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ON THE DISTRIBUTION OF MULTIPLICITIES OF ZEROS OF RIEMANN ZETA FUNCTION

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1. Formulation of results

In the present paper, studying the classical Hardy-Littlewood theorem on the distribution of zeros of the function $\zeta\left(\frac{1}{2}+it\right)$ we obtain new results concerning the distribution of multiplicities of zeros of the function $\zeta(s)$, $s=\sigma+it$ in some rectangles of the critical strip.

In 1918 Hardy and Littlewood proved a theorem (see [2], pp. 177–184) improving a theorem of Hardy from 1914 (see [1]), which can be formulated in the following way: for every sufficiently large $T>0$ the interval

$$(T, T+T^{\frac{1}{4}+2\omega})$$

($0<\omega$ an arbitrarily small number) contains a zero $t=\tilde{\gamma}$ of the function $\zeta\left(\frac{1}{2}+it\right)$ for which

$$2 \nmid n(\tilde{\gamma})$$

where $n(\tilde{\gamma})$ is the multiplicity of the zero $t=\tilde{\gamma}$.

This form of the Hardy-Littlewood theorem makes it possible to extend the theorem from the critical straight line to the critical strip in the following way.

Let $n(\varrho)$ denote the multiplicity of a nontrivial zero $\varrho=\beta+i\gamma$ of the function $\zeta(s)$. Then the following theorem holds.

Theorem. *If*

$$\zeta(s)=O(t^{a+\omega/2}), \quad \frac{1}{2}\leqslant\sigma, \quad 0\leqslant a\leqslant\frac{1}{6}, \tag{1}$$

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then for all sufficiently large $T > 0$ the rectangle

$$Q_k(a,\omega) = \{s\colon \sigma \in (0,1), t \in (T, T + H_k(a,\omega))\},\quad H_k(a,\omega) = T^{(\frac{1}{4}+a+2\omega)\frac{1}{k}} \tag{2}$$

contains a zero $\tilde{\varrho}$ satisfying

$$2k \nmid n(\tilde{\varrho}), \tag{3}$$

where

$$\begin{aligned} k &= q, q+1, \dots, k_0, k_0 = \left[\frac{\ln T}{\psi(T) \ln\ln T}\right], \\ g = q_k(a) &= \begin{cases} 3 + [8a], & \text{if } 8a \text{ is noninteger}, \\ 2 + 8a, & \text{if } 8a \text{ is integer} \end{cases} \end{aligned} \tag{4}$$

(q is obtained from the condition $\left(\frac{1}{4} + a\right)\frac{1}{k} \leqslant \frac{1}{8}$) and $0 < \psi(t)$ is a function growing arbitrarily slowly to ∞ for $T \to \infty$.

Since we have (see [7], pp. 97, 109)

$$\zeta(s) = O(^{1/6+\omega/2}), \quad \frac{1}{2} \leqslant \sigma, \tag{5}$$

i.e. $a = 1/6$ (see (1)) independently of any hypothesis, Theorem yields

Corollary 1. *For every sufficiently large $T > 0$ the rectangle*

$$Q_k\left(\frac{1}{6},\omega\right) = \left\{s\colon \sigma \in (0,1), t \in \left(T, T + H_k\left(\frac{1}{6},\omega\right)\right)\right\},\quad H_k\left(\frac{1}{6},\omega\right) = T^{\frac{5}{12k}+\frac{2\omega}{k}} \tag{6}$$

contains a zero $\tilde{\varrho}$ satisfying $2k \nmid n(\tilde{\varrho})$, where

$$k = 4, 5, \dots, k_0. \tag{7}$$

R e m a r k 1. Until now, the existing values of a in (1) represent no essential improvement of the value 1/6 which we have used in (5).

Consequently, Corollary 1 implies:
the rectangle

$$Q_4\Big(\frac{1}{6},\omega\Big) = \{s\colon \sigma \in (0,1), t \in (T, T + T^{\frac{5}{48}+\frac{\omega}{2}})\}$$

contains a zero $\tilde{\varrho}$ satisfying $8 \nmid n(\tilde{\varrho})$,
the rectangle

$$Q_5\Big(\frac{1}{6},\omega\Big) = \{s\colon \sigma \in (0,1), t \in (T, T + T^{\frac{1}{12}+\frac{2\omega}{5}})\}$$

contains a zero $\tilde{\varrho}$ satisfying $10 \nmid n(\tilde{\varrho})$, etc.

Taking into account Lindellöf's conjecture (see [7], p. 323)

$$\zeta(s) = O(t^{\omega/2}), \quad \frac{1}{2} \leqslant \sigma,$$

i.e. $a = 0$ (see (1)), Theorem implies

Corollary 2. *By Lindellöf's conjecture, for every sufficiently large $T > 0$ the rectangle*

$$Q_k(0,\omega) = \{s\colon \sigma \in (0,1), t \in (T, T + T^{\frac{1}{4k}+\frac{2\omega}{k}})\} \tag{8}$$

contains a zero $\tilde{\varrho}$ satisfying $2k \nmid n(\tilde{\varrho})$, where

$$k = 2, 3, \dots, k_0. \tag{9}$$

Consequently, using Corollary 2 we have:
the rectangle

$$Q_2(0,\omega) = \{s\colon \sigma \in (0,1), t \in (T, T + T^{\frac{1}{8}+\omega})\}$$

contains a zero $\tilde{\varrho}$ satisfying $4 \nmid n(\tilde{\varrho})$, etc.

R e m a r k 2. Let us explicitly point out the influence of Lindellöf's conjecture on the initial values k (cf. (7), (9)).

Further (see (2), (7)), we have

$$H_k(a,\omega) \in \big\langle (\ln T)^{\{\frac{1}{4}+a+2\omega+o(1)\}\psi(T)}, T^{\frac{5}{48}+\frac{\omega}{2}} \big\rangle.$$

R e m a r k 3. Consequently, Theorem is valid even for rectangles $Q_k(a,\omega)$ of "very small" height $H_k(a,\omega)$ (in the context of the results on the critical line), say (using $H_{\bar{k}}$ from (6))

$$H_{\bar{k}} \sim (\ln T)^{\ln\ln\ln T}, \quad \bar{k} \sim \left(\frac{5}{12} + 2\omega\right)\frac{\ln T}{\ln\ln T \cdot \ln\ln\ln T}.$$

It seems probable that the method of trigonometric sums does not extend to intervals of length $(\ln T)^{\psi(T)}$. It would be interesting to compare this result with a remark by I. M. Vinogradov (see [4], p. 13, lines 4-8 from below) concerning the possibilities of the method of trigonometric sums for an estimate of the remainder in the law of distribution of prime numbers.

Let us further recall that for the distribution of multiplicities of zeros of the function $\zeta(s)$ we have the estimate (see [7], p. 209)

$$1 \leqslant n(\varrho) \leqslant A \ln T. \tag{10}$$

The condition (10) has been, until now, the only information concerning the distribution of multiplicities $n(\varrho)$ in rectangles $Q_k(a,\omega)$ (see (2), (6), (8)). Nonetheless, the condition (10) offers a great number of possibilities for the distribution of multiplicities. In connection with this fact we make

R e m a r k 4. The above theorem (see also Corollaries 1,2) excludes a great number of types of the distribution of multiplicities of zeros of the function $\zeta(s)$, $s \in Q_k(a,\omega)$ which are permitted by the condition (10).

Finally, let us note that the crucial moment of the proof of Theorem is the application of the properties of univalent analytic branches of multivalued functions (cf. [5], [6])

$$\sqrt[k]{\zeta(s)}, \quad \sqrt[k]{G(s)}$$

where (see [7], pp. 81, 94)

$$G(s) = \{\chi(s)\}^{-1/2}\zeta(s), \quad \chi(s) = \frac{2^{s-1}\pi^s}{\Gamma(s)\cos\frac{\pi s}{2}}. \tag{11}$$

Here the zeros of the function $\zeta(s)$ are the branching points of the multivalued functions.

The subsequent sections of the paper contain the proof of Theorem.

Lemma 1. *Let*

$$\varphi(t;k,n)=\frac{\vartheta_1(t)}{k}-t\ln n,\quad \vartheta_1(t)=\frac{t}{2}\ln\frac{t}{2\pi}-\frac{t}{2}-\frac{\pi}{8}. \tag{12}$$

Then for an arbitrary sufficiently large $T>0$ we have

$$\int_T^{T+H_k} \mathrm{e}^{\mathrm{i}\varphi(t;k,n)}\,\mathrm{d}t = \begin{cases} o(1), & n=1,\\ O\Big(\frac{1}{\ln\frac{\tau}{n}}\Big), & 2\leqslant n<[P^{1/k}]=\tau,\quad P=\sqrt{\frac{T}{2\pi}},\\ O\Big(\frac{1}{\ln\frac{n}{\tau'}}\Big), & n>\tau+1=\tau',\end{cases} \tag{13}$$

where $k=2,3,\ldots,k_0$, $H_k=H_k(a,\omega)$ (see (2), (4)).

Proof. (A) Since

$$\varphi_t'(t;,k,1)=\frac{1}{2k}\ln\frac{t}{2\pi}\geqslant\frac{1}{2k}\ln\frac{T}{2\pi}=\frac{1}{k}\ln P,\quad \varphi_{t^2}''=\frac{1}{2kt}>0$$

for $t\in\langle T,T+H_k\rangle$, we obtain (see [7], p. 73, Lemma 1) an estimate

$$\int_T^{T+H_k} \mathrm{e}^{\mathrm{i}\varphi(t;k,1)}\,\mathrm{d}t = O\Big(\frac{k}{\ln P}\Big)=O\Big(\frac{k_0}{\ln P}\Big)=o(1)$$

(see (4)), which is the first estimate in (13).

(B) If $2\leqslant n<[P^{1/k}]=\tau$, then

$$\varphi_t'(t;k,n)\geqslant\frac{1}{k}\ln P-\ln n=\ln\frac{P^{1/k}}{n}\geqslant\ln\frac{\tau}{n}>0,\quad \varphi_{t^2}''>0, \tag{14}$$

and in the same way as in the case (A) we obtain an estimate

$$\int_T^{T+H_k} \mathrm{e}^{\mathrm{i}\varphi(t;k,n)}\,\mathrm{d}t = O\Big(\frac{1}{\ln\frac{\tau}{n}}\Big),$$

i.e. the second estimate in (13).

(C) If $n > \tau + 1 = \tau'$ then, since

$$\ln \frac{n}{\tau+1} \geqslant \ln \frac{\tau+2}{\tau+1} = \ln\left(1 + \frac{1}{\tau+1}\right) \sim \left(\frac{T}{2\pi}\right)^{-1/2k},$$
$$\frac{H_k}{kT} = O(T^{-1+\frac{5}{12k}+\frac{2\omega}{k}}) = O(T^{-\frac{7}{12}+2\omega}),$$

we obtain (cf. (14))

$$\begin{aligned}-\varphi_t'(t;k,n) &= \ln n - \frac{1}{2k}\ln\frac{t}{2\pi} \geqslant \ln n - \frac{1}{2k}\ln\frac{T+H_k}{2\pi}\\ &= \ln n - \frac{1}{2k}\ln\frac{T}{2\pi} - \frac{1}{2k}\ln\left(1+\frac{H_k}{T}\right) = \ln\frac{n}{P^{1/k}} + O\left(\frac{H_k}{kT}\right)\\ &> \ln\frac{n}{[P^{1/k}]+1} + O\left(\frac{H_k}{kT}\right) = \ln\frac{n}{\tau+1} + O\left(\frac{H_k}{kT}\right) > \frac{1}{2}\ln\frac{n}{\tau+1} > 0,\\ -\varphi_{t^2}'' &< 0.\end{aligned}$$

Consequently, the estimate

$$\int_T^{T+H_k} \mathrm{e}^{\mathrm{i}\varphi(t;k,n)}\,\mathrm{d}t = O\left(\frac{1}{\ln\frac{n}{\tau'}}\right).$$

holds, which is the last estimate in (13). □

3. Infinite sum of trigonometric integrals

For $\sigma > 1$ let us put (cf. [3], p. 11, [5], [6])

(15) $$\{\zeta(s)\}^{\frac{1}{k}} = \sum_{n=1}^{\infty}\frac{\alpha_n(k)}{n^s}, \quad \alpha_1(k) = 1, \quad k = 2,3,\ldots,k_0,$$

where we fix the branch of the multivalued function for which $\{\zeta(\sigma)\}^{1/k} > 0$.

Using the Euler product we obtain

$$\{\zeta(s)\}^{\frac{1}{k}} = \prod_p (1-p^{-s})^{-\frac{1}{k}} = \prod_p \sum_{m=0}^{\infty}(-1)^m\binom{-\frac{1}{k}}{m}p^{-ms}$$

(where p ranges over prime numbers). Since

$$0 < (-1)^m\binom{-\frac{1}{k}}{m} \leqslant 1,$$

we conclude that

$$\alpha_n(k) = (-1)^{m_1+\ldots+m_j} \binom{-\frac{1}{k}}{m_1} \ldots \binom{-\frac{1}{k}}{m_j}, \quad n = 2, 3, \ldots .$$

for $n = p_1^{m_1} \cdot p_2^{m_2} \ldots p_j^{m_j}$. Consequently,

$$0 < \alpha_n(k) \leqslant 1, \quad n = 1, 2, \ldots . \tag{16}$$

Lemma 2.

$$\sum_{n=1}^{\infty} \frac{\alpha_n(k)}{n^{1+\omega}} \cdot \int_T^{T+H_k} e^{i\varphi(t;k,n)}\, dt = O(1), \quad k = 2, 3, \ldots, k_0. \tag{17}$$

Proof. First of all, by virtue of (13), (15) we have

$$\alpha_1(k) \cdot \int_T^{T+H_k} e^{i\varphi(t;k,1)}\, dt = o(1). \tag{18}$$

Further (see (13), the second estimate),

$$\mathcal{I}_1 = \sum_{2\leqslant n<\tau} \frac{\alpha_n(k)}{n^{1+\omega} \ln \frac{\tau}{n}} = \sum_{2\leqslant n<\frac{\tau}{2}} + \sum_{\frac{\tau}{2}\leqslant n<\tau} = \mathcal{I}_2 + \mathcal{I}_3.$$

Evidently (see (16)),

$$\mathcal{I}_2 = O\left(\sum_{n=2}^{\infty} \frac{1}{n^{1+\omega}}\right) = O(1).$$

For $\mathcal{I}_3$ we have

$$\ln \frac{\tau}{n} = \ln \frac{\tau}{\tau - l} = -\ln\left(1 - \frac{l}{\tau}\right) > A\frac{l}{\tau}, \quad l = 1, \ldots, L, \quad L \leqslant \frac{\tau}{2}, \tag{19}$$

and, consequently (see (16)),

$$\mathcal{I}_3 = O\left(\tau \cdot \sum_{1\leqslant l\leqslant\frac{\tau}{2}} \frac{1}{\tau^{1+\omega} \cdot l}\right) = O\left(\frac{\ln T}{\tau^{\omega}}\right) = O(T^{-\frac{\omega}{2k}} \cdot \ln T).$$

Using (4) we obtain

$$\mathcal{I}_1 = O(1) + O(T^{-\frac{\omega}{2k}} \cdot \ln T) = O(1) + O(T^{-\frac{\omega}{2k_0}} \cdot \ln T) = O(1). \tag{20}$$

Now let us set (see (13), the last estimate)

$$\mathcal{I}_4 = \sum_{\tau'<n} \frac{\alpha_n(k)}{n^{1+\omega} \ln \frac{n}{\tau'}} = \sum_{\tau'<n\leqslant\frac{3}{2}\tau'} + \sum_{\frac{3}{2}\tau'<n} = \mathcal{I}_5 + \mathcal{I}_6.$$

For I_5 we have (cf. (19))

$$\ln \frac{n}{\tau'} = \ln \frac{l+\tau'}{\tau'} = \ln\Bigl(1+\frac{l}{\tau'}\Bigr) > A\frac{l}{\tau'}, \quad l = 1, \dots, L_1, \quad L_1 \leqslant \frac{\tau'}{2}.$$

This implies (cf. $\mathcal{I}_3$)

$$\mathcal{I}_5 = O(T^{-\frac{\omega}{2k}} \ln T) = O(T^{-\frac{\omega}{2k_0}} \ln T) = o(1).$$

Since evidently $\mathcal{I}_6 = O(1)$, we have

$$\mathcal{I}_4 = O(1). \tag{21}$$

For the other terms of the infinite sum we have

$$\sum_{\tau\leqslant n\leqslant\tau+1} \frac{\alpha_n(k)}{n^{1+\omega}} \cdot \int\limits_{T}^{T+H_k} \mathrm{e}^{\mathrm{i}\varphi(t;k,n)}\,\mathrm{d}t = O\Bigl(\frac{H_k}{\tau^{1+\omega}}\Bigr) \tag{22}$$
$$= O(H_k \cdot T^{-\frac{1}{2k}-\frac{\omega}{2k}}) = O(T^{-(\frac{1}{4}-a-2\omega)\frac{1}{k}})$$
$$= O(T^{-(\frac{1}{12}-2\omega)\frac{1}{k_0}}) = o(1).$$

Finally, by virtue of (18), (20), (22) we arrive at the estimate (17). □

4. The choice of the univalent branch of the function $\{\zeta(s)\}^{1/k}$

Let us denote by Π_k the rectangle with vertices at the points

$$\frac{1}{2} + \mathrm{i}T, \quad \frac{1}{2} + \mathrm{i}(T+H_k), \quad 1+\omega+\mathrm{i}(T+H_k), \quad 1+\omega+\mathrm{i}T.$$

Without loss of generality we may assume that the horizontal segments joining the points $1/2+\mathrm{i}T$, $1+\omega+\mathrm{i}T$ and $1/2+\mathrm{i}(T+H_k)$, $1+\omega+\mathrm{i}(T+H_k)$, respectively, do not contain zeros of the function $\zeta(s)$.

Let $\varrho_{r,l}$ be zeros of the function $\zeta(s)$ lying in the rectangle Π_k, i.e.

$$\varrho_{r,l} = \beta_r + \mathrm{i}g_{r,l}, \quad r = 0, 1, \dots, m, \quad l = 1, \dots, p_r$$

(of course, $p_r = p_r(a, k, \omega)$), where

$$\frac{1}{2} \leqslant \beta_r < 1, \quad T < \gamma_{r,l} < T + H_k \quad \left(\beta_0 = \frac{1}{2}\right).$$

The zeros $\varrho_{r,l}$ are the branching points of the multivalued function $\{\zeta(s)\}^{1/k}$.

Let us define a contour $C_k(\varepsilon) \subset \Pi_k$ in the following way:

$$C_k(\varepsilon) = L_0^k(\varepsilon) \cup M_0^{k,1} \cup \left\{ \bigcup_{r=1}^{m} \left(L_r^k(\varepsilon) \cup M_r^{k,1}\right) \right\} \cup L_{m+1}^k \cup M_2,$$

where

(A) $L_0^k(\varepsilon)$ is the segment joining the points $\frac{1}{2} + \mathrm{i}T$, $\frac{1}{2} + \mathrm{i}(T + H_k)$ modified by semicircles lying in Π_k; we circumvent the zero $\varrho_{0,l}$, $l = 1, \ldots, p_0$ along a semicircle with center at $\varrho_{0,l}$ and radius ε ($0 < \varepsilon$ an arbitrarily small number),

(B) $L_r^k(\varepsilon) = L_r^{k,1}(\varepsilon) \cup L_r^{k,2}(\varepsilon)$, $r = 1, \ldots, m$ where $L_r^{k,1}(\varepsilon)$ is the left "bank" of the cut joining the points

$$\beta_r + \mathrm{i}(T + H_k), \quad \beta_k + \mathrm{i}\gamma_{r,p_r} = \varrho_{r,p_r},$$

modified by small semicircles (we circumvent a point $\varrho_{r,l}$, $l = 1, \ldots, p_r$ along the left semicircle with center at this point and radius ε) and $L_r^{k,2}(\varepsilon)$ is the right "bank" of the cut joining the points

$$\varrho_{r,p_r}, \quad \beta_r + \mathrm{i}(T + H_k)$$

(now the point $\varrho_{r,l}$, $l = 1, \ldots, p_r$ is circumvented along a small right semicircle),

(C) L_{m+1}^k is the segment joining the points

$$1 + \omega + \mathrm{i}(T + H_k), \quad 1 + \omega + \mathrm{i}T,$$

(D) $M_0^{k,1}$ is the segment joining the points

$$\frac{1}{2} + \mathrm{i}(T + H_k), \quad \beta_1 + \mathrm{i}(T + H_k),$$

$M_r^{k,1}$ is the segment joining the points

$$\beta_r + \mathrm{i}(T + H_k), \quad \beta_{r+1} + \mathrm{i}(T + H_k), \quad r = 1, \ldots, m - 1,$$

$M_m^{k,1}$ is the segment joining the points

$$\beta_m + \mathrm{i}(T + H_k), \quad 1 + \omega + \mathrm{i}(T + H_k),$$

(E) M_2 is the point joining the points

$$1 + \omega + \mathrm{i}T, \quad \frac{1}{2} + \mathrm{i}T$$

and closing the contour $C_k(\varepsilon)$.

Let us denote by $D_k(\varepsilon)$ the domain bounded by the contour $C_k(\varepsilon)$. Since

$$\zeta(s) \neq 0, \quad s \in D_k(\varepsilon), \tag{23}$$

the multivalued function $\{\zeta(s)\}^{1/k}$ splits in $D_k(\varepsilon)$ into k univalent analytic branches.

Now we fix the desired analytic branch—let us denote it by $\{\zeta(s)\}_0^{1/k}$—by the condition $(\zeta(\frac{1}{2} + \mathrm{i}T) \neq 0)$

$$\left\{\zeta\left(\frac{1}{2} + \mathrm{i}T\right)\right\}^{\frac{1}{k}} = \left|\zeta\left(\frac{1}{2} + \mathrm{i}T\right)\right|^{\frac{1}{k}} \cdot \exp\left\{\mathrm{i}\,\frac{\arg\zeta(\frac{1}{2} + \mathrm{i}T)}{k}\right\}; \tag{24}$$

doing so we have to observe continuous change of the argument of the function $\{\zeta(s)\}_0^{1/k}$ along the contour $C_k(\varepsilon)$.

Finally, let us introduce a contour

$$C_k = \lim_{\varepsilon \to 0} C_k(\varepsilon). \tag{25}$$

5. Integrals of the function $\{\zeta(s)\}_0^{1/k}$ along the segment of the contour C_k

Let $n_{r,q} = n(\varrho_{r,q})$ be the multiplicity of the zero $\varrho_{r,q}$. Let us set

$$\Delta_{r,l} = \frac{\pi}{k}\sum_{q=1}^{l} n_{r,q}, \quad l = 1, \dots, p_r, \tag{26}$$

$$\Delta^r = \Delta_{0,p_0} + 2\sum_{l=1}^{r} \Delta_{l,p_r}, \quad r = 0, 1, \dots, m$$

(of course, $\sum_1^0 = 0$).

Further, let (see (25))

$$L_r^k = \lim_{\varepsilon \to 0} L_r^k(\varepsilon), \quad r = 0, 1, \dots, m. \tag{27}$$

Lemma 3.

$$\int_{L_0^k} \{\zeta(s)\}_0^{\frac{1}{k}}\, \mathrm{d}s = \mathrm{i} \int_T^{\gamma_{0,1}} \{\zeta(\beta_0 + \mathrm{i}T)\}^{\frac{1}{k}}\, \mathrm{d}t + \mathrm{i} \sum_{l=1}^{p_0-1} \mathrm{e}^{\mathrm{i}\Delta_{0,l}} \int_{\gamma_{0,l}}^{\gamma_{0,l+1}} + \mathrm{i}\mathrm{e}^{\mathrm{i}\Delta^0} \cdot \int_{\gamma_{0,p_0}}^{T+H_k}, \tag{28}$$

$$\begin{aligned}\int_{L_r^k} \{\zeta(s)\}_0^{\frac{1}{k}}\, \mathrm{d}s &= \mathrm{i}\mathrm{e}^{\mathrm{i}\Delta^{r-1}} \cdot (\mathrm{e}^{\mathrm{i}2\Delta_{r,p_r}} - 1) \cdot \int_{\gamma_{r,1}}^{T+H_k} \{\zeta(\beta_r + \mathrm{i}t)\}^{\frac{1}{k}}\, \mathrm{d}t \\ &+ \mathrm{i}\mathrm{e}^{\mathrm{i}(\Delta^{r-1} + \Delta_{r,1})} \cdot (\mathrm{e}^{\mathrm{i}2(\Delta_{r,p_r} - \Delta_{r,1})} - 1) \cdot \int_{\gamma_{r,2}}^{\gamma_{r,1}} + \dots \\ &+ \mathrm{i}\mathrm{e}^{\mathrm{i}(\Delta^{r-1} + \Delta_{r,p_r-1})} \cdot (\mathrm{e}^{\mathrm{i}2(\Delta_{r,p_r} - \Delta_{r,p_r-1})} - 1) \cdot \int_{\gamma_{r,p_r}}^{\gamma_{r,p_r-1}},\end{aligned} \tag{29}$$

for $r = 1, \dots, m$,

$$\int_{L_{m+1}^k} \{\zeta(s)\}_0^{\frac{1}{k}}\, \mathrm{d}s = -\mathrm{i}\mathrm{e}^{\mathrm{i}\Delta^m} \cdot \int_T^{T+H_k} \{\zeta(1 + \omega + \mathrm{i}t)\}^{\frac{1}{k}}\, \mathrm{d}t, \tag{30}$$

$$\int_{M^{k,1}} \{\zeta(s)\}_0^{\frac{1}{k}}\, \mathrm{d}s = \mathrm{e}^{\mathrm{i}\Delta^0} \cdot \int_{1/2}^{\beta} \{\zeta(\sigma + \mathrm{i}(T + H_k))\}_0^{\frac{1}{k}}\, \mathrm{d}\sigma + \sum_{r=1}^{m} \mathrm{e}^{\mathrm{i}\Delta^r} \cdot \int_{\beta_{r-1}}^{\beta_r} + \mathrm{e}^{\mathrm{i}\Delta^m} \cdot \int_{\beta_m}^{1+\omega}, \tag{31}$$

where $M^{k,1} = \bigcup_{r=0}^{m} M_r^{k,1}$, *and finally*

$$\int_{M_2} \{\zeta(s)\}_0^{\frac{1}{k}}\, \mathrm{d}s = -\mathrm{i}\mathrm{e}^{\mathrm{i}\Delta^m} \cdot \int_{1/2}^{1+\omega} \{\zeta(\sigma + \mathrm{i}T)\}^{\frac{1}{k}}\, \mathrm{d}\sigma. \tag{32}$$

Proof. (A) First of all, we have

$$\{\zeta(s)\}_0^{\frac{1}{k}} = \left\{\zeta\left(\frac{1}{2} + \mathrm{i}t\right)\right\}^{\frac{1}{k}}, \quad t \in \langle T, \gamma_{0,1} - \varepsilon\rangle, \tag{33}$$

where the value at the point $\frac{1}{2} + \mathrm{i}T$ is given by the condition (24).

In the neighborhood of the zero $\varrho_{0,1} = \frac{1}{2} + \mathrm{i}\gamma_{0,1}$ of multiplicity $n_{0,1}$ we have

$$\zeta(s) = (s - \varrho_{0,1})^{n_{0,1}} \cdot F(s), \quad F(\varrho_{0,1}) \neq 0,$$

where $F(s)$ is an analytic function. Consequently,

$$\arg\zeta(s) = n_{0,1} \cdot \arg(s - \varrho_{0,1}) + \arg F(s). \tag{34}$$

Since on the semicircle joining the points $\frac{1}{2}+\mathrm{i}(\gamma_{0,1}-\varepsilon)$, $\frac{1}{2}+\mathrm{i}(\gamma_{0,1}+\varepsilon)$ the increments of the arguments standing on the right hand side of (34) satisfy

$$\Delta\arg(s - \varrho_{0,1}) = \pi, \quad \Delta\arg F(s) = o(1), \quad \varepsilon \to 0$$

we have

$$\Delta\arg\zeta(s) = \pi n_{0,1} + o(1),$$

and hence

$$\Delta\arg\{\zeta(s)\}_0^{\frac{1}{k}} = \frac{\pi}{k} n_{0,1} + o(1).$$

Consequently (cf. (33)),

$$\{\zeta(s)\}_0^{\frac{1}{k}} = \left\{\zeta\left(\frac{1}{2} + \mathrm{i}t\right)\right\}^{\frac{1}{k}} \cdot \exp\left\{\mathrm{i}\frac{\pi}{k} n_{0,1} + o(1)\right\},$$
$$t \in \langle\gamma_{0,1} + \varepsilon, \gamma_{0,2} - \varepsilon\rangle,$$

which for $\varepsilon \to 0$ yields (see also (26))

$$\{\zeta(s)\}_0^{\frac{1}{k}} = \left\{\zeta\left(\frac{1}{2} + \mathrm{i}t\right)\right\}^{\frac{1}{k}} \cdot \mathrm{e}^{\mathrm{i}\Delta_{0,1}}, \quad t \in (\gamma_{0,1}, \gamma_{0,2}). \tag{35}$$

Analogously we obtain

$$\{\zeta(s)\}_0^{\frac{1}{k}} = \left\{\zeta\left(\frac{1}{2} + \mathrm{i}t\right)\right\}^{\frac{1}{k}} \cdot \mathrm{e}^{\mathrm{i}\Delta_{0,2}}, \quad t \in (\gamma_{0,2}, \gamma_{0,3}), \dots, \tag{36}$$
$$\{\zeta(s)\}_0^{\frac{1}{k}} = \left\{\zeta\left(\frac{1}{2} + \mathrm{i}t\right)\right\}^{\frac{1}{k}} \cdot \mathrm{e}^{\mathrm{i}\Delta^0}, \quad t \in (\gamma_{0,p_0}, T + H_k).$$

□

Remark 5. In (35), (36) as well as in all other analogous cases we explicitly write the factor (constant in the corresponding interval) related to the increment of the argument corresponding to one circumvention of the branching point.

Taking into account the above argument, in virtue of (26), (27), (33), (35), (36), ... we obtain (28).

(B) By virtue of (A) we arrive at the point $\beta_1 + \mathrm{i}(T + H_k)$ with the value

$$\{\zeta(s)\}_0^{\frac{1}{k}} = \left\{\zeta\left(\beta_1 + \mathrm{i}(T + H_k)\right)\right\}^{\frac{1}{k}} \cdot \mathrm{e}^{\mathrm{i}\Delta^0}. \tag{37}$$

Further (see Sec. 4, point (B)), we obtain quite analogously to (A) the relations

$$\begin{aligned}\int_{L_1^{k,1}} \{\zeta(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s &= \lim_{\varepsilon\to 0}\int_{L_1^{k,1}(\varepsilon)} \{\zeta(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s \\ &= -\mathrm{i}\mathrm{e}^{\mathrm{i}\Delta^0}\cdot\int_{\gamma_{1,1}}^{T+H_k}\{\zeta(\beta_1+\mathrm{i}t)\}^{\frac{1}{k}}\,\mathrm{d}t - \mathrm{i}\mathrm{e}^{\mathrm{i}\Delta^0}\cdot\mathrm{e}^{\mathrm{i}\Delta_{1,1}}\cdot\int_{\gamma_{1,2}}^{\gamma_{1,1}} - \dots \\ &\quad -\mathrm{i}\mathrm{e}^{\mathrm{i}\Delta^0}\cdot\mathrm{e}^{\mathrm{i}\Delta_{1,p_1-2}}\cdot\int_{\gamma_{1,p_1-1}}^{\gamma_{1,p_1-2}} - \mathrm{i}\mathrm{e}^{\mathrm{i}\Delta^0}\cdot\mathrm{e}^{\mathrm{i}\Delta_{1,p_1-1}}\cdot\int_{\gamma_{1,p_1}}^{\gamma_{1,p_1-1}},\end{aligned} \tag{38}$$

$$\begin{aligned}\int_{L_1^{k,2}} \{\zeta(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s &= \lim_{\varepsilon\to 0}\int_{L_1^{k,2}(\varepsilon)} \{\zeta(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s \\ &= \mathrm{i}\mathrm{e}^{\mathrm{i}(\Delta^0+\Delta_{1,p_1-1})}\cdot\mathrm{e}^{\mathrm{i}\frac{2\pi}{k}n_{1,p_1}}\cdot\int_{\gamma_{1,p_1}}^{\gamma_{1,p_1-1}} \\ &\quad + \mathrm{i}\mathrm{e}^{\mathrm{i}(\Delta^0+\Delta_{1,p_1-1})}\cdot\mathrm{e}^{\mathrm{i}\frac{2\pi}{k}n_{1,p_1}+\mathrm{i}\frac{\pi}{k}n_{1,p_1-1}}\cdot\int_{\gamma_{1,p_1-1}}^{\gamma_{1,p_1-2}} + \dots \\ &\quad + \mathrm{i}\mathrm{e}^{\mathrm{i}(\Delta^0+\Delta_{1,p_1-1})}\cdot\mathrm{e}^{\mathrm{i}(\frac{2\pi}{k}n_{1,p_1}+\Delta_{1,p_1-1}-\Delta_{1,1})}\cdot\int_{\gamma_{1,2}}^{\gamma_{1,1}} \\ &\quad + \mathrm{i}\mathrm{e}^{\mathrm{i}(\Delta^0+\Delta_{1,p_1-1})}\cdot\mathrm{e}^{\mathrm{i}(\frac{2\pi}{k}n_{1,p_1}+\Delta_{1,p_1-1})}\cdot\int_{\gamma_{1,1}}^{T+H_k}.\end{aligned} \tag{39}$$

As, for example,

$$\Delta^0 + \Delta_{1,p_1-1} + \frac{2\pi}{k}n_{1,p_1} + \Delta_{1,p_1-1} = \Delta^0 + 2\Delta_{1,p_1},$$

we have by virtue of (38), (39)

$$\int_{L_1^k} \{\zeta(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s = \int_{L_1^{k,1}} + \int_{L_1^{k,2}} \tag{40}$$

$$= \mathrm{i}\mathrm{e}^{\mathrm{i}\Delta^0} \cdot (\mathrm{e}^{\mathrm{i}2\Delta_{1,p_1}} - 1) \cdot \int_{\gamma_{1,1}}^{T+H_k}$$

$$+ \mathrm{i}\mathrm{e}^{\mathrm{i}(\Delta^0+\Delta_{1,1})} \cdot (\mathrm{e}^{\mathrm{i}2(\Delta_{1,p_1}-\Delta_{1,1})} - 1) \cdot \int_{\gamma_{1,2}}^{\gamma_{1,1}} + \dots$$

$$+ \mathrm{i}\mathrm{e}^{\mathrm{i}(\Delta^0+\Delta_{1,p_1-2})} \cdot (\mathrm{e}^{\mathrm{i}2(\Delta_{1,p_1}-\Delta_{1,p_1-2})} - 1) \cdot \int_{\gamma_{1,p_1-1}}^{\gamma_{1,p_1-2}}$$

$$+ \mathrm{i}\mathrm{e}^{\mathrm{i}(\Delta^0+\Delta_{1,p_1-1})} \cdot (\mathrm{e}^{\mathrm{i}2(\Delta_{1,p_1}-\Delta_{1,p_1-1})} - 1) \cdot \int_{\gamma_{1,p_1}}^{\gamma_{1,p_1-1}},$$

which actually is (29), $r = 1$. Note that we start from the point $\beta_1 + \mathrm{i}(T + H_k)$ with the value (cf. (37))

$$\{\zeta(\beta_1 + \mathrm{i}(T + H_k))\}_0^{\frac{1}{k}} = \{\zeta(\beta_1 + \mathrm{i}(T + H_k))\}^{\frac{1}{k}} \cdot \mathrm{e}^{\mathrm{i}\Delta^1}. \tag{41}$$

Now it is already clear that the substitutions

$$\Delta^0 \to \Delta^1 \to \Delta^2 \to \dots,$$
$$(\Delta_{1,1}, \dots, \Delta_{1,p_1}) \to (\Delta_{2,1}, \dots, \Delta_{2,p_2}) \to (\Delta_{3,1}, \dots, \Delta_{3,p_3}) \to \dots,$$
$$(\gamma_{1,1}, \dots, \gamma_{1,p_1}) \to (\gamma_{2,1}, \dots, \gamma_{2,p_2}) \to (\gamma_{3,1}, \dots, \gamma_{3,p_3}) \to \dots$$

in (40) successively yield all the other relations in (29), $r = 2, \dots, m$. Here we start from the point $\beta_r + \mathrm{i}(T + H_k)$ with the value (cf. (41))

$$\{\zeta(\beta_r + \mathrm{i}(T + H_k))\}_0^{\frac{1}{k}} = \{\zeta(\beta_r + \mathrm{i}(T + H_k))\}^{\frac{1}{k}} \cdot \mathrm{e}^{\mathrm{i}\Delta^r}. \tag{42}$$

(C) The relations (30)–(32) are evident (see Sec. 4, points (C)–(E) and also (37), (42)).

Let us recall that (see [7], pp. 94, 383)

$$(43)\qquad Z(t)=\left\{\chi\left(\frac{1}{2}+\mathrm{i}t\right)\right\}^{-1/2}\zeta\left(\frac{1}{2}+\mathrm{i}t\right),\quad \left\{\chi\left(\frac{1}{2}+\mathrm{i}t\right)\right\}^{-1/2}=\mathrm{e}^{\mathrm{i}\vartheta(t)},$$
$$\vartheta=\vartheta_1+O\left(\frac{1}{t}\right)$$

(concerning ϑ_1 see (12)), where $Z(t)$ is real for real t.

Lemma A. *In the case* (1) *for* $T\to\infty$ *we have*

$$(44)\qquad \int_T^{T+H_k}|Z(t)|^{\frac{1}{k}}\,\mathrm{d}t\geqslant\{1+o(1)\}H_k-\left|\sum_{r=1}^{m}\int_{L_r^k}\{\zeta(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s\right|,\quad k=2,3,\dots,k_0.$$

P r o o f. By virtue of (1) we obtain the following estimates for the integrals along the horizontal segments $M^{k,1}$, M_2 (see (2), (31), (32):

$$(45)\qquad \int_{M^{k,1}}\{\zeta(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s,\quad \int_{M_2}\{\zeta(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s=O(T^{(a+\frac{\omega}{2})\cdot\frac{1}{k}})=o(H_k).$$

Further (see (15), (16), (30))

$$(46)\qquad \int_{L_{m+1}^k}\{\zeta(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s=-\mathrm{i}\mathrm{e}^{\mathrm{i}\Delta'''}\cdot\int_T^{T+H_k}\{\zeta(1+\omega+\mathrm{i}t)\}^{\frac{1}{k}}\,\mathrm{d}t$$
$$=-\mathrm{i}\mathrm{e}^{\mathrm{i}\Delta'''}\cdot\int_T^{T+H_k}\left(1+\sum_{n=2}^{\infty}\frac{\alpha_n(k)}{n^{1+\omega}}n^{-\mathrm{i}t}\right)\mathrm{d}t$$
$$=-\mathrm{i}\mathrm{e}^{\mathrm{i}\Delta'''}\cdot H_k+\mathrm{e}^{\mathrm{i}\Delta'''}\cdot\sum_{n=2}^{\infty}\frac{\alpha_n(k)}{n^{1+\omega}\ln n}\{n^{-\mathrm{i}(T+H_k)}-n^{-\mathrm{i}T}\}$$
$$=-\mathrm{i}\mathrm{e}^{\mathrm{i}\Delta'''}\cdot H_k+O(1).$$

However, $\{\zeta(s)\}_0^{1/k}$ is a univalent analytic branch in the domain $D_k(\varepsilon)$ bounded by the contour $C_k(\varepsilon)$. Consequently, the Cauchy theorem implies that

$$\int_{C_k(\varepsilon)}\{\zeta(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s=0$$

and thus (for $\varepsilon \to 0$, see (25))

$$\int_{C_k} \{\zeta(s)\}_0^{\frac{1}{k}} \, \mathrm{d}s = 0. \tag{47}$$

By virtue of (45), (46) this implies

$$\int_{L_0^k} \{\zeta(s)\}_0^{\frac{1}{k}} \, \mathrm{d}s = \mathrm{i}e^{\mathrm{i}\Delta^m} \cdot H_k + o(H_k) - \sum_{r=1}^{m} \int_{L_r^k} \{\zeta(s)\}_0^{\frac{1}{k}} \, \mathrm{d}s \,. \tag{48}$$

Finally, passing to absolute values in (48) and using the identity $|Z(t)| = \left|\zeta\left(\frac{1}{2}+\mathrm{i}T\right)\right|$ (see (43)) we obtain (44). □

7. The second fundamental lemma

For $\chi(s)$ (see (11)) we have for $t \to \infty$ in an arbitrary fixed strip $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ (see [7], p. 81)

$$\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+\,\mathrm{i}t-\frac{1}{2}} \cdot e^{\mathrm{i}(t+\frac{\pi}{4})} \cdot \left\{1 + O\left(\frac{1}{t}\right)\right\}. \tag{49}$$

Let us consider a multivalued function $\{G(s)\}^{1/k}$ (see (11)). Since $\zeta(s) \neq 0$, $s \in D_k(\varepsilon)$ (see (23)) and, by (49), $\{\chi(s)\}^{-1/2} \neq 0$, $s \in D_k(\varepsilon)$, we have $G(s) \neq 0$, $s \in D_k(\varepsilon)$. Consequently, $\{G(s)\}^{1/k}$ splits in the domain $D_k(\varepsilon)$ into k univalent analytic branches.

Further, let us recall that (see (11), (43))

$$G\left(\frac{1}{2} + \mathrm{i}t\right) = Z(t). \tag{50}$$

In what follows we will study the case when $Z(t)$ does not change sign in the interval $\langle T, T + H_k\rangle$.

(a) If

$$Z(t) \geqslant 0, \quad t \in \langle T, T + H_k\rangle\,, \quad Z(T) > 0, \tag{51}$$

then we fix the desired univalent branch of the function $\{G(s)\}^{1/k}$—let us denote it by $\{G(s)\}_0^{1/k}$—by the condition (cf. (24))

$$\left\{G\left(\frac{1}{2} + \mathrm{i}T\right)\right\}_0^{\frac{1}{k}} = \{Z(T)\}^{\frac{1}{k}} > 0. \tag{52}$$

(b) If $Z(t) \leqslant 0$, $t \in \langle T, T+H_k\rangle$ then we deal with $\{-G(s)\}_0^{1/k}$ and fix the branch $\{G(s)\}_0^{1/k}$ by the condition

$$\Big\{-G\Big(\frac{1}{2}+\mathrm{i}T\Big)\Big\}_0^{\frac{1}{k}} = \{-Z(T)\}^{\frac{1}{k}} > 0.$$

Lemma B. *If the estimate* (1) *is valid and* $Z(t)$ *does not change its sign in the interval* $\langle T, T+H_k\rangle$ *then for* $T\to\infty$ *we have*

$$\int_T^{\gamma_{0,1}} |Z(t)|^{\frac{1}{k}}\,\mathrm{d}t + \sum_{l=1}^{p_0} \mathrm{e}^{\mathrm{i}\Delta_{0,l}}\cdot \int_{\gamma_{0,l}}^{\gamma_{0,l+1}} |Z(t)|^{\frac{1}{k}}\,\mathrm{d}t + \mathrm{e}^{\mathrm{i}\Delta^0}\cdot \int_{\gamma_{0,p_0}}^{T+H_k} |Z(t)|^{\frac{1}{k}}\,\mathrm{d}t \tag{53}$$

$$= o(H_k) + \mathrm{i}\sum_{r=1}^{m} \int_{L_r^k} \{G(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s\,, \quad k=2,3,\ldots,k_0.$$

P r o o f. By virtue of (a), (b) it is sufficient to prove the case (51), (52). First of all we note that (see (47))

$$\int_{C_k} \{G(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s = 0. \tag{54}$$

Since (see (42), (43), (49))

$$\{\chi(s)\}^{-\frac{1}{2}} = \Big(\frac{t}{2\pi}\Big)^{\frac{1}{2}(\sigma-\frac{1}{2})}\cdot \mathrm{e}^{\mathrm{i}\vartheta_1(t)}\cdot\Big\{1+O\Big(\frac{1}{t}\Big)\Big\}, \tag{55}$$

we have (see (1), (2), (11); $\sigma \leqslant 1+\omega$)

$$\begin{aligned}\{G(s)\}_0^{\frac{1}{k}} &= O(T^{\frac{1}{2k}(\sigma-\frac{1}{2})+\frac{1}{k}(a+\frac{\omega}{2})})\\ &= O(T^{(\frac{1}{4}+a+\omega)\frac{1}{k}}) = o(H_k), \quad s\in D_k.\end{aligned}$$

Consequently, the integrals along the horizontal segments satisfy (cf. (45))

$$\int_{M^{k,1}} \{G(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s\,,\quad \int_{M_2} \{G(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s = o(H_k). \tag{56}$$

Further, since

$$\Big(\frac{t}{2\pi}\Big)^{\frac{1}{4}+\frac{\omega}{2}} = \Big(\frac{T}{2\pi}\Big)^{\frac{1}{4}+\frac{\omega}{2}} + O\Big(\frac{T^{\frac{1}{4}+\frac{\omega}{2}}\cdot H_k}{T}\Big),\quad t\in\langle T,T+H_k\rangle\,,$$

we have (see (55))

$$(57)\qquad \{\chi(1+\omega+\mathrm{i}t)\}^{-\frac{1}{2k}} = \left(\frac{T}{2\pi}\right)^{\frac{1}{4k}+\frac{\omega}{2k}} \cdot \mathrm{e}^{\mathrm{i}\frac{\vartheta_1(t)}{k}} \cdot \left\{1+O\left(\frac{H_k}{T}\right)\right\}$$

and (see (11), (15), (57))

$$(58)\qquad \{G(1+\omega+\mathrm{i}t)\}_0^{\frac{1}{k}} = \mathrm{e}^{\mathrm{i}\Delta'} \cdot \left(\frac{T}{2\pi}\right)^{\frac{1}{4k}+\frac{\omega}{2k}} \cdot \sum_{n=1}^{\infty} \frac{\alpha_n(k)}{n^{1+\omega}} \mathrm{e}^{\mathrm{i}\{\frac{\vartheta_1(t)}{k}-t\ln n\}} + O(T^{-1+\frac{1}{4k}+\frac{\omega}{k}} \cdot H_k)$$

(Δ' is connected with the choice of the branch). Consequently (see (17), (58)),

$$(59)\qquad \int_{L^k_{m+1}} \{G(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s = -\mathrm{i}\int_T^{T+H_k} \{G(1+\omega+\mathrm{i}t)\}_0^{\frac{1}{k}}\,\mathrm{d}t$$
$$= -\mathrm{i}\mathrm{e}^{\varepsilon\Delta'} \cdot \left(\frac{T}{2\pi}\right)^{\frac{1}{4k}+\frac{\omega}{k}} \cdot O(1) + O(T^{-1-\frac{1}{4k}+\frac{\omega}{k}} \cdot H_k^2) = o(H_k).$$

□

Remark 6. Evidently, $\{G(s)\}_0^{1/k}$ satisfies an analogue of Lemma 3—let us denote it by Lemma 3′—which is obtained by the substitution $\{\zeta(s)\}_0^{1/k} \to \{G(s)\}_0^{1/k}$ since the sets of zeros and of multiplicities of the functions $\zeta(s)$ and $G(s)$ in Π_k coincide. We denote the analogues of the relations (28)–(32) by (28′)–(32′).

From (54), by virtue of (56), (59) we now obtain

$$(60)\qquad \int_{L^k_0} \{G(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s = o(H_k) - \sum_{r=1}^{m} \int_{L^k_r} \{G(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s,$$

where (by (28′))

$$(61)\qquad \int_{L^k_0} \{G(s)\}_0^{\frac{1}{k}}\,\mathrm{d}s = \mathrm{i}\int_T^{\gamma_{0,1}} \left\{G\left(\frac{1}{2}+\mathrm{i}t\right)\right\}_0^{\frac{1}{k}} \mathrm{d}t + \mathrm{i}\sum_{l=1}^{p_0-1} \mathrm{e}^{\mathrm{i}\Delta_{0,l}} \cdot \int_{\gamma_{0,l}}^{\gamma_{0,l+1}} + \mathrm{i}\mathrm{e}^{\mathrm{i}\Delta^0} \cdot \int_{\gamma_{0,p_0}}^{T+H_k}.$$

However, if (52) holds then

$$\arg\left\{G\left(\frac{1}{2}+\mathrm{i}t\right)\right\}_0^{\frac{1}{k}} = 0, \quad \left|\left\{G\left(\frac{1}{2}+\mathrm{i}t\right)\right\}_0^{\frac{1}{k}}\right| = \{Z(t)\}^{\frac{1}{k}}, \quad t \in \langle T, \gamma_{0,1}),$$

i.e.

$$\left\{G\left(\frac{1}{2}+\mathrm{i}t\right)\right\}_0^{\frac{1}{k}} = \{Z(t)\}^{\frac{1}{k}}, \quad t \in \langle T, \gamma_{0,1}),$$

and analogously to the case (35), (36) we conclude that

$$\left\{G\left(\frac{1}{2}+\mathrm{i}t\right)\right\}_0^{\frac{1}{k}} = \{Z(t)\}^{\frac{1}{k}} \cdot \mathrm{e}^{\mathrm{i}\Delta_{0,1}}, \quad t \in (\gamma_{0,1}, \gamma_{0,2}), \dots .$$

Hence (see (61))

$$\int_{L_0^k} \{G(s)\}_0^{\frac{1}{k}} \,\mathrm{d}s = \mathrm{i} \int_T^{\gamma_{0,1}} \{Z(t)\}^{\frac{1}{k}} \,\mathrm{d}t \tag{62}$$

$$+ \mathrm{i} \sum_{l=1}^{p_0} \mathrm{e}^{\mathrm{i}\Delta_{0,l}} \cdot \int_{\gamma_{0,l}}^{\gamma_{0,l+1}} \{Z(t)\}^{\frac{1}{k}} \,\mathrm{d}t + \mathrm{i}\mathrm{e}^{\mathrm{i}\Delta^0} \cdot \int_{\gamma_{0,p_0}}^{T+H_k} \{Z(t)\}^{\frac{1}{k}} \,\mathrm{d}t .$$

Finally, by virtue of (62), (60) implies (53). □

8. Proof of Theorem—conclusion

Since the zeros of the function $\zeta(s)$ are distributed symmetrically with respect to the line $\sigma = \frac{1}{2}$ (see [7], p. 40), it suffices to consider rectangles (cf. (2))

$$\overline{Q}_k = \left\{s \colon \sigma \in \left\langle \frac{1}{2}, 1\right), t \in (T, T + H_k)\right\};$$

evidently $\overline{Q}_k \subset \Pi_k$ (see Sec. 4).

Let all zeros $\varrho \in \overline{Q}_k$ satisfy the relation (cf. (3))

$$2k \mid n(\varrho). \tag{63}$$

On the one hand, using Lemma 3 (see (26), (29)) we obtain

$$\int_{L_r^k} \{\zeta(s)\}_0^{\frac{1}{k}} \,\mathrm{d}s = 0, \quad r = 1, \dots, m, \quad k = 2, 3, \dots, k_0$$

and consequently, by Lemma A (see (4), (44)),

$$\int_T^{T+H_k} |Z(t)|^{\frac{1}{k}} \,\mathrm{d}t \geqslant \frac{1}{2} H_k, \quad k = q, q+1, \dots, k_0. \tag{64}$$

On the other hand, by virtue of Lemma 3′, relation (29′) (see Remark 6), (63) implies

$$\int_{L_r^k} \{G(s)\}_0^{\frac{1}{k}} \, \mathrm{d}s = 0, \quad r = 0, 1, \dots, m, \quad k = q, q+1, \dots, k_0. \tag{65}$$

It follows from (63) that $Z(t)$ does not change sign in the interval $\langle T, T + H_k \rangle$. Moreover,

$$\mathrm{e}^{\mathrm{i}\Delta_{0,t}} = 1, \quad \mathrm{e}^{\mathrm{i}\Delta^0} = 1, \tag{66}$$

and, by virtue of (50), (65), (66), Lemma B (see (53)) yields

$$\int_T^{T+H_k} |Z(t)|^{\frac{1}{k}} \, \mathrm{d}t = o(H_k),$$

a contradiction (see (64)). □

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References

[1] *Hardy, G. H.*: Sur les zèros de la fonction $\zeta(s)$ de Riemann. C. R. Acad. Sci. *158* (1914), 1012–1014.
[2] *Hardy, G. H., Littelwood, J. E.*: Contributions to the theory of the distribution of primes. Acta Math. *41* (1918), 119–196.
[3] *Selberg, A.*: On the zeros of Riemann's zeta-function. Skr. Norske Vid. Akad., Oslo, 1942, No. 10.
[4] *Vinogradov, I.M.*: Method of Trigonometric Sums in Number Theory. Nauka, Moscow, 1980. (In Russian.)
[5] *Moser, J.*: Riemann conjecture and the behavior of some sets on the critical line. Acta Math. Univ. Comen. *48-49* (1986), 81–91. (In Russian.)
[6] *Moser, J.*: Riemann conjecture and the behavior of the function $Z(t)$ on short intervals. Acta Arithmetica *59* (1991), 157–169. (In Russian.)
[7] *Titchmarsh E. K.*: Theory of Riemann Zeta Function. IL, Moscow, 1953. (In Russian.)

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