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ANNIHILATORS AND IDEALS IN ORDERED SETS

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M. Mandelker [5] introduced and studied the concept of an annihilator in a lattice. If L is a lattice, $a, b \in L$, then the annihilator of a relative to b is the set

$$\langle a, b \rangle = \{x \in L; a \wedge x \leq b\},$$

the dual annihilator is

$$\langle a, b \rangle_d = \{x \in L; a \vee x \geq b\}.$$

He proved that L is distributive iff each annihilator of L is an ideal. Further, he found a necessary and sufficient condition for a distributive lattice to satisfy the identity $\langle a, b \rangle \vee \langle b, a \rangle_d = L$, where the symbol \vee denotes the join in the lattice of all ideals of L .

Moreover, Davey and Nieminen [1] studied connections between modularity of L and the so called prime annihilator conditions.

The aim of this paper is to study analogous connections in the case of ordered sets. Let us recall some basic notions. Let (S, \leq) be an ordered set, $X \subseteq S$; then we denote $U(X) = \{y \in S; y \geq x \text{ for all } x \in X\}$, $L(X) = \{y \in S; y \leq x \text{ for all } x \in X\}$. A subset $I \subseteq S$ is called an ideal (filter) of S if $LU(x, y) \subseteq I$ ($UL(x, y) \subseteq I$) whenever $x, y \in I$. An ideal (filter) $I \neq \emptyset$, $I \neq S$, is called prime if $L(x, y) \subseteq I$ ($U(x, y) \subseteq I$) implies $x \in I$ or $y \in I$. If an ideal (filter) is an up (down) directed set then it is called a u -ideal (l -filter). The set of all ideals (filters) of S forms a lattice $\text{Id}(S)$ ($\text{Fil}(S)$) with respect to set inclusion, see [2].

Recall from [4] that an ordered set S is called

distributive if $\forall a, b, c \in S: L(U(a, b), c) = LU(L(a, c), L(b, c))$,

modular if $\forall a, b, c \in S: a \leq c \Rightarrow L(U(a, b), c) = LU(a, L(b, c))$.

For $a, b \in S$ the annihilator of a relative to b is the set

$$\langle a, b \rangle = \{x \in S; UL(a, x) \supseteq U(b)\},$$

the dual annihilator is

$$\langle a, b \rangle_d = \{x \in S; LU(a, x) \supseteq L(b)\}.$$

The annihilator $\langle a, b \rangle$ of S is called prime if

- (i) $\langle a, b \rangle \cap \langle b, a \rangle_d = \emptyset$,
- (ii) $\langle a, b \rangle \cup \langle b, a \rangle_d = S$.

Proposition 1. *Let F, G be l -filters of an ordered set S . Then $F \vee G = \bigcup\{UL(a, b); a \in F, b \in G\}$, where the symbol \vee denotes the join in $\text{Fil}(S)$.*

Proof. See [2], Theorem 3. □

Lemma 1. *Let S be a distributive set and F an l -filter of S satisfying the condition*

$$(*) \quad \forall x \in F \forall a, b \in S: U(a, b) \subseteq F \Rightarrow UL(x, U(a, b)) \subseteq F.$$

If the set of all filters of S containing F forms a chain, then F is a prime filter.

Proof. Let $a, b \in S, U(a, b) \subseteq F$. Let us denote $G = F \vee U(a), H = F \vee U(b)$. Since filters containing F form a chain, we have e.g. $H \subseteq G$. Since both F and $U(a)$ are l -filters, by Proposition 1 we have $b \in UL(x, a)$ for some $x \in F$. Hence we obtain $U(b) \subseteq UL(x, a)$ and, by the condition $(*)$,

$$UL(x, U(a, b)) \subseteq F.$$

Using distributivity, we can derive

$$L(x, U(a, b)) = LU(L(a, x), L(b, x)) = L(UL(a, x) \cap UL(b, x)) \subseteq L(b).$$

Finally, we have $F \supseteq UL(x, U(a, b)) \supseteq U(b)$, hence $b \in F$ and F is prime. □

Remark. In a finite set S every l -filter F satisfies the condition $(*)$.

Lemma 2. *Let S be a distributive set satisfying the condition*

$$(**) \quad \text{for every } a, b \in S \text{ there exists } x \in F \text{ such that the sets } \\ UL(a, x), UL(b, x) \text{ are comparable,}$$

where F is a filter of S . Then the set of all filters of S containing F forms a chain.

Proof. Let $a, b \in S, x \in F, G, H \in \text{Fil}(S), G, H \supseteq F$. If $G \parallel H$, then let $a \in G \setminus H, b \in H \setminus G$. Further, let $UL(a, x) \supseteq UL(b, x)$. Since $a, x \in G$ we have $UL(a, x) \subseteq G$, thus $G \supseteq UL(a, x) \supseteq UL(b, x) \supseteq U(b)$, therefore $b \in G$, a contradiction with $b \notin G$. □

Now, by Lemmas 1 and 2 we infer the following

Theorem 1. *Let S be a distributive set and F a l -filter of S satisfying the condition $(*)$. Let us consider the conditions*

- (1) $\forall a, b \in S \exists x \in F: UL(a, x)$ and $UL(b, x)$ are comparable;
- (2) filters of S containing F form a chain;
- (3) F is a prime filter;
- (4) F contains a prime filter.

Then (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).

Lemma 3. *Let S be a distributive set, $a, b \in S$. Then $(***)$ implies $(****)$, where*

(***) both $\langle a, b \rangle$ and $\langle b, a \rangle$ are up directed sets and
 $\langle a, b \rangle \vee \langle b, a \rangle = S$,

(****) if F is a filter of S containing a prime filter P ,
then for $a, b \in S$ there exists $x \in F$ such that the sets
 $UL(a, x)$, $UL(b, x)$ are comparable.

Proof. Let $P \subseteq F$, $z \in P$. Then $z \in S = \langle a, b \rangle \vee \langle b, a \rangle$ and since both $\langle a, b \rangle$, $\langle b, a \rangle$ are u -ideals, we obtain by Proposition 1 that $z \in LU(x, y)$ for some $x \in \langle a, b \rangle$, $y \in \langle b, a \rangle$. This implies $z \leq k$ for every $k \in U(x, y)$. Since P is a filter and $z \in P$, we have $U(x, y) \subseteq P$ and as P is prime, $x \in P$ or $y \in P$. Suppose $x \in F$. Since $x \in \langle a, b \rangle$ we observe that $L(a, x) \subseteq L(b)$, $L(a, x) \subseteq L(b, x)$ and finally,

$$UL(a, x) \supseteq UL(b, x).$$

□

Definition. An ordered set S is called s -distributive if it satisfies the condition

$$L(U(a, b), U(c, d)) = LU(L(a, U(c, d)), L(b, U(c, d)))$$

for all $a, b, c, d \in S$.

Theorem 2. *Let S be an s -distributive set, $I \in \text{Id}(S)$, $D \in \text{Fil}(S)$, $D \cap I = \emptyset$. Let I_D be a maximal ideal of S satisfying the conditions $I_D \supseteq I$, $I_D \cap D = \emptyset$. If an ideal I_D satisfies also the conditions*

- (i) I_D is a u -ideal,

(ii) $\forall a, b \notin I_D \forall x, x' \in I_D$:

$$L(a, b) \subseteq I_D \Rightarrow U(L(a, b), L(a, x'), L(b, x), L(x, x')) \cap I_D \neq \emptyset,$$

then I_D is a prime ideal.

Proof. According to (i), I_D is a u -ideal. Let I_D be not prime, i.e. $a, b \notin I_D$ but $L(a, b) \subseteq I_D$ for some $a, b \in S$. Since the ideals $L(a)$, $L(b)$ are u -ideals, by Proposition 1 we obtain in $\text{Id}(S)$:

$$\begin{aligned} I_D \vee L(a) &= \cup\{LU(a, x); x \in I_D\}, \\ I_D \vee L(b) &= \cup\{LU(b, y); y \in I_D\}. \end{aligned}$$

Obviously, $(I_D \vee L(a)) \cap D \neq \emptyset$ and $(I_D \vee L(b)) \cap D \neq \emptyset$. Thus there exists $x, x' \in I_D$, $p, q \in D$ such that

$$p \in LU(a, x), \quad q \in LU(b, x').$$

Then we have $L(p) \subseteq LU(a, x)$, $L(q) \subseteq LU(b, x')$, therefore

$$UL(p, q) \supseteq U(LU(a, x) \cap LU(b, x')) = UL(U(a, x), U(b, x')).$$

Since S is an s -distributive set, we can derive

$$\begin{aligned} UL(U(a, x), U(b, x')) &= ULU(L(a, U(b, x')), L(x, U(b, x'))) \\ &= U(LU(L(a, b), L(a, x')), LU(L(b, x), L(x, x'))) \\ &= U(L(a, b), L(a, x'), L(b, x), L(x, x')). \end{aligned}$$

By the condition (ii) we have $U(L(a, b), L(a, x'), L(b, x), L(x, x')) \cap I_D \neq \emptyset$. Since $UL(U(b, x'), U(a, x)) \subseteq UL(p, q) \subseteq D$, we obtain $D \cap I_D \neq \emptyset$, a contradiction. \square

Remark. If S is a finite set, then the condition (i) implies the condition (ii): $X = L(a, b) \cup L(a, x') \cup L(b, x) \cup L(x, x')$ is a finite subset of I_D , hence $I_D \cap U(X) \neq \emptyset$.

Example 1. Let us consider an ordered set S whose diagram is visualized in Fig. 1.

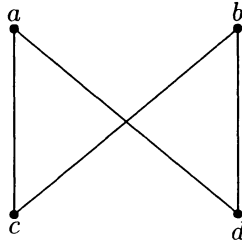


Fig. 1

It is an s -distributive set, $D = \{a, b\}$ is a filter of S , $I_D = I = \{c, d\}$. We can verify that neither I_D is a u -ideal nor I_D is prime.

Definition. Let S be an ordered set, $I \in \text{Id}(S)$, $D \in \text{Fil}(S)$, $I \cap D = \emptyset$. If I_D is the maximal ideal with $I_D \supseteq I$, $I_D \cap D = \emptyset$ satisfying the conditions (i) and (ii) of Theorem 2, then I is called a D -strong ideal.

Theorem 3. Let S be a distributive set, $a, b \in S$, $I = \langle a, b \rangle \vee \langle b, a \rangle$. If there exists an element $z \in S \setminus I$ such that the filter $U(z)$ is I -strong, then the condition (****) of Lemma 3 implies $I = S$.

Proof. Let I be a proper ideal of S . According to theorem dual to Theorem 2, there exists a prime filter F containing z such that $F \cap I = \emptyset$. By the condition (****) of Lemma 3 we have that there exists $x \in F$ such that $UL(a, x) \supseteq UL(b, x) \supseteq U(b)$, hence $x \in \langle a, b \rangle$, $x \in I$, thus $F \cap I \neq \emptyset$, a contradiction. \square

Now, we shall show connection between prime annihilators and prime ideals in ordered sets.

Definition. An ordered set S is called 3-distributive if

$$U(b, L(a, x, y)) = UL(U(a, b), U(b, x), U(b, y))$$

holds for all $a, b, x, y \in S$.

Theorem 4. Let S be a distributive and 3-distributive set. Then every prime annihilator of S is a prime ideal.

Proof. In a distributive set every prime annihilator is an ideal (see [3]). Let $\langle a, b \rangle$ be a prime annihilator. Let $L(x, y) \subseteq \langle a, b \rangle$ but $x, y \notin \langle a, b \rangle$, i.e. $x, y \in \langle b, a \rangle_d$. Then we obtain

$$(1) \quad \begin{aligned} U(x, b) &\subseteq U(a), \\ U(y, b) &\subseteq U(a), \\ L(z, a) &\subseteq L(b) \text{ for every } z \in L(x, y). \end{aligned}$$

We shall prove that $L(a, x, y) \subseteq L(b)$. Let $z^* \in L(a, x, y) \subseteq L(x, y)$. By (1) we have $L(z^*, a) \subseteq L(b)$. But $z^* \leq a$, hence $L(z^*) \subseteq L(b)$, thus $L(a, x, y) \subseteq L(b)$. Further, $U(b, L(a, x, y)) = U(b)$. By 3-distributivity we can derive

$$UL(U(b, x), U(b, a), U(b, y)) = U(b, L(a, x, y)) = U(b).$$

The inclusions (1) imply

$$UL(U(a), U(b, a), U(a)) = U(a) \supseteq U(b),$$

hence $a \leq b$. Then $\langle a, b \rangle = \langle b, a \rangle_d = S$, a contradiction with $\langle a, b \rangle$ being prime. \square

Theorem 5. Let S be an ordered set, $a, b \in S$, $a > b$. Then $\langle a, b \rangle \cup \langle b, a \rangle_d = S$ if and only if $a \succ b$ and the following condition (P) is satisfied:

$$(P) \quad \forall x \in S: (\exists z \in U(b, L(a, x)), z \parallel a) \Rightarrow x \in \langle a, b \rangle.$$

Proof. Let $\langle a, b \rangle \cup \langle b, a \rangle_d = S$, $z \in U(b, L(a, x))$, $z \parallel a$. Then $z \notin \langle b, a \rangle_d$ since $U(b, z) = U(z) \not\subseteq U(a)$; hence $z \in \langle a, b \rangle$, i.e. $L(a, z) \subseteq L(b)$. But the opposite inclusion is trivially valid, thus $L(a, z) = L(b)$, i.e. there exists an infimum $a \wedge z = b$. Since $L(a, x) \subseteq L(a, z)$, we have $L(a, x) \subseteq L(b)$, thus $x \in \langle a, b \rangle$. If there exists $q \in S$ such that $a > q > b$, then $q \notin \langle a, b \rangle$ and $q \notin \langle b, a \rangle_d$, a contradiction.

Conversely, let the condition (P) be valid and $a \succ b$. Let $x \in S$, $x \notin \langle a, b \rangle$. We shall prove that $U(b, L(a, x)) = U(a)$. Obviously, $U(b, L(a, x)) \supseteq U(a)$. If $z \in U(b, L(a, x))$, then $z \neq b$ since $x \notin \langle a, b \rangle$. Now, we have the following possibilities:

(1) $b < z < a$ — it can not occur since $a \succ b$,

(2) $z \parallel a$ — it can not occur since using the condition (P) we get $x \in \langle a, b \rangle$, a contradiction with the choice of x ,

(3) $z \geq a$.

So we have proved that $U(b, L(a, x)) \subseteq U(a)$, thus

$$U(L(a, x), b) = U(a).$$

Further, this implies

$$U(x, b) = U(x, L(a, x), b) = U(b, L(a, x)) \cap U(x) = U(a, x) \subseteq U(a),$$

i.e. $LU(b, x) \supseteq L(a)$, i.e. $x \in \langle b, a \rangle_d$. □

Proposition 2. Let S be a modular set, $a, b \in S$, $a > b$. Then

$$\langle a, b \rangle \cap \langle b, a \rangle_d = \emptyset.$$

Proof. Let $x \in \langle a, b \rangle \cap \langle b, a \rangle_d$. Then $L(a, x) \subseteq L(b)$, $U(b, x) \subseteq U(a)$. Now, by modularity of S we obtain

$$L(a, U(b, x)) = L(a) = LU(b, L(a, x)) = LU(L(b)) = L(b),$$

hence $a = b$, a contradiction. □

By Proposition 2 and Theorem 5 we have the following consequence:

Corollary. Let S be a modular set, $a, b \in S$, $a > b$. Then $\langle a, b \rangle$ is a prime annihilator iff $a \succ b$ and S satisfies the condition (P).

Example 2. Let the diagram of S be given in Fig. 2.

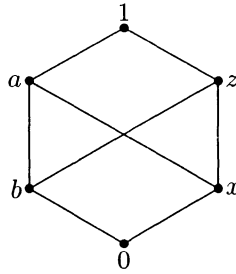


Fig. 2

It can be easily verified that $L(b)$ is prime ideal in the distributive set S . It is not a prime annihilator since $L(b) = \langle a, b \rangle = \langle z, b \rangle$, but these annihilators do not satisfy the condition (P) of Theorem 5: $z \in U(b, L(a, x))$, $z \parallel a$, but $x \notin \langle a, b \rangle$.

Theorem 6. Let S be a modular set, $m \in S$ such that there exists a unique minimal element m^* in $U(m) \setminus \{m\}$. Then $L(m) = \langle m^*, m \rangle$.

Proof. Let m^* be the unique minimal element in $U(m) \setminus \{m\}$. Obviously, $L(m) \subseteq \langle m^*, m \rangle$. Let $x \in \langle m^*, m \rangle$, $x \not\leq m$. Then $L(x, m^*) \subseteq L(m)$, $U(x, m) \subsetneq U(m)$, hence for $z \in U(x, m)$ we get $z \geq m^*$, $U(z) \subseteq U(m^*)$, thus $m^* \in LU(m, x)$. This implies $m^* \in L(m^*, U(m, x)) = LU(m, L(x, m^*))$ by modularity of S . Finally, we can derive $m^* \in LU(m, L(x, m^*)) \subseteq LU(m, L(m)) \subseteq L(m)$, thus $m^* \leq m$, a contradiction with the choice of m^* . \square

Now, if m^* is the unique minimal element in $U(m) \setminus \{m\}$, then obviously $m^* \succ m$ and for the elements m^*, m the condition (P) is valid. Hence we obtain the next corollary:

Corollary. Let S be a modular set, let $m \in S$ be such that there exists a unique minimal element in $U(m) \setminus \{m\}$. Then $L(m)$ is a prime annihilator.

Definition. An ordered set S is called complemented if for each $a \in S$ there exists $a' \in S$ such that $UL(a, a') = LU(a, a') = S$. A distributive and complemented set is called boolean. An element $q \in S$ is called a coatom if either q is a maximal element of S if S has no greatest element or $1 \succ q$ whenever S has the greatest element 1.

Proposition 3. Let S be a finite boolean set. Then every prime ideal of S is of the form $L(x)$, where x is a coatom of S .

Proof. See [2]. \square

By Proposition 3 and the preceding corollary we infer

Corollary. *Let S be a finite boolean set with the greatest element 1. Then every prime ideal of S is a prime annihilator.*

Proof. Prime ideals of S are of the form $L(m)$, where m is a maximal element of S . We put $m^* = 1$. Obviously, $m^* \succ m$ and by the preceding Corollary we have $L(m) = \langle 1, m \rangle$. Moreover, $\langle 1, m \rangle$ is a prime annihilator. \square

Remark. If an ordered set is distributive only, the last corollary need not be true. It suffices to consider the set shown in Fig. 2.

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