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ON GRAPHS WITH PRESCRIBED SUBGRAPHS OF ORDER k, AND A THEOREM OF KELLY AND MERRIELL

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Let n, k, ℓ be integers such that $n \geqslant k+1$, $k > \ell$, $k > \ell \geqslant 0$, $2k - \ell \leqslant n$, $k \geqslant 2$. We say that a graph G on n vertices has property $-(n,k,\ell)$ if for every two k-subset A and B, $|A \cap B| = \ell$ it follows that $e\langle A\rangle = e\langle B\rangle$, where $e\langle A\rangle$ is the number of edges in the induced subgraph on the vertex-set A.

Theorem. (i) *If $(n,k,\ell) \notin \{(2k,k,0),(k+1,k,k-1)\}$ then G has property $-(n,k,\ell)$ iff $G \in \{K_n, \overline{K}_n\}$.*

(ii) *If $(n,k,\ell) \in \{(2k,k,0),(k+1,k,k-1)\}$ then G has property $-(n,k,\ell)$ iff G is a regular graph.*

This theorem is closely related to an old theorem of Kelly and Merriell concerning partition of the vertex set of a graph into two isomorphic subgraphs.

1. INTRODUCTION

In 1960 Paul Kelly and David Merriell [KM] proved the following theorem, with the convention that $\langle A\rangle$ is the induced subgraph on the vertex set A.

Theorem A. *Let G be a graph on $2n$ vertices such that for every n-subset A, $\langle A\rangle \simeq \langle V/A\rangle$. Then G belongs to the class*

$$\{K_{2n}, K_{n,n}, nK_2, K_n \times K_2, 2C_4, \text{ and their complements}\}.$$

Although this theorem carries the flavour of classical graph theory, it has been long forgotten and no elaborations of this elegant result can be found in the literature. Yet it is linked, in some sense, to the recent researches of Alon and Bollobas [AB], Erdös and Hajnal [EH] and Graham and Chung [CG], who considered graphs with a small number of distinct induced subgraphs, and it follows from these researches that such graphs must contain large independent set or large complete-subgraphs.

Our main result is that much weaker condition than the isomorphism condition in the Kelly-Merriell theorem already implies, in almost all cases, that G is either complete or empty graph.

The Kelly-Merriell theorem raises some interesting questions.

1) What about the related edge-partition problem?

2) What if $\langle A\rangle \simeq \langle B\rangle$ whenever $|A| = |B|$ and $|A \cap B| = \ell$?

3) What is the role of the "isomorphism condition" and could it be replaced by weaker conditions with essentially the same conclusions? etc. etc.

We shall try to supply a complete solution to the above problems and to suggest a more general view on this subject.

Our notation is standard following Harary [H] and Bollobas [B]. In particular $\langle A\rangle$ is the induced subgraph on the vertex-set A, and $e(G)$ denotes the number of edges of G.

2. Edge-Partition

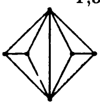
The solution of the edge-partition analogue of the Kelly-Merriell theorem is based upon Beineke's characterization of line-graphs [BE].

Theorem 1. *Let G be a graph on $2m$ edges without isolated vertices, and such that for every m-subset $A \subset E(G)$ it follows that $A \simeq E \setminus A$. Then*

$$G \in \{K_{1,2m}, 2K_{1,m}, 2mK_2, mK_{1,2}, K_{2,m}, 2C_4, 2K_3, C_6\}.$$

Proof. Recall that $L(G) \simeq L(H) \Rightarrow G \simeq H$ unless G and H contains components isomorphic to $K_{1,3}$ and K_3, (because $L(K_{1,3}) = L(K_3) = K_3$). A proof of this statement can be found in [H, p. 72]. Observe that if G has the required edge-partition property then its line graph $L(G)$ has the Kelly-Merriell partition property, and it remains only to consider which of the Kelly-Merriell graphs is a line-graph and to check the presence of $K_{1,3}$ vs. K_3 components in the source graph. Thus $L(K_{1,2m}) = K_{2m}$, $L(2K_{1,m}) = 2K_m$ and also $L(2K_3) = 2K_3$, $L(2mK_2) = \overline{K}_{2m}$, $L(2C_4) = 2C_4$, $L(mK_{1,2}) = mK_2$ and $L(K_{2,m}) = K_m \times K_2$. Of course $L(K_{1,3} \cup K_3) = 2K_3$ but this is a forbidden choice. It remains to find ancestors to $K_{n,n}$, $\overline{K_n \times K_2}$, $\overline{nK_2}$, and $\overline{2C_4}$. As line-graphs are $K_{1,3}$-free graphs, the only possible cases for an ancessors to $K_{n,n}$ are for $n \leqslant 2$ yielding nothing new. Similar argument shows that $\overline{K_n \times K_2}$ is a line graph only for $n \leqslant 3$ yielding the case $\overline{K_3 \times K_2} = C_6$.

Line graphs are also $K_5 \setminus \{e\}$ free hence $\overline{nK_2}$ is a line graph for $n \leqslant 3$. Thus $L(2K_2) = \overline{K_2}$, $L(C_4) = \overline{2K_2}$ are already in the list and $L(K_4) = \overline{3K_2}$ but K_4 doesn't have the edge-partition property because of the partition of $E(K_4)$ into a triangle-K_3 and star-$K_{1,3}$. Lastly $\overline{2C_4}$ is not a line graph as it contains an induced subgraph

$H =$  which is forbidden by Beineke's theorem. $\square$

Remark. One should remember that in 1960 the Beineke's characterization of line graphs, has not yet been born, and probably the lack of this criterion prevented Kelly and Merriell of formulating the edge-partition result.

3. On k-subgraphs with equal number of edges

Our aim in this section is to show that except for two cases, one of them is the Kelly-Merriell case, a much weaker condition than isomorphisms, imposed on pairs of k-subgraphs is sufficient to characterize G. Our main tool is a connectivity result, theorem 3 below, concerning the celebrated Kneser's graphs (see e.g., [AFL], [B]) with an additional lemma concerning k-subgraphs with equal number of edges. We emphasize here that lemma 2 below, already appeared as an advanced exercise (see e.g. [BM], exercise 1.4.5), without a proof which we supply here to keep this paper self contained.

Lemma 2. *Let G be a graph on n vertices, and let $n > k \geqslant 2$ be an integer. Suppose every k-subset of $V(G)$ induced the same number of edges.*

(i) *if $n = k + 1$ then G is regular.*

(ii) *if $n \geqslant k + 2$ then $G \in \{K_n, \overline{K}_n\}$.*

Proof. (i) Suppose $n = k + 1$. Let $A \subset V(G)$, $|A| = k$ and $\{v\} = V \setminus A$. By assumption $e\langle A\rangle$ dictates the number of edges in every k-subset. Consider $u \in A$ and $B = (A \setminus u) \cup \{v\}$. Then $e\langle B\rangle = e\langle A\rangle$. However $e\langle B\rangle = e\langle A\rangle - \deg u + \deg v = e\langle A\rangle$ holds regardless of whether $(u, v) \in E(G)$ or not so. Hence G is regular.

(ii) Suppose first $n = k + 2$. Consider $A \subset V$, $|A| = k$ and $\{u, v\} = V \setminus A$. Since all k-subsets induced the same number of edges, this is true for all k-subsets of $A \cup \{u\}$ and $A \cup \{v\}$.

Hence by (i) $\langle A \cup \{u\}\rangle$ is d-regular and $\langle A \cup \{u\}\rangle$ is r-regular. It follows that $e\langle A\rangle = e\langle A \cup \{u\}\rangle - d = e\langle A \cup \{u\}\rangle - r$, hence $\frac{(k+1)d}{2} - d = \frac{(k+1)r}{2} - r$, hence $\frac{(k+1)(d-r)}{2} = d - r$ which is possible if either $d - r = 0$ or $k = 1$, but as $k \geqslant 2$ we conclude that $d = r$.

Case 1. $d = r = 0$.

Then $e\langle A\rangle = 0$ and all the k-subsets are empty and $G = \overline{K_{k+2}}$.

Case 2. $d = r > 0$.

Consider the neighbourhood $N_A(u)$ of u in A, and assume there exists $x \in N_A(u) \setminus N_A(v)$. Then we would have

$$e\langle (A \setminus x) \cup \{v\}\rangle = e\langle A\rangle - \deg_A x + |N_A(v)| = e\langle A\rangle - (d-1) + d = e\langle A\rangle + 1$$

which is impossible by the assumption on k-subsets.

Hence $N_A(u) \subseteq N_A(v)$ and by symmetry we obtain $N_A(u) = N_A(v)$. It is clear now that for every two vertices u, v we must have $N_A(u) = N_A(v)$ where $A = V \setminus \{u, v\}$.

Now suppose the degree d satisfies $0 < d < k$. Then there exists $w \in A$ adjacents neither to u nor to v, and there exists another vertex $y \in A$ adjacents to both u and v.

But now we have $N_A(w) \neq N_A(y)$, $A = V - \{w, y\}$, and this is forbidden. Hence $d = k$ and $\langle A\rangle = K_k$, hence all k-subsets are K_k and $G = K_{k+2}$.

Now for $n \geqslant k+3$, as all k-subsets have the same number of edges it follows that all $k+2$ subsets are either K_{k+2} or K_{k+2} which implies that $G \in \{K_n, \overline{K}_n\}$. □

Recall the definition of the Kneser graph $K(n, k, \ell)$ whose vertex set is the k-subsets of n, namely $[n]^k$, two vertices being adjacent if the corresponding k-subsets intersect in exactly ℓ elements. Clearly $n > k > \ell \geqslant 0$ and we shall call a triple (n, k, l), trivial if either $2k - \ell > n$ in which case $K(n, k, \ell)$ contains no edges, or $(n, k, \ell) = (2k, k, 0)$ in which case $K(2k, k, 0)$ is a matching, and this is the exceptional case related to the Kelly-Merriell theorem.

Theorem 3. *$K(n, k, \ell)$ is connected iff (n, k, ℓ) is not a trivial triple.*

Proof. Clearly the trivial triples belong to the non-connected cases. Assume (n, k, ℓ) is a non-trivial triple.

Case 1. Assume $\ell = 0$. Clearly $n \geqslant 2k+1$.

Claim: if A and B are k-subsets with Hamming distance 2, then their distance in $K(n, k, 0)$ is at most 2. Indeed $|A \cup B| = k+1$ and as $n \geqslant 2k+1$ there are k elements forming a k-subset D such that $|A \cap D| = 0$ and $|B \cap D| = 0$. Hence (A, D), $(D, B) \in E\big(K(n, k, 0)\big)$ as claimed.

Now consider arbitrary k-subsets A and B such that $A \cap B = D$, $|D| = d$. If $d = 0$ then $(A, B) \in E(K)$ otherwise $n - |A \cup B| = n - 2k + d > d$. Let $\{x_1, x_2, \dots, x_d\} \in V \setminus (A \cup B)$ and $\{y_1, \dots, y_d\} = D$. Define $A_1 = A$ and $A_{i+1} = (A_i \setminus y_i) \cup \{x_i\}$. The Hamming distance between A_i and A_{i+1} is 2 and by the claim above A_i, A_{i+1} are a distance at most 2 in K. Moreover $|A_{i+1} \cap B| = |A_i \cap B| - 1$. Hence $|A_{d+1} \cap B| = 0$ and $(A_{d+1}, B) \in E(K)$, hence A and B are connected by a path in K.

Case 2. Assume $\ell = 1$. Clearly $n \geqslant 2k - 1$.

Claim: if A and B are k-subsets with Hamming distance 2, then their distance in $K(n,k,1)$ is at most 2. Indeed $|A \cup B| = k + 1$ and, as $n \geqslant 2k - 1$, there are $k - 2$ elements disjoint from $A \cup B$. Add to them the two elements of $A \cup B \setminus A \cap B$ to obtain a k-subset D such that $|A \cap D| = 1$, $|B \cap D| = 1$ hence (A, D), $(B, D) \in E(K)$.

Now consider arbitrary k-subsets A and B such that $A \cap B = D$, $|D| = d$. If $d = 1$ then $(A, B) \in E(K)$ and we are done.

If $d = 0$ then for some $a \in A$, $b \in B$ define $D = (B \setminus b) \cup \{a\}$. Then clearly $|D \cap A| = 1$ and the Hamming distance of B and D is 2 and we are done. If $d > 1$ apply the chain argument of case 1, this time stopping the chain at A_d instead of A_{d+1}.

Case 3. Assume $\ell \geqslant 2$. Clearly $n \geqslant 2k - \ell$.

Consider all the k-subsets containing some fixed $\ell - 1$ elements. By the result for the case $\ell = 1$, applied on $K(n-\ell+1, k-\ell+1, 1)$, all these k-tuples form a connected component in $K(n,k,\ell)$. On the other hand, as by induction on ℓ, $K(n,k,\ell-1)$ is connected it follows that if A and B are in distinct components in $K(n,k,\ell)$ there is still a chain of k-tuples $A_1 = A, \ldots, A_{t-1}, A_t = B$ such that $|A_i \cap A_{i+1}| = \ell - 1$, because $K(n,k,\ell-1)$ is connected. But then A_i and A_{i+1} must belong to the same component as they contain an $\ell - 1$ elements in common. This shows that in fact A and B lies in the same component and $K(n,k,\ell)$ is connected. □

We are now ready to present the "completion" to the Kelly-Merriell theorem, namely

Theorem 4. *Let $n > k > \ell \geqslant 0$ be integers such that $2k - \ell \leqslant n$. Let G be a graph on n vertices having property—(n,k,ℓ), (namely: if A, $B \subset V(G)$, $|A| = |B| = k \geqslant 2$, $|A \cap B| = \ell$ then $e\langle A\rangle = e\langle B\rangle$), then the following hold:*

(i) *if $(n,k,\ell) \in \{(k+1,k,k-1),(2k,k,0)\}$ then G is regular.*

(ii) *if $(n,k,\ell) \notin \{(k+1,k,k-1),(2k,k,0)\}$ then $G\{K_n, \overline{K}_n\}$.*

Proof. By theorem 3, $K(n,k,\ell)$ is connected for non-trivial triples. Hence if $e\langle A\rangle = e\langle B\rangle$ for k-subsets A, B such that $|A \cap B| = \ell$ then the connectivity of $K(n,k,\ell)$ implies that all the k-subsets of $V(G)$ have the same number of edges. By Lemma 2 if $n = k+1$ then G is regular and if $n \geqslant k+2$ then $G \in \{K_n, \overline{K_n}\}$. Hence (ii) and the case $(k+1,k,k-1)$ are proved. The only case that remains is $(2k,k,0)$ but a simple argument, already presented in Kelly-Merriell's original paper, shows that this case is possible iff G is regular.

One should observe that the other trivial triples are of no interest here because in these cases $K(n,k,\ell)$ is the empty graph and we can conclude nothing by comparing pairs. □

At this point it is natural to ask: what is so special about the number of edges in the k-subgraphs? What about other parameters of the k-subgraphs? This is the content of the next chapter.

4. Parameters imposed on k-subgraphs

Let $P(G)$ be a graph parameter, e.g., number of edges, chromatic number, independent number, domination number, etc. etc.

We say that $P(G)$ is a complete-parameter if for every $k \geqslant 2$ there exists two real numbers $a_k \leqslant b_k$ such that if $V(G) = k$ then $P(G) \in \{a_k, b_k\}$ iff $G \in \{K_k, \overline{K_k}\}$. Thus $e(G)$ is a complete parameter with $a_k = 0$ and $b_k = \binom{k}{2}$. The chromatic number is a complete parameter with $a_k = k$ and $b_k = 1$, and so are the independent number and the clique number. On the other hand the dominating number $d(G)$ is not complete and so is the matching number and so on.

Theorem 5. *Let $P(G)$ be a complete parameter, and let $k \geqslant 2$ be fixed. Suppose G is a graph on n vertices such that for every k-subsets A and B satisfying $|A \cap B| = \ell$ the following equality holds: $P\langle A\rangle = P\langle B\rangle$.*

Then for $n \geqslant N(k)$, $G \in \{K_n, \overline{K}_n\}$.

Proof. Suppose $n \geqslant R(k,k)$ the Ramsey number for K_k.

Then G must contain a k-subset A such that $\langle A\rangle \in \{K_k, \overline{K}_k\}$. If $R(k,k) > 2k$ then (n,k,ℓ) is a non-trivial triple and by theorem 3, $K(n,k,\ell)$ is connected. Hence, as $P(G)$ is complete parameter it follows that $P(A) \in \{a_k, b_k\}$ for every k-subset A and hence $G \in \{K_n, \overline{K}_n\}$.

Observe that for $k \geqslant 4$, $R(k,k) > 2k$ and for $k = 2$ we may take $n = 5$ and for $k = 3$ we may take $n = 7$, proving the theorem. $\square$

Remark. We know that for a complete parameter $P(G)$, we can take $N(k) = R(k,k)$ for $k \geqslant 4$ and $N(2) = 5$, $N(3) = 7$ in order to ensure the conclusion of Theorem 5. However from theorem 4 we infer that for $P(G) = |E(G)|$, $N(k) = 2k+1$ would suffice for every triple (n,k,ℓ). This is a large gap and it is worth considering for every complete parameter $P(G)$ the best possible value of $N(k) = N_{P(G)}(k)$ as defined in theorem 5.

So in closing this paper we propose the following problems.

Problem 1. Let $P(G)$ be a complete parameter. Is it true that $N(k) = 2k+1$ is always a valid choice in theorem 5?

Problem 2. Generalize theorem 2 from graphs to r-uniform hypergraphs. In particular would $k+r$ be a valid choice in part (ii) of theorem 2?

Problem 3. What kind of general result can be formulated concerning incomplete parameters, with respect to theorem 5?

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