\delta^{-1}R_\alpha(t). \tag{3.57}$$

To get the estimate of the term: θ_{xxt}, we multiply (3.53) with $\ell = t$ by θ_{xxt}, and then we have

$$(Q\theta_{xxt}, \theta_{xxt}) = -(g_8 + g_9, \theta_{xxt}) - (((T_\infty + \theta)(\eta_T\theta_{tt} + \eta_F u_{xtt}))_x, \theta_{xt}), \tag{3.58}$$

where we have used integration by parts and the fact that $\theta_{tx} = 0$ for $x \in \partial\Omega$ to get the last term. Since we already know the estimates of the terms θ_{xtt} and θ_{xt}, to treat the last term we may observe the following relation only:

$$\begin{aligned}((T_\infty + \theta)\eta_F u_{xxtt}, \theta_{xt}) &= \frac{d}{dt}((T_\infty + \theta)\eta_F u_{xxt}, \theta_{xt}) \\ &\quad - (((T_\infty + \theta)\eta_F)_t u_{xxt}, \theta_{xt}) - ((T_\infty + \theta)\eta_F u_{xxt}, \theta_{xtt}).\end{aligned} \tag{3.59}$$

Combining (3.58) and (3.59), multiplying the resulting formula by $e^{2\alpha t}$, integrating the resulting formula over $(0,t)$ and using (3.10) imply

$$\begin{aligned}\int_0^t e^{2\alpha s}\|\theta_{xxt}(s,\cdot)\|^2\,ds &\leqslant C\{N_\alpha(t)^2 + E_0^2 + (\sigma + \alpha)M_\alpha(t)^2\} \\ &\quad + \delta\int_0^t e^{2\alpha s}\|u_{xxt}(s,\cdot)\|^2\,ds + C\delta^{-1}\int_0^t e^{2\alpha s}\|\theta_{xtt}(s,\cdot)\|^2\,ds,\end{aligned}$$

which combined with (3.43) implies that

$$\int_0^t e^{2\alpha s}\|\theta_{xxt}(s,\cdot)\|^2\,ds \leqslant \delta\int_0^t e^{2\alpha s}\|u_{xxt}(s,\cdot)\|^2\,ds + C\delta^{-1}R_\alpha(t). \tag{3.60}$$

Combining (3.57) and (3.60) implies (3.55).

Step 7. *We verify the relation*

$$\int_0^t e^{2\alpha s}\|D^3u(s,\cdot)\|^2\,ds \leqslant C\left\{\int_0^t e^{2\alpha s}\|u_{xxt}(s,\cdot)\|^2\,ds + R_\alpha(t)\right\}. \tag{3.61}$$

Since

$$S_t = S_F u_{xt} + S_T\theta_t \text{ and } \theta_t = ((T_\infty + \theta)\eta_T)^{-1}\{(Q\theta_x)_x - (T_\infty + \theta)\eta_F u_{xt}\},$$

the second formula of which follows from (1.7) and (1.5), inserting the representation of θ_t into the right hand side of the first formula, we have

$$S_t = Vu_{xt} + ((T_\infty + \theta)\eta_T)^{-1} S_T (Q\theta_x)_x \tag{3.62}$$

where

$$V = \eta_T^{-1} M_\eta (X'_\infty + u_x, T_\infty + \theta) \quad \text{(cf. (3.5))}.$$

Note that there exists a $c_2 > 0$ such that

$$V \geqslant c_2, \tag{3.63}$$

which follows from (3.4) and (3.6). Differentiating (3.62) once with respect to t implies that

$$S_{tt} - (Vu_{xt})_t + (((T_\infty + \theta)\eta_T)^{-1} S_T (Q\theta_x)_x)_t = 0. \tag{3.64}$$

Multiplying (3.64) by u_{xtt}, we have

$$\begin{aligned} 0 = {} & (S_{tt}, u_{xtt}) - (Vu_{xtt}, u_{xtt}) - (V_t u_{xt}, u_{xtt}) \\ & + (((T_\infty + \theta)\eta_T)^{-1} S_T Q \theta_{xxt}, u_{xtt}) + ((((T_\infty + \theta)\eta_T)^{-1} S_T Q)_t \theta_{xx}, u_{xtt}) \\ & + ((((T_\infty + \theta)\eta_T)^{-1} S_T Q_x \theta_x)_t, u_{xtt}). \end{aligned} \tag{3.65}$$

Since $S_t = 0$ for $x \in \partial\Omega$ which follows from the fact that $S = 0$ for $x \in \partial\Omega$, we see that

$$(S_{tt}, u_{xtt}) = \frac{\mathrm{d}}{\mathrm{d}t}(S_t, u_{xtt}) + (S_{xt}, u_{ttt}) = \frac{\mathrm{d}}{\mathrm{d}t}(S_t, u_{xtt}) + (S_{xt}, \varrho^{-1} S_{xt}) \tag{3.66}$$

where we have used the relation:

$$u_{ttt} = \varrho^{-1}(S_x + f)_t = \varrho^{-1} S_{xt}, \tag{3.67}$$

which follows from (1.1) and (1.5). Since

$$|(S_{xt}, \varrho^{-1} S_{xt})| \leqslant C \left\{ \|(u_{xxt}, \theta_{xt})\|^2 + \sigma \|(u_{xt}, \theta_t)\|^2 \right\} \tag{3.68}$$

as follows from a direct calculation and (3.10) and since

$$|(((T_\infty + \theta)\eta_T)^{-1} S_T Q \theta_{xxt}, u_{xtt})| \leqslant \frac{c_2}{2} \|u_{xtt}\|^2 + C \|\theta_{xxt}\|^2, \tag{3.69}$$

combining (3.63), (3.65), (3.66), (3.67) and (3.68) implies that

$$
(3.70)\quad \frac{c_2}{2}\|u_{xtt}\|^2 - \frac{\mathrm{d}}{\mathrm{d}t}(S_t, u_{xtt}) \leqslant C\{\|u_{xxt}\|^2 + \|(\theta_{xxt}, \theta_{xt})\|^2 + \sigma\|(u_{xt}, \theta_t, u_{xtt}, \theta_{xx}, \theta_{xt}, u_{xxt})\|^2\},
$$

where we have used the following estimations:

$$
\begin{aligned}
&\|(((T_\infty + \theta)\eta_T)^{-1} S_T Q)_t \theta_{xx}\| \leqslant C\sigma\|\theta_{xx}\|;\\
&\|(((T_\infty + \theta)\eta_T)^{-1} S_T Q_x \theta_x)_t\|\\
&\quad \leqslant C\{\|Q_x\|_\infty \|\theta_{xt}\| + \|Q_{xt}\|\|\theta_x\|_\infty + \|(((T_\infty + \theta)\eta_T)^{-1} S_T)_t\|_\infty \|Q_x\|_\infty \|\theta_x\|\}\\
&\quad \leqslant C\sigma\|(\theta_x, \theta_{xx}, \theta_{xt})\|,
\end{aligned}
$$

which follows from (3.10), (3.12) and Sobolev's inequality. Multiplying (3.70) by $\mathrm{e}^{2\alpha t}$, integrating the resulting inequality over $(0, t)$ and using (3.55) with $\delta = 1$, (3.43) and (3.44), we have

$$
(3.71)\qquad \int_0^t \mathrm{e}^{2\alpha s}\|u_{xtt}(s, \cdot)\|^2 \,\mathrm{d}s \leqslant C\left\{\int_0^t \mathrm{e}^{2\alpha s}\|u_{xxt}(s, \cdot)\|^2 \,\mathrm{d}s + R_\alpha(t)\right\},
$$

where we have also used the estimate:

$$
|(S_t, u_{xtt})| \leqslant C\|(u_{xtt}, u_{xt}, \theta_t)\|^2.
$$

By (3.67), (3.68) and (3.43), we have

$$
(3.72)\qquad \int_0^t \mathrm{e}^{2\alpha s}\|u_{ttt}(s, \cdot)\|^2 \,\mathrm{d}s \leqslant C\left\{\int_0^t \mathrm{e}^{2\alpha s}\|u_{xxt}(s, \cdot)\|^2 \,\mathrm{d}s + R_\alpha(t)\right\}.
$$

By (3.51) with $\ell = x$ and the estimations of $\|g_6\|$ and $\|g_7\|$, we have

$$
(3.73)\qquad \|u_{xxx}\| \leqslant C\left\{\|u_{xtt}\| + \|u_{tt}\| + \|\theta_{xx}\| + \sigma\|(u_x, u_t, \theta)_x\|\right\}.
$$

Since

$$
\int_0^1 \varrho(x) u_{tt}(t, x) \,\mathrm{d}x = 0
$$

as follows from (1.1), (1.3) and (2.18), by Poincaré's inequality (3.16) with $p(x) = \varrho(x)$ we have

$$
(3.74)\qquad \|u_{tt}\| \leqslant C\|u_{xtt}\|.
$$

Inserting (3.74) into (3.73), multiplying the resulting inequality by $e^{2\alpha t}$, integrating the resulting inequality over $(0,t)$ and using (3.71) and (3.55) with $\delta = 1$, we have

$$\int_0^t e^{2\alpha s}\|u_{xxx}(s,\cdot)\|^2\,ds \leqslant C\left\{\int_0^t e^{2\alpha s}\|u_{xxt}(s,\cdot)\|^2\,ds + R_\alpha(t)\right\},$$

which combined with (3.71) and (3.72) implies (3.61).

Step 8. *For $\mu > 0$ small enough, we verify the following relation:*

$$\begin{aligned}(3.75)\qquad &\int_0^t e^{2\alpha s}\|(D^3u,\theta_{xx},\theta_{xxx},\theta_{xxt})(s,\cdot)\|^2\,ds\\ &\qquad\leqslant \mu\int_0^t e^{2\alpha s} < u_{xxt}(s,\cdot) >^2\,ds + C\mu^{-4}R_\alpha(t).\end{aligned}$$

To get (3.75), we shall consider the multiplication of (3.53) with $\ell = x$ by u_{xxt}. To do this, we observe the following relation:

$$\begin{aligned}(3.76)\qquad (Q\theta_{xxx},u_{xxt}) = {} & \langle Q\theta_{xx},u_{xxt}\rangle - (Q_x\theta_{xx},u_{xxt})\\ & - \frac{d}{dt}(Q\theta_{xx},u_{xxx}) + (Q_t\theta_{xx},u_{xxx}) + (Q\theta_{xxt},u_{xxx}).\end{aligned}$$

Multiplying (3.53) with $\ell = x$ by $u_{xxt}e^{2\alpha t}$, integrating the resulting equation on $(0,t)\times\Omega$ and using (3.76), (3.43) and the estimations of $\|g_8\|$ and $\|g_9\|$, we have

$$\begin{aligned}(3.77)\qquad \int_0^t e^{2\alpha s}\|u_{xxt}(s,\cdot)\|^2\,ds \leqslant {} & \int_0^t e^{2\alpha s}|\langle Q\theta_{xx},u_{xxt}\rangle|\,ds\\ & + \int_0^t e^{2\alpha s}|(Q\theta_{xxt},u_{xxx})|\,ds + CR_\alpha(t).\end{aligned}$$

By Schwarz's inequality, (3.55) and (3.61), we have

$$(3.78)\qquad \int_0^t e^{2\alpha s}|(Q\theta_{xxt},u_{xxx})|\,ds \leqslant C\left\{\delta^{\frac{1}{2}}\int_0^t e^{2\alpha s}\|u_{xxt}(s,\cdot)\|^2\,ds + \delta^{-\frac{3}{2}}R_\alpha(t)\right\}$$

for any $\delta > 0$ small enough. To estimate the first term in the right-hand side of (3.77), we use the following relation:

$$\begin{aligned}\langle\theta_{xx}(s,\cdot)\rangle^2 &\leqslant \|\theta_{xx}(s,\cdot)^2\|_\infty \leqslant C\int_0^1 |\bar\partial_x^1\theta_{xx}(s,\cdot)^2|\,dx\\ &\leqslant C\left\{\|\theta_{xx}(s,\cdot)\|^2 + \|\theta_{xx}(s,\cdot)\|\|\theta_{xxx}(s,\cdot)\|\right\}\end{aligned}$$

where to get the second inequality we have used Sobolev's inequality:

$$\|v\|_\infty \leqslant C \int_0^1 |\bar{\partial}_x^1 v(x)|\,\mathrm{d}x,$$

in one dimensional case. And then, we have

(3.79)

$$\begin{aligned}
&\int_0^t e^{2\alpha s} |\langle Q\theta_{xx}, u_{xxt}\rangle|\,\mathrm{d}s \\
&\quad\leqslant \mu \int_0^t e^{2\alpha s} \langle u_{xxt}(s,\cdot)\rangle^2\,\mathrm{d}s + C\mu^{-1} \int_0^t e^{2\alpha s} \langle \theta_{xx}(s,\cdot)\rangle^2\,\mathrm{d}s \\
&\quad\leqslant \mu \int_0^t e^{2\alpha s} \langle u_{xxt}(s,\cdot)\rangle^2\,\mathrm{d}s + C\mu^{-1}(1+\delta^{-\frac{1}{2}}) \int_0^t e^{2\alpha s}\|\theta_{xx}(s,\cdot)\|^2\,\mathrm{d}s \\
&\qquad + C\mu^{-1}\delta^{\frac{1}{2}} \int_0^t e^{2\alpha s}\|\theta_{xxx}(s,\cdot)\|^2\,\mathrm{d}s,
\end{aligned}$$

using (3.54) and (3.55),

$$\begin{aligned}
&\leqslant \mu \int_0^t e^{2\alpha s} \langle u_{xxt}(s,\cdot)\rangle^2\,\mathrm{d}s + C\{\mu^{-1}(1+\delta^{-\frac{1}{2}})\delta + \mu^{-1}\delta^{\frac{1}{2}}\} \int_0^t e^{2\alpha s}\|u_{xxt}(s,\cdot)\|^2\,\mathrm{d}s \\
&\quad + C\{\mu^{-1}(1+\delta^{-\frac{1}{2}})\delta^{-1} + \mu^{-\frac{1}{2}}\delta^{\frac{1}{2}}\}R_\alpha(t).
\end{aligned}$$

Combining (3.77), (3.78) and (3.79) implies that

$$\begin{aligned}
\int_0^t e^{2\alpha s}\|u_{xxt}(s,\cdot)\|^2\,\mathrm{d}s &\leqslant \mu \int_0^t e^{2\alpha s} \langle u_{xxt}(s,\cdot)\rangle^2\,\mathrm{d}s \\
&\quad + C\mu^{-1}\delta^{\frac{1}{2}} \int_0^t e^{2\alpha s}\|u_{xxt}(s,\cdot)\|^2\,\mathrm{d}s + C\mu^{-1}\delta^{-\frac{3}{2}} R_\alpha(t)
\end{aligned}$$

for $0 < \delta$, $\mu < 1$. Therefore, choosing $\delta > 0$ in such a way that

$$C\mu^{-1}\delta^{\frac{1}{2}} = \frac{1}{2},$$

we have

$$\int_0^t e^{2\alpha s}\|u_{xxt}(s,\cdot)\|^2\,\mathrm{d}s \leqslant 2\mu \int_0^t e^{2\alpha s} \langle u_{xxt}(s,\cdot)\rangle^2\,\mathrm{d}s + C\mu^{-4}R_\alpha(t),$$

which combined with (3.61), (3.54) and (3.55) with $\delta = 1$ implies (3.75).

Step 9. *We verify the relation*

$$\begin{aligned}
\text{(3.80)}\quad &\int_0^t e^{2\alpha s} \langle u_{xxt}(s,\cdot)\rangle^2\,\mathrm{d}s \\
&\qquad\leqslant C\left\{\int_0^t e^{2\alpha s}\|(D^3u, \theta_{xx}, \theta_{xxx}, \theta_{xxt})(s,\cdot)\|^2\,\mathrm{d}s + R_\alpha(t)\right\}.
\end{aligned}$$

Differentiating (3.67) once with respect to x implies that

$$u_{xxxt} = S_F^{-1}\left\{(\varrho u_{ttt})_x - S_T\theta_{xxt} - g_{10}\right\}, \tag{3.81}$$

where

$$g_{10} = 2\left((S_F)_x u_{xxt} + (S_T)_x\theta_{xt}\right) + (S_F)_{xx}u_{xt} + (S_T)_{xx}\theta_t.$$

By (3.10), (3.13) and Sobolev's inequality, we have

$$\begin{aligned}\|g_{10}\| &\leqslant C\left\{\|(S_F, S_T)_x\|_\infty \|(u_{xxt}, \theta_{xt})\| + \|(S_F, S_T)_{xx}\|\|(u_{xt}, \theta_t)\|_\infty\right\}\\ &\leqslant C\sigma\|(u_{xxt}, u_{xt}, \theta_{xt}, \theta_t)\|.\end{aligned}$$

Put $q(x) = x - \frac{1}{2}$, and then we have

$$(u_{xxxt}, qu_{xxt}) = \frac{1}{4}\langle u_{xxt}\rangle^2 - \frac{1}{2}\|u_{xxt}\|^2. \tag{3.82}$$

On the other hand, we have

$$\begin{aligned}&(S_F^{-1}(\varrho u_{ttt})_x, qu_{xxt}) \\ &\quad = (S_F^{-1}\varrho' u_{ttt}, qu_{xxt}) + \frac{\mathrm{d}}{\mathrm{d}t}(S_F^{-1}\varrho u_{xtt}, qu_{xxt}) - ((S_F)^{-1})_t\varrho u_{xtt}, qu_{xxt}) \\ &\quad + \frac{1}{2}((S_F^{-1}\varrho q)_x u_{xtt}, u_{xtt}) - \frac{1}{4}\left\{(S_F^{-1}\varrho u_{xtt}^2)(t,1) + (S_F^{-1}\varrho u_{xtt}^2)(t,0)\right\}.\end{aligned} \tag{3.83}$$

Multiplying (3.81) by qu_{xxt} and using (3.10), (3.82) and (3.83), we have

$$\begin{aligned}&\frac{1}{4}\left\{\langle u_{xxt}\rangle^2 + \langle S_F^{-1}\varrho u_{xtt}\rangle^2\right\} \\ &\qquad \leqslant C\left\{\|\theta_{xxt}\|^2 + \|D^3u\|^2 + \|g_{10}\|^2\right\} + \frac{\mathrm{d}}{\mathrm{d}t}(S_F^{-1}\varrho u_{xtt}, qu_{xxt}).\end{aligned} \tag{3.84}$$

Multiplying (3.84) by $\mathrm{e}^{2\alpha t}$, integrating the resulting equation over $(0,t)$ and using (3.44), we have (3.80).

Combining (3.75) and (3.80) and choosing $\mu > 0$ small enough, we arrive at the relation:

$$\int_0^t \mathrm{e}^{2\alpha s}\|(D^3u, \theta_{xx}, \theta_{xxx}, \theta_{xxt})(s,\cdot)\|^2\,\mathrm{d}s \leqslant CR_\alpha(t). \tag{3.85}$$

In view of (3.43) and (3.85), our task is now to estimate the terms: D^2u, θ_t and θ_{tt}.

Step 10. *We verify the relation*

$$\int_0^t \mathrm{e}^{2\alpha s}\|(D^2u, \theta_t, \theta_{tt})(s,\cdot)\|^2\,\mathrm{d}s \leqslant CR_\alpha(t). \tag{3.86}$$

Since $u_{xx} = (S_F^0)^{-1}\{\varrho u_{tt} - S_T^0\theta_x - (S_F^0)_x u_x - (S_T^0)_x\theta\}$ as follows from (3.47), by (3.47), (3.10), (3.15), (3.43) and (3.85) we have

$$\int_0^t e^{2\alpha s}\|(u_{tt}, u_{xx})(s,\cdot)\|^2 \, ds \leqslant CR_\alpha(t). \tag{3.87}$$

Since $S_t = 0$ for $x \in \partial\Omega$, by Poincaré's inequality (3.17) with $v = S_t$ and (3.10), we have

$$\|S_t\|^2 \leqslant C\|S_{tx}\|^2 \leqslant C\left\{\|(u_{xxt}, \theta_{xt})(t,\cdot)\|^2 + \sigma\|(u_{xt}, \theta_t)(t,\cdot)\|^2\right\},$$

which combined with (3.62), (3.63), (3.43) and (3.85) implies that

$$\int_0^t e^{2\alpha s}\|u_{xt}(s,\cdot)\|^2 \, ds \leqslant CR_\alpha(t). \tag{3.88}$$

By (3.49) and (3.53) with $\ell = t$, we have

$$\|(\theta_t, \theta_{tt})(t,\cdot)\|^2 \leqslant C\left\{\|(\theta_{xx}, u_{xt}, u_{xtt}, \theta_{xxt})(t,\cdot)\|^2 + \sigma\|(\theta_x, \theta_{xx}, \theta_{xt}, u_{xt}, \theta_t)(t,\cdot)\|^2\right\},$$

which combined with (3.85) and (3.88) implies that

$$\int_0^t e^{2\alpha s}\|(\theta_t, \theta_{tt})(s,\cdot)\|^2 \, ds \leqslant CR_\alpha(t). \tag{3.89}$$

Combining (3.87)–(3.89), we establish (3.86).

Therefore, combining (3.43), (3.44), (3.85) and (3.86) implies that

$$N_\alpha(t)^2 + M_\alpha(t)^2 \leqslant C\left\{E_0^2 + (\sigma + \alpha)M_\alpha(t)^2 + \sigma N_\alpha(t)^2\right\}. \tag{3.90}$$

Choosing σ and α finally in such a way that

$$C(\sigma + \alpha) < \frac{1}{4} \quad \text{and} \quad C\sigma < \frac{1}{4},$$

we establish (3.1), which completes the proof of Theorem 2.5.

APPENDIX. A PROOF OF LEMMA 2.3.

It is sufficient to prove the lemma in case that $N = 3$, because the higher regularity of $X_\infty(x)$ follows from the relation:

$$\frac{\partial S}{\partial F}(X'_\infty(x), T_\infty)X''_\infty(x) = -F'(x) = -f(x).$$

For $(V(x), p) \in H^2 \times \mathbb{R}$, let us define its norm by the formula: $|(V,p)| = \|V\|_2 + |p|$ and also let us define the map Φ from $H^2 \times \mathbb{R}$ into itself by the formula:

$$\Phi(V,p) = (S(V(x),p), \int_0^1 \{\varepsilon(V(x),p) + V(x)F(x)\}\,\mathrm{d}x).$$

We shall show that the map Φ is a local homeomorphism near the point $(1, \tau_0) \in H^2 \times \mathbb{R}$. To do this, it is sufficient to prove that the differentiation Ψ of Φ at $(1, \tau_0)$ is bijective, where Ψ is given by the formula:

$$\Psi(V(x),p) = \frac{\mathrm{d}}{\mathrm{d}\theta}\Phi((1,\tau_0) + \theta(V(x),p))\Big|_{\theta=0} = (\Psi_1(V(x),p), \Psi_2(V(x),p))$$

where

$$\Psi_1(V(x),p) = \frac{\partial S}{\partial F}(1,\tau_0)V(x) + \frac{\partial S}{\partial T}(1,\tau_0)p,$$
$$\Psi_2(V(x),p) = \int_0^1 \left(\frac{\partial \varepsilon}{\partial F}(1,\tau_0)V(x) + \frac{\partial \varepsilon}{\partial T}(1,\tau_0)p\right)\mathrm{d}x + \int_0^1 V(x)F(x)\,\mathrm{d}x.$$

First of all, we shall show that Ψ is surjective, that is, we shall solve the equation:

$$\Psi_1(V(x),p) = W(x) \text{ and } \Psi_2(V(x),p) = q \tag{Ap.1}$$

for given $(W(x), q) \in H^2 \times \mathbb{R}$. The first equation of (Ap.1) becomes the following formula:

$$V(x) = \frac{\partial S}{\partial F}(1,\tau_0)^{-1}\Big\{W(x) - \frac{\partial S}{\partial T}(1,\tau_0)p\Big\}, \tag{Ap.2}$$

and then inserting (Ap.2) into the second equation of (Ap.1) implies that

$$q = \frac{\partial S}{\partial F}(1,\tau_0)^{-1}\Big\{\frac{\partial \varepsilon}{\partial F}(1,\tau_0)\int_0^1 W(x)\,\mathrm{d}x + \int_0^1 W(x)F(x)\,\mathrm{d}x + Lp\Big\}, \tag{Ap.3}$$

where

$$L = M_\varepsilon(1,\tau_0) - \frac{\partial S}{\partial T}(1,\tau_0)\int_0^1 F(x)\,\mathrm{d}x$$

and $M_\varepsilon(1,\tau_0)$ is defined by (3.5) with $g=\varepsilon$. In view of (3.7) with $g=\varepsilon$, we have $L>\frac{1}{2}M_\varepsilon(1,\tau_0)$ provided that

$$\|f\| \leqslant \left|\frac{\partial S}{\partial T}(1,\tau_0)\right|^{-1} M_\varepsilon(1,\tau_0)/2, \tag{Ap.4}$$

because $|\int_0^1 F(x)\,\mathrm{d}x| \leqslant \|F\| \leqslant \|f\|$. Therefore, Ψ is surjective if $\|f\|$ is small enough.

If $(W(x),q)=(0,0)$, by (Ap.2) and (Ap.3) we see that $(V(x),p)$ is also equal to $(0,0)$, which means that Ψ is injective, and then Ψ is bijective. Therefore, the implicit function theorem yields that there exist neighborhoods U_1 of $(1,\tau_0)$ and U_2 of $\Phi(1,\tau_0)$ such that Φ is homeomorphic from U_1 onto U_2. Since

$$\begin{aligned}
&|(-F(x),e_0)-\Phi(1,\tau_0)|\\
&\leqslant \|F\|_2 + \left|\int_0^1 \left\{\varepsilon(X_0'(x),T_0(x))-\varepsilon(1,\tau_0)+\frac{\varrho(x)}{2}X_1(x)^2+(X_0'(x)-1)F(x)\right\}\mathrm{d}x\right|\\
&\leqslant \|f\|_1 + C\|(X_0',T_0)-(1,\tau_0)\|_\infty + \frac{\|\varrho\|_\infty}{2}\|X_1\|^2+\|X_0'-1\|_\infty\|f\|
\end{aligned}$$

where C is a constant such that

$$\left|\frac{\partial\varepsilon}{\partial F}(F,T)\right|,\ \left|\frac{\partial\varepsilon}{\partial T}(F,T)\right| \leqslant C \quad \text{for } (F,T)\in G(\delta),$$

there exists a $\kappa>0$ such that $(-F(x),e_0)\in U_2$ provided that

$$\|(X_0',T_0)-(1,\tau_0)\|_\infty + \|X_1\| + \|f\|_1 < \kappa,$$

which implies the unique existence of $(V(x),T_\infty)\in U_1\subset H^2\times\mathbb{R}$ satisfying the equation: $\Phi(V,T_\infty)=(-F(x),e_0)$. If we put $X_\infty(x)=\int_0^x V(y)\,\mathrm{d}y$, then $X_\infty(x)$ and T_∞ satisfy the required properties, because the inverse of Φ is also a continuous map from U_2 onto U_1.

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