Czechoslovak Mathematical Journal

Gary Chartrand; Heather Gavlas; Frank Harary; Michelle Schultz

On signed degrees in signed graphs

Czechoslovak Mathematical Journal, Vol. 44 (1994), No. 4, 677–690

Persistent URL: http://dml.cz/dmlcz/128488

Terms of use:

© Institute of Mathematics AS CR, 1994

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

ON SIGNED DEGREES IN SIGNED GRAPHS

GARY CHARTRAND,[1] Western Michigan University,
HEATHER GAVLAS, Western Michigan University,
FRANK HARARY,[2] New Mexico State University,
MICHELLE SCHULTZ, Western Michigan University

(Received November 18, 1992)

1. INTRODUCTION

A *signed graph* S consists of a graph G with a designation of its edges as either positive or negative. These were first discovered in [3]. The *(signed) degree* $\operatorname{sdeg} v$ of a vertex v of S is the number of positive edges incident with v less the number of negative edges incident with v. Thus, if v is incident with d^+ positive edges and d^- negative edges, then $\operatorname{sdeg} v = d^+ - d^-$. However, in the graph G, $\deg v = d^+ + d^-$. Consequently, the degree of a vertex in S and of the same vertex in G are of the same parity.

For a vertex v in a signed graph S, $\operatorname{sdeg} v$ may be positive, negative, or zero. For example, the signed graph S of Figure 1 has two vertices of degree 2, one of degree 0, and two of degree -1.

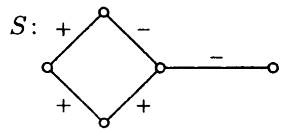

Figure 1. A signed graph S.

The *degree sequence* of a signed graph S has the signed degrees in nonincreasing order. The degree sequence of the signed graph S of Figure 1 is, therefore, 2, 2, 0, -1, -1. A finite sequence σ of integers is *graphical* if σ is a degree sequence of some signed graph. Certainly, then, 2, 2, 0, -1, -1 is graphical.

[1] Research supported in part by Office of Naval Research Contract N00014-91-J-1060.
[2] Research supported in part by Office of Naval Research Contract N00014-90-J-1860.

It is easy to characterize degree sequences of paths and stars.

Theorem 1. *Let $\sigma\colon d_1, d_2, \ldots, d_p$ be a sequence of integers such that exactly two terms d_i $(1 \leqslant i \leqslant p)$ are 1 or -1 and the remaining terms are -2, 0, or 2. Then σ is the degree sequence of a signed path if and only if one of the following is satisfied:*

(1) if there are exactly two integers i $(1 \leqslant i \leqslant p)$ such that $d_i = 1$ or exactly two integers i such that $d_i = -1$, then σ contains an even number of zeros, or

(2) if there is one integer i such that $d_i = 1$ and one integer j such that $d_j = -1$, then σ contains an odd number of zeros.

Theorem 2. *Let $\sigma\colon d_1, d_2, \ldots, d_p$ be a sequence of integers such that $d_i = 1$ or $d_i = -1$ for $i = 1, 2, \ldots, p-1$. Then σ is the degree sequence of a signed star if and only if $d_p = \sum\limits_{i=1}^{p-1} d_i$.*

A *double star* (first defined in Grossman, Harary and Klawe [1]) is a tree containing exactly two vertices that are not end-vertices. These two vertices are the *centers* of the double star. For example, the graph G in Figure 2 is a double star with centers u and v. Of course, if we designate each edge of G as positive or negative, then we have a *signed double star*. One such signed double star S is shown in Figure 2.

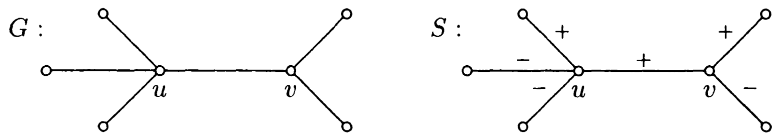

Figure 2. A double star G and a signed double star S.

The next result gives a characterization of degree sequences of signed double stars.

Theorem 3. *Let $\sigma\colon d_1, d_2, \ldots, d_p$ be a sequence of integers with $d_i = 1$ or $d_i = -1$ for $i = 1, 2, \ldots, p-2$. Let a denote the number of integers i $(1 \leqslant i \leqslant p-2)$ such that $d_i = 1$ and let b be the number of integers i $(1 \leqslant i \leqslant p-2)$ such that $d_i = -1$. Further, let $d = a - b$. Then σ is the degree sequence of a signed double star if and only if one of the following is satisfied:*

(1) $d_{p-1} + d_p = d + 2$ and $-b + 1 \leqslant d_{p-1}, d_p \leqslant a + 1$,

(2) $d_{p-1} + d_p = d - 2$ and $-b - 1 \leqslant d_{p-1}, d_p \leqslant a - 1$.

Proof. First let $\sigma\colon d_1, d_2, \ldots, d_p$ be the degree sequence of a signed double star T. So T has order p, where $p - 2$ of the vertices of T are end-vertices. Since each end-vertex must have degree 1 or -1, we may assume, without loss of generality, that $d_i = 1$ or $d_i = -1$ for $i = 1, 2, \ldots, p-2$. Let u and v be the centers of T, say $\operatorname{sdeg} u = d_{p-1}$ and $\operatorname{sdeg} v = d_p$. Now the edge joining u and v is either positive or negative. First consider uv as a positive edge.

Let a be the number of integers i $(1 \leqslant i \leqslant p-2)$ such that $d_i = 1$ and b integers i $(1 \leqslant i \leqslant p-2)$ such that $d_i = -1$. Then $-b+1 \leqslant d_{p-1} \leqslant a+1$. Similarly, $-b+1 \leqslant d_p \leqslant a+1$. Also, observe that if $d = a-b$, then

$$\operatorname{sdeg} u + \operatorname{sdeg} v = d_{p-1} + d_p = a - b + 2 = d + 2.$$

The proof is similar when uv is a negative edge.

For the converse, let the sequence $\sigma\colon d_1, d_2, \ldots, d_p$ satisfy condition (1). We show that there exists a signed double star T with degree sequence σ. Let T'' be the signed graph consisting of a single positive edge uv (see Figure 3). Now $d_p > 0$ or $d_p \leqslant 0$. Since the proofs of these two cases are similar, we only consider the first.

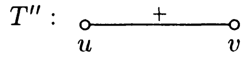

Figure 3. The signed graph T'' with one positive edge uv.

Given $d_p > 0$, we begin by adding edges to the graph T'' to produce a new graph T' with $\operatorname{sdeg}_{T'} v = d_p$. In particular, we add $d_p - 1$ vertices to T'' and join each new vertex to v with a positive edge. The graph T' is shown in Figure 4. Note that at this time, we have one vertex of degree d_p, and d_p vertices of degree 1. Also, since $a = p-2-b$, it follows that

$$a - (d_p - 1) = (p - 2 - b) - (d_p - 1) = p - (d_p + 1) - b.$$

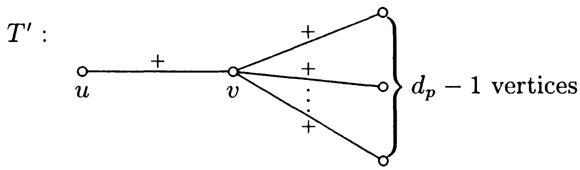

Figure 4. A signed graph T' with all positive edges.

To complete the construction of T, we add $p-(d_p+1)$ vertices to T' and join b of the new vertices to u by negative edges and join the remaining $p-(d_p+1)-b = a-(d_p-1)$ new vertices to u by positive edges. The resulting signed graph T is depicted in Figure 5.

Observe that T is a double star of order p containing a vertices of degree 1 and b vertices of degree -1. Also $\operatorname{sdeg} v = d_p$, and

$$\operatorname{sdeg} u = a - (d_p - 1) - b + 1 = (a - b) - d_p + 2 = d_{p-1}.$$

Thus, T has the desired properties. The proof is similar when σ satisfies condition (2). $\square$

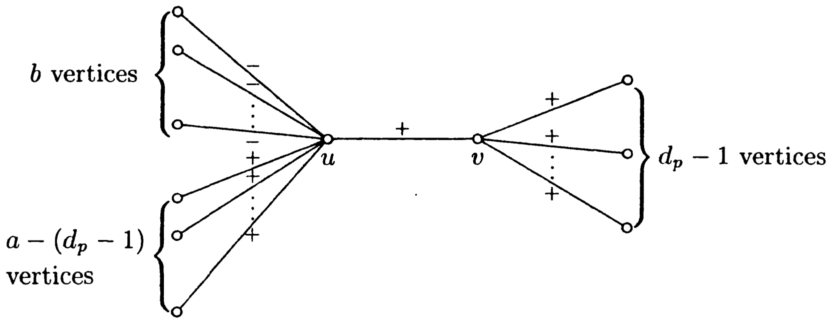

Figure 5. A signed double star T with degree sequence σ.

A few facts about graphical sequences are described next.

Theorem 4. *If S is a signed graph of order p and size q, then*

$$k = \sum \operatorname{sdeg} v = 2q \pmod 4,$$

and the number of positive edges of S is $\frac{1}{4}(2q + k)$ while the number of negative edges of S is $\frac{1}{4}(2q - k)$.

Proof. Suppose that S is obtained by designating each edge of a graph G as positive or negative and $V(S) = \{v_1, v_2, \ldots, v_p\}$. Suppose, further, that v_i $(1 \leqslant i \leqslant p)$ is incident with d_i^+ positive edges and d_i^- negative edges, so that $\operatorname{sdeg} v_i = d_i^+ - d_i^-$ while $\deg v_i = d_i^+ + d_i^-$. Of course, $\sum \deg v_i = 2q$.

Let S have a positive edges and b negative edges. Then $q = a + b$, $\sum d_i^+ = 2a$, and $\sum d_i^- = 2b$. Consequently,

$$k = \sum \operatorname{sdeg} v_i = 2a - 2b = 2q - 4b$$

so that $k \equiv 2q \pmod 4$. Solving for a and b, we have $a = \frac{1}{4}(2q+k)$ and $b = \frac{1}{4}(2q-k)$. □

Corollary 4a. *A necessary condition for a sequence d_1, d_2, …, d_p of integers to be graphical is that $\sum d_i$ is even.*

Of course, another necessary condition for d_1, d_2, …, d_p to be graphical is that each $|d_i| < p$. A *zero sequence* is a finite sequence every term of which is 0. Clearly, every zero sequence is graphical. If σ is the sequence d_1, d_2, …, d_p, then the *negative* $-\sigma$ of σ is the sequence $-d_1$, $-d_2$, …, $-d_p$. Obviously, a sequence σ is graphical if and only if $-\sigma$ is graphical.

When considering a nonzero sequence $\sigma\colon d_1, d_2, \ldots, d_p$, we may assume, without loss of generality, that σ is nonincreasing and $|d_1| \geqslant |d_p|$, for we may always replace

σ by $-\sigma$ if necessary. We say that a nonzero sequence $\sigma\colon d_1, d_2, \ldots, d_p$ is a *standard sequence* if σ is nonincreasing, $\sum d_i$ is even, $d_1 > 0$, each $|d_i| < p$, and $|d_1| \geqslant |d_p|$.

2. Degree sequences of complete signed graphs

If every edge of a signed graph S is designated as positive, then, in effect, S is a graph and the signed degree of each vertex of S is its degree. For graphs, Havel [5] and, later, Hakimi [2] independently showed that a nonincreasing nonzero sequence $\sigma\colon d_1, d_2, \ldots, d_p$ of nonnegative integers is graphical if and only if the sequence $\sigma'\colon d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_p$ is graphical. This result was also discovered and proved by the author of [4], but he found the Havel reference and hence did not publish it.

We now show that there is an analogue to the Havel-Hakimi-Harary Theorem for complete signed graphs.

Theorem 5. *Let $\sigma\colon d_1, d_2, \ldots, d_p$ be a standard sequence and let $r = \frac{1}{2}(d_1 + p - 1)$. Then σ is a degree sequence of a complete signed graph if and only if $\sigma'\colon d_2 - 1, d_3 - 1, \ldots, d_{r+1} - 1, d_{r+2} + 1, d_{r+3} + 1, \ldots, d_p + 1$ is a degree sequence of a complete signed graph.*

Proof. Let S' be a complete signed graph with degree sequence σ'. Label the vertices of S' as v_2, v_3, …, v_p so that

$$\operatorname{sdeg} v_i = \begin{cases} d_i - 1, & 2 \leqslant i \leqslant r+1, \\ d_i + 1, & r+2 \leqslant i \leqslant p. \end{cases}$$

Now construct a new complete signed graph S by adding a vertex v_1 to S' and joining v_1 by positive edges to the vertices v_2, v_3, …, v_{r+1} and by negative edges to the vertices v_{r+2}, v_{r+3}, …, v_p. Then observe that

$$\operatorname{sdeg} v_1 = r - (p - r - 1) = 2r - (p-1) = d_1.$$

Also, it is clear that $\operatorname{sdeg} v_i = d_i$ for $i = 2, 3, \ldots, p$. Hence, S is a complete signed graph with degree sequence σ.

For the converse, let σ be the degree sequence of a complete signed graph. For any complete signed graph with degree sequence σ, we may assume that the vertices are labeled v_1, v_2, …, v_p such that $\operatorname{sdeg} v_i = d_i$ for $i = 1, 2, \ldots, p$. Among all complete signed graphs having σ as a degree sequence, let S be one with the property that the sum m of the (signed) degrees of the vertices joined to v_1 by positive edges is maximum. If d^+ denotes the number of positive edges incident with v_1 and d^- is the number of negative edges incident with v_1, then $\operatorname{sdeg} v_1 = d_1 = d^+ - d^-$. Further, since S is a complete signed graph, $\deg v_1 = d^+ + d^- = p - 1$. From this it follows that $d^+ = \frac{1}{2}(d_1 + p - 1) = r$. We claim that v_1 must be joined by positive edges to vertices having the degrees $d_2, d_3, \ldots, d_{r+1}$. Assume that this is not the case. Then there exist vertices v_i and v_j with $j > i$ such that the edge $v_1 v_i$ is negative and the edge $v_1 v_j$ is positive.

Since σ is a standard sequence, $\operatorname{sdeg} v_i > \operatorname{sdeg} v_j$, or $d_i > d_j$. Hence, there exists a vertex v_n of S distinct from v_1, v_i, and v_j such that $v_n v_i$ is a positive edge and $v_n v_j$ is a negative edge. But if we now change the signs of these edges so that $v_1 v_j$ and $v_n v_i$ are negative and $v_1 v_i$ and $v_n v_j$ are positive (see Figure 6), then we produce a complete signed graph with degree sequence σ in which the sum of the degrees of the vertices joined to v_1 by positive edges exceeds m, which is a contradiction.

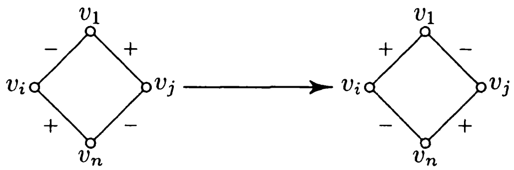

Figure 6. A change of signs of edges in S.

Thus, we may assume that v_1 is joined by positive edges to the vertices $v_2, v_3, \ldots, v_{r+1}$ and by negative edges to the vertices $v_{r+2}, v_{r+3}, \ldots, v_p$. The complete signed graph $S - v_1$ thus has σ' as a degree sequence. □

As we will see in the next section, it is possible to generalize this idea to obtain a characterization of degree sequences of signed graphs in general. But first we illustrate the algorithm that is suggested by the previous theorem. Consider the sequence $\sigma_1\colon 3, 1, 1, -1, -1, -1$. Note that $p = 6$ and $d_1 = 3$, so $r = 4$. Applying Theorem 5 to σ_1, we obtain $\sigma_2'\colon 0, 0, -2, -2, 0$. Now if we write σ_2' in standard form, we have $\sigma_2\colon 2, 2, 0, 0, 0$. For σ_2, then, we have $p = 5$ and $d_1 = 2$, and so $r = 3$. Applying Theorem 5 to σ_2 gives $\sigma_3'\colon 1, -1, -1, 1$. Now observe that the complete signed graph S shown in Figure 7 has degree sequence σ_3'. So, by Theorem 5, it follows that σ_2 (and σ_2') is the degree sequence of a complete signed graph. Hence, σ_1 is also the degree sequence of a complete signed graph.

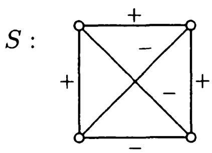

Figure 7. A complete signed graph S with degree sequence σ_3'.

Now we present a necessary and sufficient condition for a sequence of integers to be graphical.

Theorem 6. *Let $\sigma\colon d_1, d_2, \ldots, d_p$ be a standard sequence. Then σ is graphical if and only if there exist integers r, s with $d_1 = r - s$ and $0 \leqslant s \leqslant \frac{1}{2}(p-1-d_1)$ such that*

$$\sigma'\colon d_2 - 1, d_3 - 1, \ldots, d_{r+1} - 1, d_{r+2}, d_{r+3}, \ldots, d_{p-s}, d_{p-s+1} + 1, \ldots, d_p + 1$$

is graphical.

Proof. Let r and s be integers with $d_1 = r - s$ and $0 \leqslant s \leqslant \frac{1}{2}(p-1-d_1)$ such that σ' is the degree sequence of a signed graph. Then there exists a signed graph S' having degree sequence σ'. We may assume that the vertices of S' are labeled v_2, v_3, …, v_p so that

$$\operatorname{sdeg} v_i = \begin{cases} d_i - 1, & 2 \leqslant i \leqslant r+1, \\ d_i, & r+2 \leqslant i \leqslant p-s, \\ d_i + 1, & p-s+1 \leqslant i \leqslant p. \end{cases}$$

Now we construct a new signed graph S by adding a vertex v_1 to S' and joining v_1 to v_i $(2 \leqslant i \leqslant r+1)$ by positive edges and to v_i $(p-s+1 \leqslant i \leqslant p)$ by negative edges. Then it is clear that S is a signed graph with degree sequence σ.

Conversely, let σ be the degree sequence of a signed graph. Then there exists a signed graph T having degree sequence σ. We may assume that the vertices of T are labeled v_1, v_2, …, v_p so that $\operatorname{sdeg} v_i = d_i$ for $i = 1, 2, \ldots, p$. So in T, if v_1 is incident to d^+ positive edges and d^- negative edges, then $d_1 = \operatorname{sdeg} v_1 = d^+ - d^-$ and the degree of v_1 in the underlying graph of T is $d^+ + d^-$, that is, $\deg v_1 = d^+ + d^-$.

We may assume that the vertices of all signed graphs having degree sequence σ are labeled v_1, v_2, …, v_p so that $\operatorname{sdeg} v_i = d_i$ for $i = 1, 2, \ldots, p$. Now consider the collection of all signed graphs having degree sequence σ such that $\deg v_1 = d^+ + d^-$. Among all such signed graphs, choose S to be one with the property that the sum of the degrees of the vertices joined to v_1 by positive edges is maximum. Let $r = d^+$ and $s = d^-$. We claim that v_1 must be joined by positive edges to the vertices of S having degrees d_2, d_3, …, d_{r+1}. For assume that this is not the case. Then there exist vertices v_i and v_j with $i < j$ such that v_1v_j is positive and either (1) v_1v_i is negative or (2) v_1 and v_i are not adjacent in S. Also, we know that $d_i > d_j$. The proofs of these two cases are similar, so we consider only (1).

First, note that if there exists a vertex v_n such that v_nv_i is positive and v_nv_j negative, then we may relabel the edges so that v_1v_i and v_nv_j are positive and v_1v_j

and $v_n v_i$ are negative. (See Figure 8.) This results in a signed graph in which the sum of the degrees of the vertices joined to v_1 by positive edges exceeds that in S, which contradicts our choice of S. Hence, we may assume that there is no such vertex v_n in S.

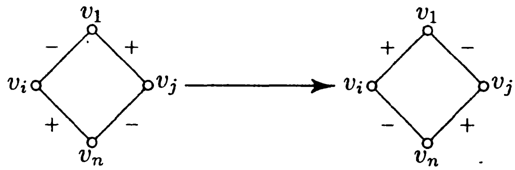

Figure 8. A relabeling of the edges in the signed graph S.

Next, assume that v_i is not incident to any positive edges. Then since $d_i > d_j$, there exist at least two negative edges $v_n v_j$ and $v_k v_j$ such that v_n and v_k are not adjacent to v_i. This situation is shown in Figure 9. If the edges are changed as shown also in Figure 9, then again a contradiction is produced. So v_i must be incident to at least one positive edge.

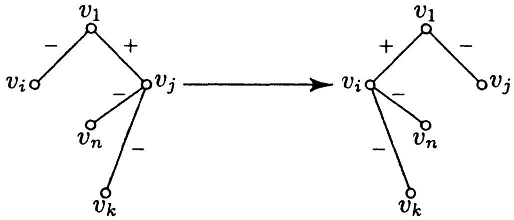

Figure 9. Constructing a new signed graph with degree sequence σ.

We claim that, in fact, there must exist at least one vertex v_m such that $v_m v_i$ is positive and v_m is not adjacent to v_j. Assume, instead, that whenever v_i is joined to a vertex by a positive edge, then v_j is also joined to this vertex by a positive edge. Since $d_i > d_j$, we have the situation illustrated in Figure 9, which as we have seen, contradicts the choice of S. Hence, as claimed, there exists a vertex v_m in S such that $v_m v_i$ is positive and v_m is not adjacent to v_j. In a similar manner, it is possible to show that there exists a vertex v_n such that $v_n v_j$ is negative and v_n is not adjacent to v_i. Hence, we now have the situation shown in Figure 10. Changing the edges as illustrated in Figure 10, we again contradict the choice of S. Thus, v_1 must be joined by positive edges to the vertices of degrees $d_2, d_3, \ldots, d_{r+1}$ in S.

Next, we claim that v_1 is joined by negative edges to the vertices of degrees d_{p-s+1}, $d_{p-s+2}, \ldots, d_p$. Actually, the proof of this fact is similar to the argument just given and so we omit it. In conclusion, observe that the signed graph $S - v_1$ has degree sequence σ'. $\square$

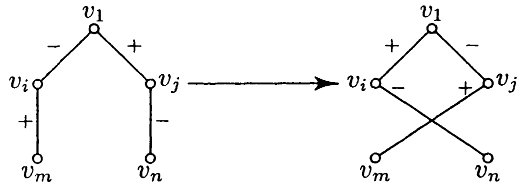

Figure 10. Constructing another signed graph with degree sequence σ.

To illustrate this theorem, consider the sequence $\sigma_1: 5, 4, 3, 0, -1, -1, -2$. Then $0 \leqslant s \leqslant \frac{1}{2}$, so we must choose $s = 0$ and $r = 5$. Applying Theorem 6, we obtain the sequence $\sigma_2: 3, 2, -1, -2, -2, -2$. Now observe that there are two possible choices for s, namely 0 or 1. If we choose $s = 1$ and $r = 4$, then we obtain the sequence $\sigma_3': 1, -2, -3, -3, -1$. Putting σ_3' into standard form gives $\sigma_3: 3, 3, 2, 1, -1$. Now, choosing $s = 0$ and $r = 3$, we obtain $\sigma_4: 2, 1, 0, -1$. Finally, if we choose $s = 0$ and $r = 2$, then we have $\sigma_5: 0, -1, -1$, which is clearly graphical. Hence, by Theorem 6, it follows that σ_1 is also graphical.

Next, consider the sequence $\sigma_1: 3, 0, -1, -2$. The only choice of s is $s = 0$ in which case we have $r = 3$. Applying Theorem 6, we obtain $\sigma_2: -1, -2, -3$. Clearly σ_2 is not graphical as there is no signed graph of order 3 containing a vertex of degree -3. Hence σ_1 is not graphical.

In both examples, we chose $s \neq 0$ only once. In fact, we would have come to the same conclusion had we chosen $s = 0$ in this one instance. This could lead us to believe that one can always choose $s = 0$ in Theorem 6. The discussion that follows addresses this question.

Let $\sigma: d_1, d_2, \ldots, d_p$ be a standard sequence, and let σ' be the sequence $d_2 - 1$, $d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_p$. Motivated by the Havel-Hakimi-Harary Theorem, we say that σ' is obtained from σ by *procedure H*.

Let σ_1 be a given sequence in standard form, and let σ_2' be the sequence obtained from σ_1 by procedure H. Furthermore, let σ_2 be the standard sequence obtained from σ_2' by rearranging the terms of σ_2' or $-\sigma_2'$, as necessary. When this is done, we will say that we have *standardized* σ_2' (to obtain σ_2). This process will be referred to as *standardization.* Continuing in this manner, we can construct a maximum number of sequences $\sigma_1, \sigma_2, \ldots, \sigma_k$, where for $i = 1, 2, \ldots, k-1$, the sequence σ_{i+1} is obtained from σ_i by procedure H and standardization. Thus σ_i is standard for $1 \leqslant i \leqslant k-1$ and σ_k is not standard. Although σ_k is not standard, we will write it uniquely in nonincreasing order whose first term in absolute value is at least as large as the last term in absolute value. Then we say that $\sigma_1, \sigma_2, \ldots, \sigma_k$ is the *H-sequence* corresponding to σ_1. Now since σ_k is not standard, either σ_k is a zero sequence or σ_k contains a term whose absolute value equals the number of terms in σ_k. The first case is addressed in our next result.

Theorem 7. *Let σ_1 be a standard sequence and let σ_1, σ_2, ..., σ_k be the H-sequence corresponding to σ_1. If σ_k is a zero sequence, then σ_1 is graphical.*

Proof. We begin by showing that if σ_{i+1} is graphical for some $i \in \{1, 2, \ldots, k-1\}$, then σ_i is graphical. Suppose that σ_i is the sequence a_1, a_2, ..., a_p. Then, performing procedure H on σ_i, we obtain the sequence $\sigma'_{i+1}: a_2-1, a_3-1, \ldots, a_{a_1+1}-1, a_{a_1+2}, \ldots, a_p$. So σ_{i+1} is either a rearrangement of σ'_{i+1} or a rearrangement of $-\sigma'_{i+1}$. Since σ_{i+1} is graphical, σ'_{i+1} is graphical, that is, there exists a signed graph S' with degree sequence σ'_{i+1}. Let the vertices of S' be labeled so that

$$\operatorname{sdeg} v_i = \begin{cases} a_i - 1, & 2 \leqslant i \leqslant a_1 + 1, \\ a_i, & a_1 + 2 \leqslant i \leqslant p. \end{cases}$$

We now construct a new signed graph S by adding a vertex v_1 to S' and the positive edges $v_1 v_i$ for $i = 2, 3, \ldots, a_1 + 1$. Then S is a signed graph having degree sequence σ_i. Hence, σ_i is graphical.

Finally, since σ_k is a zero sequence and σ_k is graphical, it follows, by the previous argument, that all sequences preceding σ_k are graphical. In particular, σ_1 is graphical. □

As an example, consider the sequence σ_1: $3, 0, 0, 0, -3$. Observe that the H-sequence corresponding to σ_1 is σ_1, σ_2, σ_3, where

$$\sigma_1: 3, 0, 0, 0, 0, -3,$$
$$\sigma_2: 3, 1, 1, 1, 0,$$
$$\sigma_3: 0, 0, 0, 0.$$

Since σ_3 is a zero sequence, σ_1 is graphical.

On the basis of Theorem 7, then, if we are given a standard sequence σ_1 and construct its corresponding H-sequence σ_1, σ_2, ..., σ_k, it follows that σ_1 is graphical if σ_k is a zero sequence. But what if σ_k is a nonzero sequence? Then σ_k must contain a term whose absolute value equals the number of terms in σ_k. Of course, σ_k is not graphical. What does this say about σ_1? In order to gain some insight into the answer to this question, we present the following necessary condition for a standard sequence to be graphical.

Theorem 8. *Let $\sigma: d_1, d_2, \ldots, d_p$ be a standard sequence. If σ is graphical, then for every pair a, b of nonnegative integers with $1 \leqslant a + b \leqslant p$ such that $d_a \geqslant 0$ when $a > 0$ and $d_{p-b+1} \leqslant 0$ when $b > 0$, it follows that*

$$\sum_{i=1}^{a} d_i - \sum_{i=p-b+1}^{p} d_i \leqslant (p-1)(a+b) - 2ab.$$

Proof. Since σ is graphical, there exists a signed graph S with vertices v_1, v_2, …, v_p such that $\operatorname{sdeg} v_i = d_i$ for $i = 1, 2, \ldots, p$. Let a and b be nonnegative integers with $1 \leqslant a + b \leqslant p$ such that $d_a \geqslant 0$ when $a > 0$ and $d_{p-b+1} \leqslant 0$ when $b > 0$.

Let $A = \{v_1, v_2, \ldots, v_a\}$ and $B = \{v_{p-b+1}, v_{p-b+2}, \ldots, v_p\}$. Furthermore, let C denote the set consisting of the remaining $p - a - b$ vertices of S. If each vertex v_i is incident with d_i^+ positive edges and d_i^- negative edges, then

$$\sum_{i=1}^{a} d_i - \sum_{i=p-b+1}^{p} d_i = \sum_{i=1}^{a} (d_i^+ - d_i^-) - \sum_{i=p-b+1}^{p} (d_i^+ - d_i^-).$$

Note that $\sum_{i=1}^{a} d_i$ is the sum of the degrees of the vertices in A, while $\sum_{i=p-b+1}^{p} d_i$ is the sum of the degrees of the vertices in B. Now if e is a positive edge joining a vertex of A to a vertex of B, then e contributes 1 to $\sum_{i=1}^{a} (d_i^+ - d_i^-)$ and also contributes 1 to $\sum_{i=p-b+1}^{p} (d_i^+ - d_i^-)$. Hence the contribution of e to the expression $\sum_{i=1}^{a} d_i - \sum_{i=p-b+1}^{p} d_i$ is 0. A similar situation results if e is a negative edge. Thus, when computing $\sum_{i=1}^{a} d_i - \sum_{i=p-b+1}^{p} d_i$, the edges joining vertices of A to vertices of B may be ignored.

Any other positive edge incident with the vertices v_i $(1 \leqslant i \leqslant a)$ either joins two vertices of A or joins a vertex of A to a vertex of C. Similarly, any other negative edge incident with the vertices of v_i $(p - b + 1 \leqslant i \leqslant p)$ joins two vertices of B or joins a vertex of B and a vertex of C. Thus

$$\begin{aligned}\sum_{i=1}^{a} d_i - \sum_{i=p-b+1}^{p} d_i &\leqslant a(a-1) + a(p-a-b) + b(b-1) + b(p-a-b) \\ &= (p-1)(a+b) - 2ab.\end{aligned}$$

□

This theorem gives us a way to check if a given sequence is *not* graphical, namely, we have the following corollary, which is simply a statement of the contrapositive of Theorem 8.

Corollary 8a. *Let $\sigma\colon d_1, d_2, \ldots, d_p$ be a standard sequence. If there exist nonnegative integers a and b with $1 \leqslant a + b \leqslant p$, where $d_a \geqslant 0$ if $a > 0$ and $d_{p-b+1} \leqslant 0$ if $b > 0$ such that*

$$\sum_{i=1}^{a} d_i - \sum_{i=p-b+1}^{p} d_i > (p-1)(a+b) - 2ab,$$

then σ is not graphical.

As in example, consider the sequence σ: $7,7,7,7,6,3,3,0$. Now choose $a = 4$ and $b = 1$, and observe that

$$\sum_{i=1}^{a} d_i - \sum_{i=p-b+1}^{p} d_i = 7+7+7+7-0 = 28$$

while $(p-1)(a+b) - 2ab = 7 \cdot 5 - 8 = 27$. Since $28 > 27$, it follows by Corollary 8a that σ is not graphical.

We now return to the question at hand, namely, if we are given a standard sequence σ_1 and its corresponding H-sequence $\sigma_1, \sigma_2, \ldots, \sigma_k$, where the first term of σ_k equals the number of terms in σ_k, then what can we say about σ_1? As our next result states, in this case, σ_{k-1} is not graphical.

Theorem 9. *Let σ_1 be a standard sequence, and let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be the H-sequence corresponding to σ_1. If the first term of σ_k equals the number of terms in σ_k, then σ_{k-1} is not graphical.*

Proof. Suppose that σ_k contains r terms. Then the first term of σ_k is r. Since σ_k was obtained by applying procedure H to σ_{k-1} and then standardization, it follows that σ_{k-1} contains $r+1$ terms and contains either (1) the terms r and $-(r-1)$, or (2) the terms $r-1$, $-(r-1)$, and $-(r-1)$. We consider these two cases.

Case 1: *The sequence σ_{k-1} contains r and $-(r-1)$.* Then choose $a = 1$ and $b = 1$ in Corollary 8a. Now observe that

$$r - \big(-(r-1)\big) = 2r - 1 > 2r - 2.$$

Hence it follows that σ_{k-1} is not graphical.

Case 2: *Thc sequence σ_{k-1} contains the terms $r-1$, $-(r-1)$, and $-(r-1)$.* Then choose $a = 1$ and $b = 2$. Then

$$r - 1 + 2(r-1) = 3r - 3 > 3r - 4.$$

Thus, again by Corollary 8a, σ_{k-1} is not graphical. □

The following result extends the previous one to say that in addition to σ_{k-1}, the two sequences σ_{k-2} and σ_{k-3} are also not graphical when the first term of σ_k equals the number of terms in σ_k. The tedious proof is similar to that of Theorem 9 and hence is omitted.

Theorem 10. *Let σ_1 be a standard sequence with corresponding H-sequence σ_1, $\sigma_2, \ldots, \sigma_k$. If the first term of σ_k equals the number of terms in σ_k, then σ_{k-1}, σ_{k-2}, and σ_{k-3} are not graphical.*

Now let us consider some examples. First, we have already seen that the sequence $\sigma_1 : 7, 7, 7, 7, 6, 3, 3, 0$ is not graphical. Observe that the H-sequence corresponding to σ_1 is σ_1, σ_2, σ_3, σ_4, σ_5, where

$$\begin{aligned}
&\sigma_1 : 7, 7, 7, 7, 6, 3, 3, 0,\\
&\sigma_2 : 6, 6, 6, 5, 2, 2, -1,\\
&\sigma_3 : 5, 5, 4, 1, 1, -2,\\
&\sigma_4 : 4, 3, 0, 0, -3,\\
&\sigma_5 : 4, 1, 1, -2.
\end{aligned}$$

Since σ_5 contains four terms, the first of which is 4, it follows by Theorem 10 that σ_2, σ_3, and σ_4 are not graphical. It is important to note, however, that we cannot use Theorem 10 to show that σ_1 is not graphical. Recall that, in fact, we used Corollary 8a to verify this.

As a second example, consider the sequence $\sigma_1 : 5, 5, 5, 5, 4, 2$ and its corresponding H-sequence σ_1, σ_2, σ_3, σ_4, σ_5. So we have

$$\begin{aligned}
&\sigma_1 : 5, 5, 5, 5, 4, 2,\\
&\sigma_2 : 4, 4, 4, 3, 1,\\
&\sigma_3 : 3, 3, 2, 0,\\
&\sigma_4 : 2, 1, -1,\\
&\sigma_5 : 2, 0.
\end{aligned}$$

Again, by Theorem 10, we know that σ_2, σ_3, and σ_4 are not graphical.We would like to determine whether σ_1 is graphical. Obviously, we cannot use Theorem 10 to show that σ_1 is not graphical. In the previous example, we used Corollary 8a. It turns out that the same approach will not work here, that is, for every pair a, b of nonnegative integers with $1 \leqslant a + b \leqslant 6$ such that $d_a \geqslant 0$ if $a > 0$ and $d_{p-b+1} \leqslant 0$ if $b > 0$, it follows that

$$\sum_{i=1}^{a} d_i - \sum_{i=p-b+1}^{p} d_i \leqslant (p-1)(a+b) - 2ab.$$

Of course, from this we will neither be able to conclude that σ_1 is graphical nor not graphical. Notice that σ_1 has no zeros or negative terms. Thus the only choice for b is 0. Therefore, the inequality above simplifies to

$$\sum_{i=1}^{a} d_i \leqslant (p-1)a,$$

which is true for each choice of a. So the sequence σ_1 cannot be shown to be not graphical by Theorem 8.

We now prove that, in fact, σ_1 is not graphical, for suppose, instead, that σ_1 is graphical. Then there exists a signed graph S with degree sequence σ_1. So four vertices of S have degree 5, that is, four vertices are joined by positive edges to every other vertex of S. These edges account for 14 of the edges of S. Since S can have at most 15 edges, it is clear that the two remaining vertices of S do not have the required degrees. So σ_1 is not graphical.

Conjecture 1. Let σ_1 be a given standard sequence and let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be its corresponding H-sequence. Then σ_1 is graphical if and only if σ_k is graphical.

The truth of this conjecture can be established from the following conjecture.

Conjecture 2. If $\sigma\colon d_1, d_2, \ldots, d_p$ is a standard graphical sequence, then there exists a signed graph containing a vertex of degree d_1 incident only with positive edges.

References

[1] *J. Grossman, F. Harary and M. Klawe*: Generalized ramsey theory for graphs X: Double stars. Discrete Math. *28* (1979), 247–254.

[2] *S. L. Hakimi*: On the realizability of a set of integers as degrees of the vertices of a graph. J. SIAM Appl. Math. *10* (1962), 496–506.

[3] *F. Harary*: On the notion of balance in a signed graph. Michigan Math. J. *2* (1953), 143–146.

[4] *F. Harary*: Graph Theory. Addison-Wesley, Reading, 1969.

[5] *V. Havel*: A remark on the existence of finite graphs. Časopis Pěst. Mat. *80* (1955), 477–480. (In Czech.)

Authors' addresses: *G. Chartrand, H. Gavlas, M. Schultz*, Western Michigan University, Dept. of Mathematics and Statistics, Kalamazoo, Michigan 49008-5152, U.S.A., *Frank Harary*, New Mexico State University, U.S.A.