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THE FEYNMAN-KAC FORMULA FOR A SYSTEM OF PARABOLIC EQUATIONS

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INTRODUCTION

Consider the Cauchy problem

$$(1.1)\qquad \begin{cases} \dfrac{\partial \vec{u}}{\partial t} = \alpha \Delta \vec{u} + c(x,t)\vec{u}, & (x,t) \in \mathbb{R}^p \times (0,T], \\ \vec{u}\big|_{t=0} = \vec{f}(x), \quad x \in \mathbb{R}^p, \end{cases}$$

where $\vec{u}$ is an m-vector: $\vec{u}(x,t) = [u^{(1)}(x,t), \ldots, u^{(m)}(x,t)]$ and $c(x,t)$ is an $m \times m$ matrix, with the diffusion coefficient α being the same in each equation of the system.

Assuming c and $\vec{f}$ to be bounded continuous functions, this problem has a unique solution in a suitable sense to be made precise below. Taking $\alpha = \frac{1}{2}$ for convenience, we are going to develop the following stochastic representation for this solution

$$(1.2)\qquad \vec{u}(x,t) = \int_\Omega Y(t,t,x,\omega)\vec{f}(x+\omega(t))\,\mathrm{d}W(\omega).$$

Here $\mathrm{d}W(w)$ represents Wiener measure in Ω the set of all continuous mappings $\omega\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^p$ with $\omega(0) = 0$; and $Y(s,t,x,\omega)$ represents for fixed t, x and ω the fundamental matrix solution of the ordinary differential equations initial value problem

$$(1.3)\qquad \begin{cases} \dfrac{\mathrm{d}Y}{\mathrm{d}s}(s,t,x,\omega) = c(x+\omega(t-s),s)Y(s,t,x,\omega), & 0 \leqslant s \leqslant t, \\ Y(0,t,x,\omega) = I. \end{cases}$$

When $m = 1$, $Y(s,t,x,\omega) = \mathrm{e}^{\int_0^s c(x+\omega(t-\sigma),\sigma)\,\mathrm{d}\sigma}$ and (1.2) is then the classical Feynman-Kac formula [10] for the solution of problem (1.1) in which the system reduces to a single equation.

Our proof of (1.2) is based on E. Nelson's idea [12] of using the Trotter-Lie-Kato formula to derive the classical Feynman-Kac formula in the special case where c only depends on x. Although the known proofs of the Trotter-Lie-Kato formula are not applicable to problem (1.1) in the generality with which we wish to consider it, the formula is nevertheless valid in this situation; and the justification for a variant of this formula under the assumption that $\vec{f}$ and c are sufficiently smooth is provided in Section 2. With this as our starting point, formula (1.2) is then established in Section 3 for the situation where $\vec{f}$ and c are sufficiently smooth. Finally in Section 4, after having defined what we mean by a solution of (1.1) in case $\vec{f}$ and c are merely continuous and bounded, we show how the representation (1.2) may be obtained for such solutions as well.

Much work has been done on stochastic representations of solutions to partial differential equations. In particular we mention the work of R. Hersh and his collaborators (see [9] for a survey of these results). Finally, for a general overview of the Feynman-Kac formula cf. B. Simon's monograph [14] as well as his expository article [15].

SECTION 2

In this section we want to describe an approximation scheme for solving (1.1) which is in the same spirit as the Trotter-Lie-Kato formula [16].

To describe the scheme, suppose our eventual aim is to evaluate the solution $\vec{u}$ of (1.1) at (x,t). Then our first step is to divide the interval $(0,t]$ into $2n$ subintervals of equal length: $I_k = (\tau_{k-1}, \tau_k]$, $k = 1, \dots, 2n$, where $\tau_k = kt/2n$, $k = 0, 1, \dots, 2n$. This being done, we then define the function $\vec{u}_n(x,\tau)$ on $\mathbb{R}^p \times [0,t]$ so as to provide us with a good approximation to $\vec{u}(x,t)$, the solution of (1.1) on $\mathbb{R}^p \times [0,t]$, by defining it piecemeal on $\mathbb{R}^p \times I_k$, $k = 1, 2, \dots, 2n$ as follows: On $\mathbb{R}^p \times I_1$ it is defined as the solution of the parabolic initial value problem

$$(2.1)\qquad \begin{cases} \dfrac{\partial \vec{u}_n}{\partial \tau}(x,\tau) = \Delta \vec{u}_n(x,\tau) & \text{in } \mathbb{R}^p \times I_1, \\ \lim\limits_{\tau \downarrow 0} \vec{u}_n(x,\tau) = \vec{f}(x), & x \in \mathbb{R}^p. \end{cases}$$

On $\mathbb{R}^p \times I_2$ is then defined as the solution of the ordinary differential equation initial value problem

$$(2.2)\qquad \begin{cases} \dfrac{\partial \vec{u}_n}{\partial \tau}(x,\tau) = 2c(x,\tau_2)\vec{u}_n(x,\tau) & \text{in } \mathbb{R}^p \times I_2, \\ \lim\limits_{\tau \downarrow \tau_1} \vec{u}_n(x,\tau) = \vec{u}_n(x,\tau_1), & x \in \mathbb{R}^p; \end{cases}$$

in effect the last condition is the requirement that the "initial value" for $\vec{u}_n(x,\tau)$ on $\mathbb{R}^p \times I_2$ be the "final value" obtained for $\vec{u}_n(x,\tau)$ on the preceding slab $\mathbb{R}^p \times I_1$.

We then continue in this vein, defining $\vec{u}_n(x,\tau)$ on each slab $\mathbb{R}^p \times I_k$, $k = 1,2,\dots,2n$ in succession, alternately as the solution of $\frac{\partial \vec{u}_n}{\partial \tau}(x,\tau) = \Delta\vec{u}_n(x,\tau)$ or $\frac{\partial \vec{u}_n}{\partial \tau}(x,\tau) = 2c(x,\tau_k)\vec{u}_n(x,\tau)$ whose "initial value" on the slab is the "final value" on the preceding slab.

More precisely, having defined $\vec{u}_n(x,\tau)$ on $\mathbb{R}^p \times I_j$ for $j = 1,2,\dots,2k$, our definition of $\vec{u}_n(x,\tau)$ on $\mathbb{R}^p \times I_{2k+1}$ is as the solution of

$$\text{(2.3)} \qquad \begin{cases} \dfrac{\partial \vec{u}_n}{\partial \tau}(x,\tau) = \Delta\vec{u}_n(x,\tau) & \text{in } \mathbb{R}^p \times I_{2k+1}, \\ \lim\limits_{\tau\downarrow\tau_{2k}} \vec{u}_n(x,\tau) = \vec{u}_n(x,\tau_{2k}), & x \in \mathbb{R}^p; \end{cases}$$

and then on $\mathbb{R}^p \times I_{2k+2}$ it is defined as the solution of

$$\text{(2.4)} \qquad \begin{cases} \dfrac{\partial \vec{u}_n}{\partial \tau}(x,\tau) = 2c(x,\tau_{2k+2})\vec{u}_n(x,\tau) & \text{in } \mathbb{R}^p \times I_{2k+2}, \\ \lim\limits_{\tau\downarrow\tau_{2k+1}} \vec{u}_n(x,\tau) = \vec{u}_n(x,\tau_{2k+1}), & x \in \mathbb{R}^p. \end{cases}$$

This defines $\vec{u}_n(x,\tau)$ on $\mathbb{R}^p \times (0,t]$. We now define it in $\mathbb{R}^p \times [0,t]$ by requiring it to be continuous there; in view of (2.1) this is tantamount to setting $\vec{u}_n(x,0) = \vec{f}(x)$.

Our aim is now to show that as $n \to \infty$, $\vec{u}_n(x,\tau)$ converges to the solution $\vec{u}(x,\tau)$ of (2.1) and to do so we need the following result.

Lemma 2.1. *Assume that the x-derivatives of $\vec{f}(x)$ and $c(x,\tau)$ of order $\leqslant j$ all exist as continuous bounded functions in $\mathbb{R}^p$ and $\mathbb{R}^p \times [0,T]$, respectively, then the sequence of functions $\{\vec{u}_n(x,\tau)\}$ in $\mathbb{R}^p \times [0,t]$ (with $t \in [0,T]$) constructed above has x-derivatives of order $\leqslant j$ which are continuous in $\mathbb{R}^p \times [0,t]$ and are bounded there, uniformly with respect to n.*

Deferring the proof of this to the appendix, we immediately apply it to establish

Theorem 2.2. *Assume that $\vec{f}(x)$ and $c(x,\tau)$ satisfy the conditions of the preceding lemma with $j = 4$, then*

$$\vec{u}_n(x,\tau) \longrightarrow \vec{u}(x,\tau) \qquad \text{as } n \longrightarrow \infty$$

uniformly on compact subsets of $\mathbb{R}^p \times [0,t]$, where $\vec{u}(x,\tau)$ is the solution of (1.1) *in $\mathbb{R}^p \times [0,t]$ (with $\alpha = \frac{1}{2}$), i.e. $\vec{u}(x,\tau)$ is the solution of*

$$\text{(2.5)} \qquad \begin{cases} \dfrac{\partial \vec{u}}{\partial \tau}(x,\tau) = \dfrac{1}{2}\Delta\vec{u}(x,\tau) + c(x,\tau)\vec{u}(x,\tau) & \text{in } \mathbb{R}^p \times (0,t], \\ \lim\limits_{\tau\downarrow 0} \vec{u}(x,\tau) = \vec{f}(x), & x \in \mathbb{R}^p, \end{cases}$$

which is continuous in the closure $\mathbb{R}^p \times [0,t]$.

Remark. Under the considered hypotheses for $\vec{f}$ and c, problem (2.5) is known to have a unique bounded classical solution (see [7]), Chapter 1).

Proof. By introducing the functions

$$\alpha_n(s) = \begin{cases} 1 \text{ on } I_{2k-1}, & k = 1,2,\dots,n, \\ 0 \text{ on } I_{2k}, & k = 1,2,\dots,n, \end{cases} \tag{2.6}$$

and

$$\beta_n(s) = 1 - \alpha_n(s) \tag{2.7}$$

on $(0,t]$, we can write the defining equations (2.3) and (2.4) for $\vec{u}_n(x,s)$ more concisely in the form

$$\frac{\partial \vec{u}_n}{\partial s}(x,s) = \alpha_n(s)\Delta\vec{u}_n(x,s) + \beta_n(s)2c_n(x,s)\vec{u}_n(x,s), \tag{2.8}$$

where c_n is defined on $\mathbb{R}^p \times [0,t]$ as follows:

$$c_n(x,s) = c(x,\tau_k), \quad (x,s) \in \mathbb{R}^p \times I_k, \quad k = 1,2,\dots,n, \tag{2.9}$$

with $c_n(x,0) = c(x,\tau_1)$; so that in view of the continuity of c

$$c_n(x,s) \longrightarrow c(x,s) \quad \text{as } n \longrightarrow \infty \tag{2.10}$$

uniformly on compact subsets of $\mathbb{R}^p \times [0,t]$.

Of course, it is understood that the derivative $\frac{\partial \vec{u}_n}{\partial s}(x,s)$ in (2.8) only exists for s in the interior of I_k, $k = 1,2,\dots,n$, in fact this equation only makes sense for $s \in \operatorname{Int}(I_k)$, $k = 1,2,\dots,n$. On the other hand, we have arranged the definition of $\vec{u}_n(x,s)$ so that it is continuous as a function of s; consequently the integrated form of (2.8):

$$\vec{u}_n(x,\tau) - \vec{u}_n(x,\sigma) = \int_\sigma^\tau [\alpha_n(s)\Delta\vec{u}_n(x,s) + \beta_n(s)2c_n(x,s)\vec{u}_n(x,s)]\, ds \tag{2.11}$$

is valid for (x,τ) and $(x,\sigma) \in \mathbb{R}^p \times [0,t]$ without restriction.

But now by Lemma 2.1, $\{\vec{u}_n(x,s)\}$ as well as $\{\Delta\vec{u}_n(x,s)\}$ are uniformly bounded with respect to n on $\mathbb{R}^p \times [0,t]$; since $\{\alpha_n\}$, $\{\beta_n\}$ and $\{c_n\}$ are also uniformly bounded, it follows from (2.11) that

$$|\vec{u}_n(x,\tau) - \vec{u}_n(x,\sigma)| \leqslant A|\tau - \sigma| \tag{2.12}$$

uniformly in n for $x \in \mathbb{R}^p$ and $\tau, \sigma \in [0, t]$, where A is a suitable constant. Because of the uniform boundedness of all the first order x-derivatives of $\vec{u}_n(x, s)$ we also have

$$(2.13) \qquad |\vec{u}_n(x, s) - \vec{u}_n(y, s)| \leqslant B|x - y|$$

holding uniformly in n for $x, y \in \mathbb{R}^p$ and $s \in [0, t]$ with B an appropriate constant. Combining (2.12) and (2.13) we arrive at the equicontinuity of $\{\vec{u}_n(x, s)\}$ jointly in x and s for $(x, s) \in \mathbb{R}^p \times [0, t]$.

Applying the operator $\left(\frac{\partial}{\partial x}\right)^\gamma$ with $|\gamma| \leqslant 2$ to both sides of (2.11) and carrying out the differentiations on the right side underneath the integral sign, we obtain an integral formula for $\left(\frac{\partial}{\partial x}\right)^\gamma \vec{u}_n(x, \tau) - \left(\frac{\partial}{\partial x}\right)^\gamma \vec{u}_n(x, \sigma)$ in terms of various x-derivatives of $\vec{u}_n(x, s)$ of orders $\leqslant 4$. Thus an argument similar to the one just carried out in the preceding paragraph allows us to conclude that $\left\{\left(\frac{\partial}{\partial x}\right)^\gamma \vec{u}_n(x, s)\right\}$ with $|\gamma| \leqslant 2$ is also equicontinuous, jointly in x and s for $(x, s) \in \mathbb{R}^p \times [0, t]$.

Because of the equicontinuity and uniform boundedness of $\{\vec{u}_n(x, s)\}$ we can extract a subsequence $\{\vec{u}_{n_k}(x, s)\}$ which converges to a function $\vec{u}(x, s)$ uniformly on compact subsets of $\mathbb{R}^p \times [0, t]$. Further, because of the equicontinuity and uniform boundedness of $\left\{\left(\frac{\partial}{\partial x}\right)^\gamma \vec{u}_n(x, s)\right\}$ for $|\gamma| \leqslant 2$ we can, by passing to a finer subsequence if need be, conclude that $\vec{u}(x, s)$ has x-derivatives of order $\leqslant 2$ and that

$$\left(\frac{\partial}{\partial x}\right)^\gamma \vec{u}_{n_k}(x, s) \longrightarrow \left(\frac{\partial}{\partial x}\right)^\gamma \vec{u}(x, s) \qquad \text{for } |\gamma| \leqslant 2$$

as $k \longrightarrow \infty$, uniformly on compact subsets of $\mathbb{R}^p \times [0, t]$. Replacing n by n_k in (2.11) and then passing to the limit as $k \longrightarrow \infty$, we therefore find that

$$(2.14) \qquad \vec{u}(x, \tau) - \vec{u}(x, \sigma) = \int_\sigma^\tau \left[\frac{1}{2}\Delta\vec{u}(x, s) + c(x, s)\vec{u}(x, s)\right] \mathrm{d}s$$

for $0 \leqslant \sigma < \tau \leqslant t$ and $x \in \mathbb{R}^p$; where to arrive at this result we have used (2.10) together with

Lemma 2.3. *If $\{\varphi_{n_k}(s)\}$ is a sequence of integrable functions which converge uniformly to $\varphi(s)$ on the interval $[0, t]$, then*

$$\lim_{k\to\infty} \int_\sigma^\tau \alpha_{n_k}(s)\varphi_{n_k}(s)\, \mathrm{d}s = \lim_{k\to\infty} \int_\sigma^\tau \beta_{n_k}(s)\varphi_{n_k}(s)\, \mathrm{d}s$$

$$= \frac{1}{2}\int_\sigma^\tau \varphi(s)\, \mathrm{d}s \qquad \text{for } 0 \leqslant \sigma < \tau \leqslant t.$$

□

The proof of Lemma 2.3 is also deferred to the appendix.

From (2.14) we conclude that $\vec{u}(x,\tau)$ has a τ-derivative with

$$\frac{\partial \vec{u}}{\partial \tau}(x,\tau) = \frac{1}{2}\Delta \vec{u}(x,\tau) + c(x,\tau)\vec{u}(x,\tau).$$

in $\mathbb{R}^p \times [0,t]$. Furthermore, since $\vec{u}(x,\tau)$ is continuous in $\mathbb{R}^p \times [0,t]$ and $\vec{u}_{n_k}(x,0) = \vec{f}(x)$ so that $\vec{u}(x,0) = \lim_{k\to\infty} \vec{u}_{n_k}(x,0) = \vec{f}(x)$, it follows that $\vec{u}(x,\tau)$ satisfies the initial condition:

$$\lim_{\tau\downarrow 0} \vec{u}(x,\tau) = \vec{f}(x).$$

Thus the subsequence $\{\vec{u}_{n_k}(x,\tau)\}$ of $\{\vec{u}_n(x,\tau)\}$ does indeed converge uniformly on compact subsets of $\mathbb{R}^p \times [0,t]$ to a solution $\vec{u}(x,\tau)$ of (2.5) in $\mathbb{R}^p \times [0,t]$ which is bounded (due to the uniform boundedness of$\{\vec{u}_n(x,\tau)\}$). Since bounded solutions of (2.5) are unique, it follows that the entire sequence $\{\vec{u}_n(x,\tau)\}$ must converge to this unique solution $\vec{u}(x,\tau)$ of (2.5) uniformly on compact subsets of $\mathbb{R}^p \times [0,t]$. Otherwise, if $\{\vec{u}_n(x,\tau)\}$ failed to so converge to $\vec{u}(x,\tau)$, we could find a compact subset G of $\mathbb{R}^p \times [0,t]$ so that for some subsequence $\{\vec{u}_{n_k}(x,\tau)\}$ and an $\varepsilon_0 > 0$ we would have

$$\sup_G |\vec{u}_{n_k}(x,\tau) - \vec{u}(x,\tau)| \geqslant \varepsilon_0 > 0$$

for $k = 1,2,\ldots$. But by the argument above we can extract a further subsequence from $\{\vec{u}_{n_k}(x,\tau)\}$ converging to $\vec{u}(x,\tau)$ uniformly on compact subsets of $\mathbb{R}^p \times [0,t]$ which is a contradiction; and this completes the proof of Theorem 2.2.

Section 3

Here we will establish formula (1.2) for the solution of (1.1) under the assumption that $\vec{f}$ and c have sufficiently many bounded continuous derivatives with respect to the space variables. By Theorem 2.2 this will ensure that the functions $\vec{u}_n(x,\tau)$ constructed in the approximation scheme described in Section 2 converge to the solution of (1.1).

To establish (1.2), first recall that $\vec{u}_n(x,\tau)$ was defined inductively as follows: Having been defined on $\mathbb{R}^p \times I_j$ for $j = 1,2,\ldots,2k-2$, it was then defined on $\mathbb{R}^p \times I_{2k-1}$ as the solution of

$$\begin{cases} \dfrac{\partial \vec{u}_n}{\partial \tau}(x,\tau) = \Delta \vec{u}_n(x,\tau) & \text{in } \mathbb{R}^p \times I_{2k-1}, \\ \lim\limits_{\tau\downarrow\tau_{2k-2}} \vec{u}_n(x,\tau) = \vec{u}_n(x,\tau_{2k-2}), & x \in \mathbb{R}^p; \end{cases}$$

and then on $\mathbb{R}^p \times I_{2k}$ it was defined as the solution of

$$\begin{cases} \dfrac{\partial \vec{u}_n}{\partial \tau}(x,\tau) = 2c(x,\tau_{2k})\vec{u}_n(x,\tau) & \text{in } \mathbb{R}^p \times I_{2k}, \\ \lim\limits_{\tau \downarrow \tau_{2k-1}} \vec{u}_n(x,\tau) = \vec{u}_n(x,\tau_{2k-1}), & x \in \mathbb{R}^p; \end{cases}$$

with the process begun by taking $\vec{u}_n(x,\tau)$ on $\mathbb{R}^p \times I_1$, as the solution of

$$\begin{cases} \dfrac{\partial \vec{u}_n}{\partial \tau}(x,\tau) = \Delta \vec{u}_n(x,\tau) & \text{in } \mathbb{R}^p \times I_1, \\ \lim\limits_{\tau \downarrow 0} \vec{u}_n(x,\tau) = \vec{f}(x), & x \in \mathbb{R}^p. \end{cases}$$

Accordingly $\vec{u}_n(x,\tau_{2k})$ can be expressed in terms of $\vec{u}_n(x,\tau_{2k-2})$ via the formula

$$\begin{aligned} \vec{u}_n(x,\tau_{2k}) &= \mathrm{e}^{h2c(x,\tau_{2k})}\vec{u}_n(x,\tau_{2k-1}) \\ &= \int_{\mathbb{R}^p} \mathrm{e}^{h2c(x,\tau_{2k})}\Big(\frac{\mathrm{e}^{-|y_k|^2/4h}}{(\pi 4h)^{p/2}} I\Big)\vec{u}_n(x+y_k,\tau_{2k-2})\,\mathrm{d}y_k\,, \qquad k = 2,3,\ldots,n; \end{aligned}$$

while

$$\begin{aligned} \vec{u}_n(x,\tau_2) &= \mathrm{e}^{h2c(x,\tau_2)}\vec{u}_n(x,\tau_1) \\ &= \int_{\mathbb{R}^p} \mathrm{e}^{h2c(x,\tau_2)}\Big(\frac{\mathrm{e}^{-|y_1|^2/4h}}{(\pi 4h)^{p/2}} I\Big)\vec{f}(x+y_1)\,\mathrm{d}y_1\,, \end{aligned}$$

where $h = \tau_j - \tau_{j-1} = t/2n$ and $\mathrm{e}^{-h2c(x,\tau_{2j})}$ denote exponential matrices.

Chaining these formulas together we obtain

$$\begin{aligned} (3.1)\ \vec{u}_n(x,t) &= \vec{u}_n(x,\tau_{2n}) \\ &= \int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} \mathrm{e}^{2hc(x,\tau_{2n})}\Big(\frac{\mathrm{e}^{-|y_n|^2/4h}}{(\pi 4h)^{p/2}} I\Big) \\ &\qquad \times \prod_{j=0}^{n-2} \mathrm{e}^{2hc(x+y_n+y_{n-1}+\cdots+y_{n-j},\tau_{2(n-j-1)})}\Big(\frac{\mathrm{e}^{-|y_{n-j-1}|^2/4h}}{(\pi 4h)^{p/2}} I\Big) \\ &\qquad\qquad \times \vec{f}(x+y_n+y_{n-1}+\cdots+y_2+y_1)\,\mathrm{d}y_1 \cdots \mathrm{d}y_n\,. \end{aligned}$$

Next we use the fact that each of the matrices $\frac{\mathrm{e}^{-|y_k|^2/4h}}{(\pi 4h)^{p/2}} I$ being a diagonal matrix with identical elements along the diagonal, commutes with all other matrices.

Availing ourselves of this together with the change of variables

$$\begin{aligned}
v_1 &= y_n,\\
v_2 &= y_n + y_{n-1},\\
v_3 &= y_n + y_{n-1} + y_{n-2},\\
&\vdots\\
v_n &= y_n + y_{n-1} + y_{n-2} + \cdots + y_1
\end{aligned}$$

as well as setting $t_k = \tau_{2k}$, $k = 0, 1, \cdots, n$, and $\Delta t = t_k - t_{k-1} = 2h$, (3.1) then becomes

$$\begin{aligned}
(3.2)\qquad \vec{u}_n(x,t) &= \vec{u}_n(x,t_n) = \vec{u}_n(x,\tau_{2n})\\
&= \int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} \mathrm{e}^{\Delta t c(x,t_n)} \left(\prod_{k=1}^{n-1} \mathrm{e}^{\Delta t c(x+v_k, t_{n-k})} \right) \vec{f}(x+v_n)\\
&\quad \times \frac{\mathrm{e}^{-|v_1|^2/2t_1}}{[2\pi t_1]^{p/2}} \left(\prod_{k=2}^{n} \frac{\mathrm{e}^{-|v_k - v_{k-1}|^2/2(t_k - t_{k-1})}}{[2\pi(t_k - t_{k-1})]^{p/2}} \right) \mathrm{d}v_1 \cdots \mathrm{d}v_n\,.
\end{aligned}$$

Identifying v_k with $\omega(t_k)$, the values at t_k of a continuous function $\omega\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^p$, with $\omega(0) = 0$, the last integral can be re-interpreted as a Wiener integral; namely, as the Wiener integral of the functional

$$\mathrm{e}^{\Delta t c(x,t_n)} \left(\prod_{k=1}^{n-1} \mathrm{e}^{\Delta t c(x+\omega(t_k), t_{n-k})} \right) \vec{f}(x+\omega(t_n));$$

thus (as $t_n = t$)

$$(3.3)\qquad \vec{u}_n(x,t) = \int_{\Omega} \mathrm{e}^{\Delta t c(x,t)} \left(\prod_{k=1}^{n-1} e^{\Delta t c(x+\omega(t_k), t_{n-k})} \right) \vec{f}(x+\omega(t))\, \mathrm{d}W(\omega),$$

$\mathrm{d}W(w)$ denoting Wiener measure over the space Ω of all continuous functions $\omega\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^p$ with $\omega(0) = 0$. (See [17] p. 443, Theorem 29.6 for a discussion of the underlying formula when $p = 1$).

We now wish to pass to the limit as $n \longrightarrow \infty$ in this formula, and for this purpose we need to identify the limit of the matrix product

$$(3.4)\qquad \mathrm{e}^{\Delta t c(x,t)} \prod_{k=1}^{n-1} \mathrm{e}^{\Delta t c(x+\omega(t_k), t_{n-k})}.$$

It turns out that this converges to $Y(t,t,x,\omega)$, where $Y(s,t,x,\omega)$ for fixed t,x and ω is defined as the fundamental matrix solution of the initial value problem

$$(3.5)\qquad \begin{cases} \dfrac{\mathrm{d}Y}{\mathrm{d}s}(s,t,x,\omega) = c(x+\omega(t-s),s)Y(s,t,x,\omega), & 0 < s \leqslant t, \\ \lim\limits_{s\downarrow 0} Y(s,t,x,\omega) = I. \end{cases}$$

To see this consider what is in effect essentially an Euler polygonal approximation scheme for solving (3.5). Divide $(0,t]$ into n equal intervals by means of the points $s_k = t_k = kt/n$, $k = 0,1,\ldots,n$, and define the function $Y_n(s,t,x,\omega)$ on $(0,t]$ by defining it successively on the intervals $(s_{k-1},s_k]$, $k = 1,2,\ldots,n$, as follows: On $(s_0,s_1] = (0,s_1]$ take it as the solution of the initial value problem

$$(3.6)\qquad \begin{cases} \dfrac{\mathrm{d}Y_n}{\mathrm{d}s}(s,t,x,\omega) = c(x+\omega(t-s_1),s_1)Y_n(s,t,x,\omega), & 0 < s \leqslant s_1, \\ \lim\limits_{s\downarrow 0} Y_n(s,t,x,\omega) = I. \end{cases}$$

Then having defined it on the intervals $(s_{j-1},s_j]$, $j = 1,\ldots,k-1$, define it on $(s_{k-1},s_k]$ as the solution of

$$(3.7)\qquad \begin{cases} \dfrac{\mathrm{d}Y_n}{\mathrm{d}s}(s,t,x,\omega) = c(x+\omega(t-s_k),s_k)Y_n(s,t,x,\omega), & s_{k-1} < s \leqslant s_k, \\ \lim\limits_{s\downarrow s_{k-1}} Y_n(s,t,x,\omega) = Y_n(s_{k-1},t,x,\omega), \end{cases}$$

i.e. the "initial value" for Y_n on $(s_{k-1},s_k]$ is the "final value" for Y_n on $(s_{k-2},s_{k-1}]$.

Solving (3.7) we have

$$Y_n(s_k,t,x,\omega) = \mathrm{e}^{(s_k-s_{k-1})c(x+\omega(t-s_k),s_k)}Y_n(s_{k-1},t,x,\omega)$$

or equivalently

$$Y_n(t_k,t,x,\omega) = \mathrm{e}^{\Delta t c(x+\omega(t_{n-k}),t_k)}Y_n(t_{k-1},t,x,\omega)$$

for $k = 2,\ldots,n$; while for $k = 1$, (3.6) gives

$$Y_n(t_1,t,x,\omega) = \mathrm{e}^{\Delta t c(x+\omega(t_{n-1}),t_1)}.$$

Chaining these together, recalling that $t_n = t$ and $\omega(0) = 0$, we obtain

$$\begin{aligned} (3.8)\qquad Y_n(t,t,x,\omega) &= Y_n(t_n,t,x,\omega) \\ &= \mathrm{e}^{\Delta t c(x,t_n)}\mathrm{e}^{\Delta t c(x+\omega(t_1),t_{n-1})}\mathrm{e}^{\Delta t c(x+\omega(t_2),t_{n-2})}\cdots\mathrm{e}^{\Delta t c(x+\omega(t_{n-1}),t_1)} \\ &= \mathrm{e}^{\Delta t c(x,t)}\prod_{k=1}^{n-1}\mathrm{e}^{\Delta t c(x+\omega(t_k),t_{n-k})} \end{aligned}$$

precisely the matrix product (3.4); and so (3.3) may be written

$$\vec{u}_n(x,t) = \int_\Omega Y_n(t,t,x,\omega)\vec{f}(x+\omega(t))\,\mathrm{d}W(\omega). \tag{3.9}$$

Finally, to prove that $Y_n \longrightarrow Y$, we integrate the defining equations (3.6) and (3.7) for Y_n, which leads to the integral equation

$$Y_n(s,t,x,\omega) = I + \int_0^s c_n(\sigma)Y_n(\sigma,t,x,\omega)\,\mathrm{d}\sigma\,, \qquad s \in (0,t]$$

where for fixed t, x and $\omega, c_n(\sigma)$ is defined by

$$c_n(\sigma) = c(x+\omega(t-s_k), s_k), \qquad \sigma \in (s_{k-1}, s_k],\ k = 1, 2 \ldots n.$$

Now compare this with the integral equation satisfied by Y:

$$Y(s,t,x,\omega) = I + \int_0^s c(x+\omega(t-\sigma),\sigma)Y(\sigma,t,x,\omega)\,\mathrm{d}\sigma\,, \qquad s \in (0,t].$$

Since

$$c_n(\sigma) \longrightarrow c(x+\omega(t-\sigma),\sigma) \qquad \text{as } n \longrightarrow \infty$$

uniformly in $(0,t]$, an application of Lemma 5.4. in the appendix allows us to conclude that

$$Y_n(s,t,x,\omega) \longrightarrow Y(s,t,x,\omega) \qquad \text{as } n \longrightarrow \infty$$

for each fixed $s \in (0,t]$ (in fact uniformly on $(0,t]$). Furthermore, on account of the boundedness of $c(x,\tau)$ in $\mathbb{R}^p \times [0,t]$, it follows from Lemma 5.3 that the Y_n's are uniformly bounded. We may therefore apply the Lebesgue dominated convergence theorem to pass to the limit in the integral on the right of (3.9); and doing so in conjunction with Theorem 2.2 we arrive at

$$\vec{u}(x,t) = \int_\Omega Y(t,t,x,\omega)\vec{f}(x+\omega(t))\,\mathrm{d}W(\omega),$$

the desired result.

In this section we will establish (1.2) for those solutions of (1.1) in which $\vec{f}$ and c are only assumed to be bounded continuous functions in $\mathbb{R}^p$ and $\mathbb{R}^p \times [0,T]$ respectively. But before doing so we need to explain what we mean by a solution of (1.1) under such circumstances.

If $\vec{f}$ and c are given bounded continuous functions in $\mathbb{R}^p \times [0,T]$ and $\mathbb{R}^p$ respectively, it is possible that (1.1) might not have a classical solution $\vec{u}(x,t)$ corresponding to them. However, if a bounded classical solution $\vec{u}(x,t)$ does exists, then it can be shown (see [11], p. 347) that it will have to satisfy the integral equation

$$\vec{u}(x,t) = \int_0^t \int_{\mathbb{R}^p} F(x-\xi, t-\tau) c(\xi,\tau) \vec{u}(\xi,\tau)\, d\xi\, d\tau + \int_{\mathbb{R}^p} F(x-\xi,t)\vec{f}(\xi)\, d\xi \tag{4.1}$$

in $\mathbb{R}^p \times [0,T]$ where

$$F(x,t) = \frac{e^{-|x|^2/2t}}{(2\pi t)^{p/2}}, \qquad t > 0. \tag{4.2}$$

Conversely, if $c(x,t)$ is sufficiently smooth, for example if $c(x,t)$ is locally Hölder continuous in x uniformly with respect to t, then any bounded continuous solution of (4.1) is also a classical solution of (1.1) (see [7], Thm. 12, p. 25). This suggests that an appropriate definition of a solution of (1.1) when $\vec{f}$ and c are merely bounded continuous functions is as a solution of the integral equation (4.1) and we adopt this as our definition.

Using the method of iteration it is easy to show that (4.1) does indeed possess a unique bounded continuous solution under these assumptions for $\vec{f}$ and c. In fact, writing (4.1) over in the form

$$\vec{u} = \vec{v} + K\vec{u}$$

where

$$\vec{v}(x,t) = \int_{\mathbb{R}^p} F(x-\xi,t)\vec{f}(\xi)\, d\xi \tag{4.3}$$

and K denotes the operator

$$K\vec{w}(x,t) = \int_0^t \int_{\mathbb{R}^p} F(x-\xi, t-\tau) c(\xi,\tau) \vec{w}(\xi,\tau)\, d\xi\, d\tau,$$

one can derive the estimates

$$\|K^n\vec{w}\| \leqslant \frac{(MT)^n}{n!}\|\vec{w}\|, \qquad n = 1, 2, \ldots$$

where M is a bound for $c(x,t)$ in $\mathbb{R}^p \times [0,T]$ and

$$\|\vec{\varphi}\| = \sup_{\mathbb{R}^p \times [0,T]} |\varphi(x,t)|; \tag{4.4}$$

from which it follows immediately that the series of iterants

$$\vec{v} + K\vec{v} + K^2\vec{v} + \cdots$$

converges uniformly in $\mathbb{R}^p \times [0,T]$ to the unique solution of $\vec{u} = \vec{v} + K\vec{u}$ in the Banach space of bounded continuous functions in $\mathbb{R}^p \times [0,T]$ normed by (4.4). (Recall that $\vec{f}$ is assumed to be bounded and continuous in $\mathbb{R}^p$ so that $\vec{v}(x,t)$ as defined by (4.3) is in the Banach space being considered. More precisely this is so for the function $\vec{v}(x,t)$ defined for $t > 0$ by (4.3) and taken to be $\vec{f}(x)$ for $t = 0$. In order to avoid some clumsiness in connection with this, we adopt the convention that if $\vec{f}(\xi)$ is a bounded continuous function in $\mathbb{R}^p$ the expression $\int_{\mathbb{R}^p} F(x-\xi,t)\vec{f}(\xi)\,\mathrm{d}\xi$ is to be interpreted as $\vec{f}(x)$ when $t = 0$.)

Having explained what we mean by a solution of (1.1) when $\vec{f}$ and c are only assumed to be bounded, continuous functions and having shown it to exist, we now turn to the proof of formula (1.2) for such solutions. This will be accomplished by approximating such solutions by solutions corresponding to smooth $\vec{f}$ and c's for which formula (1.2) has already been established. The foundation for this argument is the following result.

Lemma 4.1. *Assume that $g(x)$ is a bounded continuous function in $\mathbb{R}^p$, then the functions*

$$g_\delta(x) = \int_{\mathbb{R}^p} \frac{\mathrm{e}^{-|x-\xi|^2/4\delta}}{(4\pi\delta)^{p/2}} g(\xi)\,\mathrm{d}\xi\,, \qquad \delta > 0 \tag{4.5}$$

have continuous derivatives of all orders that are bounded in $\mathbb{R}^p$, and

$$g_\delta(x) \longrightarrow g(x) \qquad \text{as } \delta \downarrow 0 \tag{4.6}$$

pointwise and boundedly in $\mathbb{R}^p$ (in fact the convergence is uniform over compact subsets of $\mathbb{R}^p$).

Proof. Differentiating we have

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} g_\delta(x) = \int_{\mathbb{R}^p} \left(\frac{\partial}{\partial x}\right)^{\alpha} \frac{e^{-|x-\xi|^2/4\delta}}{(4\pi\delta)^{p/2}} g(\xi)\, d\xi\,, \qquad \delta > 0,$$

so that

$$\sup_{\mathbb{R}^p} \left| \left(\frac{\partial}{\partial x}\right)^{\alpha} g_\delta(x) \right| \leqslant \sup_{\mathbb{R}^p} |g(\xi)| \int_{\mathbb{R}^p} \left| \left(\frac{\partial}{\partial \xi}\right)^{\alpha} \frac{e^{-|\xi|^2/4\delta}}{(4\pi\delta)^{p/2}} \right| d\xi < \infty$$

in view of the finiteness of the integral on the right for $\delta > 0$.

To establish the convergence (4.6) we write

$$\begin{aligned} g_\delta(x) &= \int_{\mathbb{R}^p} \frac{e^{-|x-\xi|^2/4\delta}}{(4\pi\delta)^{p/2}} g(\xi)\, d\xi = \int_{\mathbb{R}^p} \frac{e^{-|\xi|^2/4\delta}}{(4\pi\delta)^{p/2}} g(x-\xi)\, d\xi \\ &= \int_{\mathbb{R}^p} \frac{e^{-|y|^2}}{(4\pi)^{p/2}}\, g(x - 2\delta^{1/2} y)\, dy \end{aligned}$$

where we have made the change of variable $y = \xi/2\delta^{1/2}$. The result (4.6) then follows by passing to the limit as $\delta \downarrow 0$ underneath the integral sign, using the Lebesgue dominated convergence theorem. Clearly the convergence is bounded since $\sup_{\mathbb{R}^p} |g_\delta(x)| \leqslant \sup_{\mathbb{R}^p} |g(x)|$.

Suppose now that for given continuous and bounded functions $\vec{f}$ and c in $\mathbb{R}^p$ and $\mathbb{R}^p \times [0,T]$ respectively, we consider the corresponding solution $\vec{u}$ of (1.1), by which we mean the solution $\vec{u}$ of the integral equation (4.1). To establish (1.2) for such $\vec{u}$, we first construct approximations to $\vec{f}$ and c by applying (4.5) to $\vec{f}$ and c respectively with $\delta = 1/n$, $n = 1, 2, \ldots$. Denoting the resulting functions by $\vec{f}_n$ and c_n, $n = 1, 2, \ldots$, we then consider the solutions $\vec{u}_n$ of (1.1) corresponding to them. Since $\vec{f}_n$ and c_n have bounded continuous derivatives of all orders in the space variables, $\vec{u}_n$ is actually a classical solution of (1.1) which will also have to be a solution of the integral equation

$$\vec{u}_n(x,t) = \int_0^t \int_{\mathbb{R}^p} F(x-\xi, t-\tau) c_n(\xi,\tau) \vec{u}_n(\xi,\tau)\, d\xi\, d\tau + \int_{\mathbb{R}^p} F(x-\xi,t) \vec{f}_n(\xi)\, d\xi \tag{4.7}$$

in $\mathbb{R}^p \times [0,T]$. Furthermore, in the light of the result established in Section 3, the representation (1.2) is valid for $\vec{u}_n$:

$$\vec{u}_n(x,t) = \int_\Omega Y_n(t,t,x,\omega) \vec{f}_n(x + \omega(t))\, dW(\omega), \tag{4.8}$$

where $Y_n(s,t,x,\omega)$ is the solution of

$$\begin{cases} \dfrac{\mathrm{d}Y_n}{\mathrm{d}s}(s,t,x,\omega) = c_n(x+\omega(t-s),s)Y_n(s,t,x,\omega), & 0 < s \leqslant t, \\ \lim\limits_{s \downarrow 0} Y_n(s,t,x,\omega) = I. \end{cases}$$

We now pass to the limit in (4.8) as $n \longrightarrow \infty$. On the right, for fixed t, x and ω, we have in view of (4.6) that $\vec{f}_n(x+\omega(t)) \longrightarrow \vec{f}(x+\omega(t))$ with the convergence being bounded. Again, because of (4.6), for fixed s, t, x, and ω with $s \in [0,t]$

$$c_n(x+\omega(t-s),s) \longrightarrow c(x+\omega(t-s),s)$$

boundedly as $n \longrightarrow \infty$; consequently by Lemmas 5.3 and 5.4 $Y_n(s,t,x,\omega) \longrightarrow Y(s,t,x,\omega)$ pointwise and boundedly for $s \in [0,t]$ as $n \longrightarrow \infty$, where $Y(s,t,x,\omega)$ is the solution of (1.3). Applying the Lebesgue dominated convergence theorem we may therefore pass to the limit on the right of (4.8) underneath the integral sign which results in

$$\lim_{n\to\infty} \vec{u}_n(x,t) = \int_\Omega Y(t,t,x,\omega)\vec{f}(x+\omega(t))\,\mathrm{d}W(\omega)$$

for $(x,t) \in \mathbb{R}^p \times [0,T]$. The proof will now be completed by showing that

$$\vec{u}_n(x,t) \longrightarrow \vec{u}(x,t) \quad \text{as } n \longrightarrow \infty \tag{4.9}$$

pointwise in $\mathbb{R}^p \times [0,T]$.

To accomplish this we note that as $\vec{u}$ and $\vec{u}_n$ satisfy equations (4.1) and (4.7) respectively, their difference may be expressed in the form

$$\begin{aligned} &\vec{u}_n(x,t) - \vec{u}(x,t) \\ &= \int_0^t \int_{\mathbb{R}^p} F(x-\xi,t-\tau)c_n(\xi,\tau)[\vec{u}_n(\xi,\tau) - \vec{u}(\xi,\tau)]\,\mathrm{d}\xi\,\mathrm{d}\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^p} F(x-\xi,t-\tau)[c_n(\xi,\tau) - c(\xi,\tau)]\vec{u}(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \\ &\quad + \int_{\mathbb{R}^p} F(x-\xi,t)[\vec{f}_n(\xi) - \vec{f}(\xi)]\,\mathrm{d}\xi \qquad \text{for } (x,t) \in \mathbb{R}^p \times [0,T]. \end{aligned}$$

Taking norms this leads to

$$\begin{aligned} &|\vec{u}_n(x,t) - \vec{u}(x,t)| \\ &\leqslant \int_0^t \int_{\mathbb{R}^p} F(x-\xi,t-\tau)M|\vec{u}_n(\xi,\tau) - \vec{u}(\xi,\tau)|\,\mathrm{d}\xi\,\mathrm{d}\tau + |\vec{\varphi}_n(x,t)|, \\ &\qquad\qquad (x,t) \in \mathbb{R}^p \times [0,T], \end{aligned} \tag{4.10}$$

where M is a bound for $c_n(\xi,\tau)$ in $\mathbb{R}^p \times [0,T]$ which is uniform with respect to n and

(4.11)
$$\vec{\varphi}_n(x,t) = \int_0^t \int_{\mathbb{R}^p} F(x-\xi,t-\tau)[c_n(\xi,\tau) - c(\xi,\tau)]\vec{u}(\xi,\tau)\,d\xi\,d\tau + \int_{\mathbb{R}^p} F(x-\xi,t)[\vec{f}_n(\xi) - \vec{f}(\xi)]\,d\xi, \quad (x,t) \in \mathbb{R}^p \times [0,T].$$

By Lemma 5.5 in the appendix, the integral inequality (4.10) satisfied by $|\vec{u}_n - \vec{u}|$ implies the following estimate for this quantity:

(4.12)
$$|\vec{u}_n(x,t) - \vec{u}(x,t)| \leqslant |\vec{\varphi}_n(x,t)| + \int_0^t \int_{\mathbb{R}^p} Me^{M(t-\tau)}F(x-\xi,t-\tau)|\vec{\varphi}_n(\xi,\tau)|\,d\xi\,d\tau$$

for $(x,t) \in \mathbb{R}^p \times [0,T]$. But now since $\vec{f}_n \longrightarrow \vec{f}$ and $c_n \longrightarrow c$ pointwise and boundedly, it follows from (4.11) that

$$\vec{\varphi}_n(x,t) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

pointwise in $\mathbb{R}^p \times [0,T]$, by virtue of the Lebesgue dominated convergence theorem; moreover this convergence is also bounded. Therefore, by one more application of the Lebesgue theorem to the integral on the right of (4.12), we are able to conclude that $\vec{u}_n(x,t) \longrightarrow \vec{u}(x,t)$ as $n \longrightarrow \infty$ pointwise in $\mathbb{R}^p \times [0,T]$; and this completes the proof. □

Appendix

Here we will give the proofs of various technical results referred to in the body of the paper. We begin with Lemma 2.1 whose statement we repeat for the convenience of the reader.

Lemma 2.1. *Assume that the x-derivatives of $\vec{f}(x)$ and $c(x,\tau)$ of order $\leqslant j$ all exist as continuous bounded functions in $\mathbb{R}^p$ and $\mathbb{R}^p \times [0,T]$, respectively, then the sequence of functions $\{\vec{u}_n(x,\tau)\}$ in $\mathbb{R}^p \times [0,t]$ (with $t \in (0,T]$) constructed above has x-derivatives of order $\leqslant j$ which are continuous in $\mathbb{R}^p \times [0,t]$ and are bounded there, uniformly with respect to n.*

For the proof of this as well as for what is to come in the sequel it will be convenient to introduce the following notations: Assuming $\vec{g}$ and q to be an m-vector and an $m \times m$ matrix respectively, $|\vec{g}|$ will denote a vector norm and $||q||$ the corresponding matrix norm. If $\vec{g} = \vec{g}(x)$ and $q = q(x)$ are functions of x, $D_x^j \vec{g}(x)$ and $D_x^j q(x)$ will denote the totality of derivatives of order $\leqslant j$ for $\vec{g}(x)$ and $q(x)$, respectively with

$$|D_x^j \vec{g}(x)| = \max_{0 \leqslant |\alpha| \leqslant j} \left| \left(\frac{\partial}{\partial x}\right)^\alpha \vec{g}(x) \right|$$

and

$$\|D_x^j q(x)\| = \max_{0 \leqslant |\alpha| \leqslant j} \left\| \left(\frac{\partial}{\partial x}\right)^\alpha q(x) \right\|.$$

(Here we are using the Schwartz notation: α denotes the multi-index $[\alpha_1, \alpha_2, \dots, \alpha_p]$ with non-negative integer components; $\left(\frac{\partial}{\partial x}\right)^\alpha$ then represents the partial differential operator $\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_p}\right)^{\alpha_p}$ of order $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_p$.)

The proof of Lemma 2.1 will be a simple consequence of the two results which follow.

Lemma 5.1. *Let $\vec{v}(x,\tau)$ be a bounded solution of*

$$\begin{cases} \dfrac{\partial \vec{v}}{\partial \tau}(x,\tau) = \Delta \vec{v}(x,\tau) & \text{in } \mathbb{R}^p \times (a,b], \\ \lim\limits_{\tau \downarrow a} \vec{v}(x,\tau) = \vec{\varphi}(x), & x \text{ in } \mathbb{R}^p \end{cases}$$

for $\vec{\varphi}(x)$ a given bounded continuous function in $\mathbb{R}^p$. Suppose that the derivatives $\left(\frac{\partial}{\partial x}\right)^\alpha \vec{\varphi}(x)$ of order $|\alpha| \leqslant j$ all exist as continuous bounded functions in $\mathbb{R}^p$, then the x-derivatives $\left(\frac{\partial}{\partial x}\right)^\alpha \vec{v}(x,\tau)$ of order $|\alpha| \leqslant j$ all exist as continuous bounded functions in $\mathbb{R}^p \times (a,b]$ and

$$\sup_{\mathbb{R}^p \times (a,b]} |D_x^j \vec{v}(x,\tau)| \leqslant \sup_{\mathbb{R}^p} |D_x^j \varphi(x)| \tag{5.1}$$

Proof. Since we may differentiate under the integral sign in the representation

$$\vec{v}(x,\tau) = \int_{\mathbb{R}^p} \frac{e^{-|\xi|^2/4(\tau-a)}}{[4\pi(\tau-a)]^{p/2}} \vec{\varphi}(x-\xi)\,d\xi, \qquad (x,\tau) \in \mathbb{R}^p \times (a,b],$$

the derivatives $\left(\frac{\partial}{\partial x}\right)^\alpha \vec{v}(x,\tau)$ clearly exist and are given by the formula

$$\left(\frac{\partial}{\partial x}\right)^\alpha \vec{v}(x,\tau) = \int_{\mathbb{R}^p} \frac{e^{-|\xi|^2/4(\tau-a)}}{[4\pi(\tau-a)]^{p/2}} \left(\frac{\partial}{\partial x}\right)^\alpha \vec{\varphi}(x-\xi)\,d\xi$$

for $(x,\tau) \in \mathbb{R}^p \times (a,b]$. From this one also sees that the derivatives are continuous. Finally, the estimate (5.1) is a result of the evaluation

$$\int_{\mathbb{R}^p} \frac{e^{-|\xi|^2/4(\tau-a)}}{[4\pi(\tau-a)]^{p/2}}\,d\xi = 1 \qquad \text{for all } \tau > a.$$

□

Lemma 5.2. *Let $w(x,\tau)$ be a solution of the initial value problem*

$$(5.2)\qquad \begin{cases} \dfrac{\partial \vec{w}}{\partial \tau}(x,\tau) = q(x)\vec{w}(x,\tau) & \text{in } \mathbb{R}^p \times (a,b], \\ \lim\limits_{\tau\downarrow a} \vec{w}(x,\tau) = \vec{\psi}(x), & x \in \mathbb{R}^p \end{cases}$$

and suppose that the derivatives $\left(\frac{\partial}{\partial x}\right)^\alpha \vec{\psi}(x)$ and $\left(\frac{\partial}{\partial x}\right)^\alpha q(x)$ of the initial function $\vec{\psi}(x)$ and $m \times m$ coefficient matrix $q(x)$, of order $|\alpha| \leqslant j$ all exist as continuous and bounded functions in $\mathbb{R}^p$. Then the derivatives $\left(\frac{\partial}{\partial x}\right)^\alpha \vec{w}(x,\tau)$ of order $|\alpha| \leqslant j$ all exist as continuous and bounded functions in $\mathbb{R}^p \times (a,b]$ and they satisfy the estimate

$$(5.3)\qquad \sup_{\mathbb{R}^p \times (a,b]} |D_x^j w(x,\tau)| \leqslant \mathrm{e}^{P(b-a)} \sup_{\mathbb{R}^p} |D_x^j \vec{\psi}(x)|$$

where P is a constant depending only on $\sup\limits_{\mathbb{R}^p} \|D_x^j q(x)\|$.

Proof. The existence of the derivatives $\left(\frac{\partial}{\partial x}\right)^\alpha \vec{w}(x,\tau)$ is a standard result from the theory of ordinary differential equations. To obtain the estimate (5.3), we note that $\left(\frac{\partial}{\partial x}\right)^\alpha \vec{w}(x,\tau)$ will have to be the solution of the initial value problem obtained by formally differentiating (5.2) with respect to x:

$$\begin{cases} \dfrac{\partial}{\partial \tau}\Big[\Big(\dfrac{\partial}{\partial x}\Big)^\alpha \vec{w}(x,\tau)\Big] = \Big(\dfrac{\partial}{\partial x}\Big)^\alpha \Big[q(x)\vec{w}(x,\tau)\Big] \\ = q(x)\Big(\dfrac{\partial}{\partial x}\Big)^\alpha \vec{w}(x,\tau) + \sum\limits_{\substack{\beta+\gamma=\alpha \\ |\beta|\geqslant 1}} \dfrac{\alpha!}{\beta!\gamma!}\Big[\Big(\dfrac{\partial}{\partial x}\Big)^\beta q(x)\Big]\Big[\Big(\dfrac{\partial}{\partial x}\Big)^\gamma \vec{w}(x,\tau)\Big] \text{ in } \mathbb{R}^p \times (a,b], \\ \lim\limits_{\tau\downarrow a} \Big(\dfrac{\partial}{\partial x}\Big)^\alpha \vec{w}(x,\tau) = \Big(\dfrac{\partial}{\partial x}\Big)^\alpha \vec{\psi}(x), \qquad x \in \mathbb{R}^p, \end{cases}$$

where we have used Leibniz's formula for the derivative of a product with $\delta!$ denoting $(\delta_1!)(\delta_2!)\cdots(\delta_p!)$ when $\delta = [\delta_1, \delta_2, \ldots, \delta_p]$.

The solution of this problem admits the following representation in terms of the exponential matrix $\mathrm{e}^{sq(x)}$:

$$\Big(\frac{\partial}{\partial x}\Big)^\alpha \vec{w}(x,\tau) = \mathrm{e}^{(\tau-a)q(x)}\Big(\frac{\partial}{\partial x}\Big)^\alpha \vec{\psi}(x)$$
$$+ \int_a^\tau e^{(\tau-\sigma)q(x)}\Big\{ \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\geqslant 1}} \frac{\alpha!}{\beta!\gamma!}\Big[\Big(\frac{\partial}{\partial x}\Big)^\beta q(x)\Big]\Big[\Big(\frac{\partial}{\partial x}\Big)^\gamma \vec{w}(x,\sigma)\Big]\Big\}\,\mathrm{d}\sigma$$

in $\mathbb{R}^p \times (a,b]$; from which we see that

$$(5.4)\quad \sup_{\mathbb{R}^p \times (a,b]} |D_x^j \vec{w}(x,\tau)| \leqslant e^{Q(b-a)} \sup_{\mathbb{R}^p} |D_x^j \vec{\psi}(x)| + (b-a)\left(e^{Q(b-a)} 2^j N\right) \sup_{\mathbb{R}^p \times (a,b]} |D_x^j \vec{w}(x,\tau)|$$

where $Q = \sup_{\mathbb{R}^p} \|q(x)\|$ and $N = \sup_{\mathbb{R}^p} \|D_x^j q(x)\|$.

Now set $L = e^Q 2^j N$ and note that $e^{Q(b-a)} 2^j N \leqslant L$ when $(b-a) \leqslant 1$. Hence from (5.4) we find that

$$[1 - L(b-a)] \sup_{\mathbb{R}^p \times (a,b]} |D_x^j \vec{w}(x,\tau)| \leqslant e^{Q(b-a)} \sup_{\mathbb{R}^p} |D_x^j \vec{\psi}(x)|$$

if $(b-a) \leqslant 1$. Making use of the inequality $e^{-2\theta} \leqslant 1 - \theta$ for $0 \leqslant \theta \leqslant \frac{1}{2}$, with $\theta = L(b-a)$, the preceding implies that

$$\sup_{\mathbb{R}^p \times (a,b]} |D_x^j \vec{w}(x,\tau)| \leqslant e^{P(b-a)} \sup_{\mathbb{R}^p} |D_x^j \vec{\psi}(x)|$$

where $P = Q + 2L = Q + 2e^Q 2^j N$, provided that $(b-a)$ is sufficiently small: $(b-a) \leqslant \min(1, 1/2L)$.

Thus (5.3) has been established under the assumption that $(b-a)$ is sufficiently small. The latter condition is not really essential as we can see by subdividing $(a,b]$ by the points $a = a_0 < a_1 < \cdots < a_n = b$, in such a way that the subintervals $(a_{\nu-1}, a_\nu]$ are so small that (5.3) is applicable to $\mathbb{R}^p \times (a_{\nu-1}, a_\nu]$:

$$\sup_{\mathbb{R}^p \times (a_{\nu-1}, a_\nu]} |D_x^j \vec{w}(x,\tau)| \leqslant e^{P(a_\nu - a_{\nu-1})} \sup_{\mathbb{R}^p} |D_x^j \vec{w}(x, a_{\nu-1})|.$$

Putting these together, we obtain

$$\begin{aligned} \sup_{\mathbb{R}^p \times (a_{k-1}, a_k]} |D_x^j \vec{w}(x,\tau)| &\leqslant \left(\prod_{\nu=1}^{k} e^{P(a_\nu - a_{\nu-1})}\right) \sup_{\mathbb{R}^p} |D_x^j \vec{w}(x, a_0)| \\ &= e^{P(a_k - a)} \sup_{\mathbb{R}^p} |D_x^j \vec{\psi}(x)| \\ &\leqslant e^{P(b-a)} \sup_{\mathbb{R}^p} |D_x^j \vec{\psi}(x)| \end{aligned}$$

for $k = 1, 2, \ldots, n$, which proves (5.3) without any smallness requirement on $(b-a)$.

□

Proof of Lemma 2.1. Recall that our approximation scheme consisted in the construction of a sequence of functions $\{\vec{u}_n(x,\tau)\}$ on $\mathbb{R}^p \times [0,t]$ in which $\vec{u}_n(x,\tau)$ was defined by alternately solving the equations

$$\frac{\partial \vec{u}_n}{\partial \tau}(x,\tau) = \Delta \vec{u}_n(x,\tau) \qquad \text{on } \mathbb{R}^p \times I_{2k+1}$$

and

$$\frac{\partial \vec{u}_n}{\partial \tau}(x,\tau) = 2c(x,\tau_{2k+2})\vec{u}_n(x,\tau) \qquad \text{on } \mathbb{R}^p \times I_{2k+2}$$

using as our "initial values" for $\vec{u}_n$ in any particular slab $\mathbb{R}^p \times I_k$, the "final values" obtained for $\vec{u}_n$ in the preceding slab $\mathbb{R}^p \times I_{k-1}$; the entire process being started by defining $\vec{u}_n$ on $\mathbb{R}^p \times I_1$ as the solution of the initial value problem

$$\begin{cases} \dfrac{\partial \vec{u}_n}{\partial \tau}(x,\tau) = \Delta \vec{u}_n(x,\tau) & \text{in } \mathbb{R}^p \times I_1 \\ \lim\limits_{\tau \downarrow 0} \vec{u}_n(x,\tau) = \vec{f}(x) & \text{in } \mathbb{R}^p. \end{cases}$$

To prove the lemma we will estimate the growth of $|D_x^j \vec{u}_n(x,\tau)|$ slab by slab. For this purpose we set

$$\sigma_k = \sup_{\mathbb{R}^p \times I_k} |D_x^j \vec{u}_n(x,\tau)|, \qquad k = 1,2,\ldots,n$$

and

$$\sigma_0 = \sup_{\mathbb{R}^p} |D_x^j \vec{u}_n(x,0)| = \sup_{\mathbb{R}^p} |D_x^j \vec{f}(x)|.$$

Then by Lemma 5.1

$$\sigma_{2k+1} \leqslant \sigma_{2k}, \qquad k = 0,1,\ldots,n-1, \tag{5.5}$$

and by Lemma 5.2

$$\sigma_{2k} \leqslant e^{Ah}\sigma_{2k-1}, \qquad k = 1,2,\ldots,n, \tag{5.6}$$

where $h = t/2n =$ the length of each I_k and A is a constant depending on $\sup\limits_{\mathbb{R}^p \times [0,t]} \|D_x^j c(x,\tau)\|$.

Combining (5.5) and (5.6) we find that

$$\sigma_{2k} \leqslant e^{Ah}\sigma_{2k-2}, \qquad k = 1,2,\ldots,n$$

which implies that

$$\sigma_{2k} \leqslant e^{1kh}\sigma_0, \qquad k = 1,2,\ldots,n.$$

Since $kh = \tau_{2k}/2 \leqslant t/2$ and by (5.5) $\sigma_{2k+1} \leqslant \sigma_{2k}$ for $k = 0, 1, \ldots, n-1$, the preceding enables us to conclude that

$$\sup_{\mathbf{R}^\nu \times [0,t]} |D_x^j \vec{u}_n(x,\tau)| \leqslant \mathrm{e}^{At/2} \sup_{\mathbf{R}^\nu} |D_x^j \vec{f}(x)|$$

independent of n; the desired result. □

Next we turn to the proof of Lemma 2.3 whose statement we also repeat for convenience.

Lemma 2.3. *If $\{\varphi_{n_k}(s)\}$ is a sequence of integrable functions which converge uniformly to $\varphi(s)$ on the interval $[0,t]$, then*

$$\lim_{k\to\infty} \int_\sigma^\tau \alpha_{n_k}(s)\varphi_{n_k}(s)\,\mathrm{d}s = \lim_{k\to\infty} \int_\sigma^\tau \beta_{n_k}(s)\varphi_{n_k}(s)\,\mathrm{d}s$$
$$= \frac{1}{2}\int_\sigma^\tau \varphi(s)\,\mathrm{d}s\,, \qquad 0 \leqslant \sigma < \tau \leqslant t.$$

Proof of Lemma 2.3. Recall the definition of α_n and β_n on $(0,t]$:

$$\alpha_n(s) = \begin{cases} 1 & \text{on } I_{2k-1},\ k = 1, \ldots, n, \\ 0 & \text{on } I_{2k},\ k = 1, 2, \ldots, n, \end{cases}$$

and

$$\beta_n(s) = 1 - \alpha_n(s).$$

First we prove the result involving the α_n's. Since $\varphi_{n_k} \longrightarrow \varphi$ uniformly on $[0,t]$,

$$\int_\sigma^\tau \alpha_{n_k}(s)[\varphi_{n_k}(s) - \varphi(s)]\,\mathrm{d}s \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty;$$

and so it is enough to prove that

$$\lim_{k\to\infty} \int_\sigma^\tau \alpha_{n_k}(s)\varphi(s)\,\mathrm{d}s = \frac{1}{2}\int_\sigma^\tau \varphi(s)\,\mathrm{d}s\,,$$

or somewhat more generally

$$\lim_{n\to\infty} \int_\sigma^\tau \alpha_n(s)\varphi(s)\,\mathrm{d}s = \frac{1}{2}\int_\sigma^\tau \varphi(s)\,\mathrm{d}s \tag{5.7}$$

for an arbitrary integrable function $\varphi(s)$ on $[0,t]$. Clearly (5.7) is true if $\varphi(s)$ is the characteristic function of an interval; since linear combinations of these are dense in the space of integrable functions on $[0,t]$, it follows that (5.7) holds for any integrable function $\varphi(s)$ on $[0,t]$. □

From this the corresponding result involving the β_n's is an immediate consequence of the relation $\beta_n = 1 - \alpha_n$.

Lemma 5.3. *Let $A(s)$ be a bounded measurable matrix valued function on $[0,t]$ and let $Y(s)$ be the matrix solution of the integral equation*

$$Y(s) = X + \int_0^s A(\sigma)\,\mathrm{d}\sigma\,, \qquad 0 \leqslant s \leqslant t,$$

for X a given matrix. Then $Y(s)$ has a bound on $[0,t]$ which depends only X, t and the bound for $A(s)$; in fact

$$\sup_{0\leqslant s\leqslant t} \|Y(s)\| \leqslant \mathrm{e}^{Kt}\|X\|, \tag{5.8}$$

where $K = \sup_{0\leqslant s\leqslant t} \|A(s)\|$.

Proof. Taking norms on both sides of the integral equation satisfied by Y we have

$$\|Y(s)\| \leqslant \|X\| + \int_0^s \|A(\sigma)\|\ \|Y(\sigma)\|\,\mathrm{d}\sigma \ \leqslant \|X\| + K\int_0^s \|Y(\sigma)\|\,\mathrm{d}\sigma\,, \qquad 0 \leqslant s \leqslant t.$$

By Gronwall's inequality this immediately implies that

$$\|Y(s)\| \leqslant \mathrm{e}^{Ks}\|X\|, \qquad 0 \leqslant s \leqslant t;$$

from which (5.8) is apparent. □

Lemma 5.4. *Let $A(s)$ and $A_n(s), n = 1,2,\ldots,$ be bounded measurable matrix valued functions on $[0,t]$ and consider the matrix solutions Y and Y_n of the corresponding integral equations*

$$Y(s) = X + \int_0^s A(\sigma)Y(\sigma)\,\mathrm{d}\sigma\,, \qquad 0 \leqslant s \leqslant t$$

and

$$Y_n(s) = X + \int_0^s A_n(\sigma)Y_n(\sigma)\,\mathrm{d}\sigma\,, \qquad 0 \leqslant s \leqslant t.$$

Suppose further that $A_n \longrightarrow A$ as $n \longrightarrow \infty$ pointwise and boundedly, the latter meaning that the A_n's are uniformly bounded on $[0,t]$. Then

$$Y_n(s) \longrightarrow Y(s) \qquad \text{as } n \longrightarrow \infty$$

uniformly on $[0,t]$.

Proof. Subtracting the equations satisfied by $Y(s)$ and $Y_n(s)$ we find that

$$Y(s)-Y_n(s) = \int_0^s [A(\sigma)-A_n(\sigma)]Y(\sigma)\,\mathrm{d}\sigma + \int_0^s A_n(\sigma)[Y(\sigma)-Y_n(\sigma)]\,\mathrm{d}\sigma\,, \quad 0 \leqslant s \leqslant t.$$

Taking norms this leads to

$$\|Y(s)-Y_n(s)\| \leqslant \int_0^s \|A(\sigma)-A_n(\sigma)\|\;\|Y(\sigma)\|\,\mathrm{d}\sigma + \int_0^s \|A_n(\sigma)\|\;\|Y(\sigma)-Y_n(\sigma)\|\,\mathrm{d}\sigma$$

$$\leqslant L\int_0^s \|A(\sigma)-A_n(\sigma)\|\,\mathrm{d}\sigma + M\int_0^s \|Y(\sigma)-Y_n(\sigma)\|d\sigma$$

where L and M are bounds for $Y(\sigma)$ and $A_n(\sigma)$ on $[0,t]$, respectively.

An application of Gronwall's inequality to the preceding then gives us

$$\|Y(s)-Y_n(s)\| \leqslant \int_0^s \mathrm{e}^{M(s-\sigma)}L\|A(\sigma)-A_n(\sigma)\|\,\mathrm{d}\sigma$$

for $s \in [0,t]$; and consequently

$$\sup_{0\leqslant s\leqslant t} \|Y(s)-Y_n(s)\| \leqslant \int_0^t \mathrm{e}^{M(t-\sigma)}L\|A(\sigma)-A_n(\sigma)\|\,\mathrm{d}\sigma\,.$$

Since $A_n(\sigma) \longrightarrow A(\sigma)$ as $n \longrightarrow \infty$, pointwise and boundedly, the result then follows from the Lebesgue dominated convergence theorem. □

Lemma 5.5. *Suppose* v *and* ψ *are bounded continuous functions in* $\mathbb{R}^p \times [0,T]$ *satisfying the inequality*

$$v(x,t) \leqslant \psi(x,t) + \int_0^t\int_{\mathbb{R}^p} MF(x-\xi,t-\tau)v(\xi,\tau)\,\mathrm{d}\xi\;\mathrm{d}\tau \tag{5.9}$$

for $(x,t) \in \mathbb{R}^p \times [0,T]$ *with* $M > 0$, *then*

$$v(x,t) \leqslant \psi(x,t) + \int_0^t\int_{\mathbb{R}^p} M e^{M(t-\tau)}F(x-\xi,t-\tau)\psi(\xi,\tau)\,\mathrm{d}\xi\;\mathrm{d}\tau \tag{5.10}$$

for $(x,t) \in \mathbb{R}^p \times [0,T]$.

Remark. Recall that according to (4.2) $f(x,t) = \mathrm{e}^{-|x|^2/2t}/(2\pi t)^{p/2}$.

P r o o f. To avoid tedious repetition in the argument that follows, it is to be understood, without any further mention, that the point (x,t) always lies in the set $\mathbb{R}^p \times [0,T]$.

To establish (5.10) we begin by writing (5.9) in the abbreviated form

$$v \leqslant \psi + Kv \tag{5.11}$$

where

$$Kw = Kw(x,t) = \int_0^t \int_{\mathbb{R}^p} MF(x-\xi, t-\tau) w(\xi,\tau)\, \mathrm{d}\xi\, \mathrm{d}\tau$$

is an operator mapping the space of bounded continuous functions in $\mathbb{R}^p \times [0,T]$ continuously into itself. Iterating (5.11) we obtain, since K maps non-negative functions into non-negative functions

$$v \leqslant \psi + K[\psi + Kv] = \psi + K\psi + K^2 v,$$
$$v \leqslant \psi + K\psi + K^2[\psi + Kv] = \psi + K\psi + K\psi + K^2\psi + K^3 v,$$

and in general

$$v \leqslant \psi + K\psi + K^2\psi + \cdots + K^n\psi + K^{n+1}v.$$

Estimating $K^j w$ in a straightforward way yields

$$|K^j w(x,t)| \leqslant \frac{(Mt)^j}{j!}\|w\|, \qquad j = 1,2,\ldots,$$

where $\|w\| = \sup_{\mathbb{R}^p \times [0,T]} |w(x,t)|$; it follows that $K^{n+1}v \longrightarrow 0$ as $n \longrightarrow \infty$, and hence that

$$v \leqslant \sum_{j=0}^{\infty} K^j \psi \tag{5.12}$$

with the series converging.

But now on the basis of the well-known identities

$$\int_{\mathbb{R}^p} F(x-\xi, t-\tau) F(\xi - \eta, \tau - s)\, \mathrm{d}\xi = F(x-\eta, t-s), \qquad s < \tau < t,$$

we can inductively establish the formulas

$$K^j\psi(x,t) = \int_0^t \int_{\mathbb{R}^p} M^j \frac{(t-\tau)^{j-1}}{(j-1)!} F(x-\xi, t-\tau)\psi(\xi,\tau)\, \mathrm{d}\xi\, \mathrm{d}\tau\,, \qquad j = 1,2,\ldots;$$

from which it follows that

$$\sum_{j=0}^{\infty} K^j\psi = \psi(x,t) + \int_0^t \int_{\mathbb{R}^p} M\mathrm{e}^{M(t-\tau)} F(x-\xi, t-\tau)\psi(\xi,\tau)\, \mathrm{d}\xi\, \mathrm{d}\tau\,.$$

Combining this with (5.12) we arrive at the desired result (5.10). □

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