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ON THE EQUIVALENCE OF VARIATIONAL PROBLEMS III

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The equivalence problem for multiple variational integrals (i.e., the problem whether two such integrals can be identified modulo contact forms by means of an invertible transformation) proves to be much more difficult than the case of one independent variable. So we shall begin with rather particular examples here. Continuing [3], the method is quite elementary: if such an integral is given, we search for other objects intrinsically related to it. They may be of very diverse nature and of a certain independent interest but our final aim is to determine an intrinsical coframe, the *Frenet coframe*. Then, since the equivalence transformations between variational integrals necessarily identify the relevant intrinsical objects of the same nature, we may interrupt the calculations. Indeed, the Frenet coframes determine (or disprove the existence of) the equivalence transformation to an analogous extent as the Maurer-Cartan forms did for the structure of Lie groups. The realization proper of equivalences, investigation of the structure of invariants (i.e., of intrinsically related differential operators, cf. [7]), and discussion of certain degenerate subcases cannot be made at this place for technical reasons.

It is to be mentioned that the common method of G-structures and the related generalized geometries in finite-dimensional underlying spaces of jets of a fixed order given in advance is of a quite different nature, see the classical works [1, 2] and the recent expositions [5, 6]. In principle, the interrelation between them should be based on the theory of infinitely prolonged Lie-Cartan pseudogroups [4], and for this reason it is not realizable here. Also the equivalence of constrained multiple integrals and order increasing transformations are tacitly passed over.

1. Preparatory results. We begin with the simplest but typical equivalence problem which will serve as a model for the next more complicated tasks. We are interested in the variational integral

$$\iint f(x,y,u,u_x,u_y)\,\mathrm{d}x \wedge \mathrm{d}y \to \text{ extremum } \quad (u = u(x,y), f \neq 0) \tag{1}$$

without any assumption on the underlying space, that is, in the space of jets of infinite order. This space is equipped with coordinates x, y, u_{ij} $(i,j = 0,1,\dots)$, contact forms $\vartheta_{ij} \equiv \mathrm{d}u_{ij} - u_{i+1,j}\,\mathrm{d}x - u_{i,j+1}\,\mathrm{d}x$, and the Lagrange density $\lambda = f\,\mathrm{d}x \wedge \mathrm{d}y$. (We shall use abbreviations like $u = u_{00}, u_x = u_{10}, u_y = u_{01}, \dots, \vartheta = \vartheta_{00}, \vartheta_x = \vartheta_{10}, \vartheta_y = \vartheta_{01}, \dots$, and occasionally $a^{00} = a, \dots$ for various coefficients through our exposition.)

The module $\Omega = \{\vartheta_{ij}; i,j = 0,1,\dots\}$ of all forms of the kind $\sum a^{ij}\vartheta_{ij}$ and the family of all forms ξ of the kind

$$\xi = \lambda + \sum (b^{ij}\,\mathrm{d}y - c^{ij}\,\mathrm{d}x + d^{ij}_{kl}\vartheta_{k\ell}) \wedge \vartheta_{ij} \tag{2}$$

(i.e., satisfying $\xi \cong \lambda \pmod{\Omega}$) are taken for primary intrinsical objects. Here both the sums are finite but of an arbitrary (uncertain) length with arbitrary (varying) coefficients $a^{ij}, \dots, d^{ij}_{k\ell}$ (each depending on a finite number of coordinates of the infinite-dimensional underlying space).

Clearly the module $\Omega^{\perp}$ of all vector fields Z satisfying $\omega(Z) \equiv 0$ $(\omega \in \Omega)$ is an intrinsical object, too. This derived intrinsical object consists of all vector fields $Z = p\partial_x + q\partial_y$ where

$$\partial_x = \partial/\partial x + \sum u_{i+1,j}\partial/\partial u_{ij}, \ \partial_y = \partial/\partial y + \sum u_{i,j+1}\partial/\partial u_{ij}$$

(infinite series) are the familiar formal derivatives and p, q are arbitrary functions. It follows that $\partial = r\partial_x \wedge \partial_y$ with variable r is an intrinsical family of bivectors. It contains a unique bivector $\bar{\partial}$ with the property $\bar{\partial} \rfloor \xi = 1$, namely $\bar{\partial} = \partial_x \wedge \partial_y / f$. This is a very simple example of the procedure called *specification* which will be repeatedly employed. (The specified objects will be denoted by upper bars but often we shall not dogmatically follow this rule for technical reasons.)

We shall prove in Section 5 that for every $\ell = 0,1,\dots$, the submodule $\Omega^{\ell} = \{\vartheta_{ij}; i + j \leqslant \ell\} \subset \Omega$ is an intrinsical object. (So the equivalences are *prolonged contact transformations.* Indeed, by the classical definition, these are just such transformations which preserve all modules $\Omega^0 = \{\vartheta\}$, $\Omega^1 = \{\vartheta, \vartheta_x, \vartheta_y\}, \dots$.) It will be moreover proved in Section 6 that the familiar *Poincaré-Cartan* $(\mathcal{PC})$ *form*

$$\bar{\xi} = \lambda - (P\,\mathrm{d}y - Q\,\mathrm{d}x) \wedge \vartheta \quad (P = \partial f/\partial x, Q = \partial f/\partial y) \tag{3}$$

is an intrinsical element of the family of forms ξ. Then the differential

$$\mathrm{d}\bar{\xi} = \{E\,\mathrm{d}x \wedge \mathrm{d}y - (L\vartheta_x + M\vartheta_y) \wedge \mathrm{d}y + (M\vartheta_x + N\vartheta_y) \wedge \mathrm{d}x\} \wedge \vartheta \tag{4}$$

is an intrinsical object, too. Here $E = \partial f/\partial u - \partial_x P - \partial_y Q$ is the familiar *Euler-Lagrange* ($\mathcal{EL}$) *operator* and we abbreviate $L = \partial^2 f/\partial u_x^2$, $M = \partial^2 f/\partial u_x \partial u_y$, $N = \partial^2 f/\partial u_y^2$. Denoting by $\omega = a\vartheta$ ($a \neq 0$), an arbitrary generator of Ω^0, the family of forms $\{\ldots\}/a$ is clearly intrinsical modulo Ω^0 (cf. (4)). There is a unique t, namely $t = E/fa$, such that $\{\ldots\}/a \cong t\bar{\xi}$ (modulo Ω). It follows that

$$\frac{1}{a}\{\ldots\} - t\bar{\xi} = \frac{1}{a}((M\vartheta_x + N\vartheta_y) \wedge \mathrm{d}x - (L\vartheta_x + M\vartheta_y) \wedge \mathrm{d}y) \tag{5}$$

is an intrinsical family modulo Ω^0. Recalling that $Z = p\partial_x + q\partial_y$, $\omega = a\vartheta$, both the families of forms

$$Z\rfloor\,\mathrm{d}\omega \cong a(p\vartheta_x + q\vartheta_y),$$
$$Z\rfloor\Big(\frac{1}{a}\{\ldots\} - t\bar{\xi}\Big) \cong \frac{1}{a}((pM - qL)\vartheta_x + (pN - qM)\vartheta_y)$$

are intrinsical modulo Ω^0. They are proportional if and only if

$$pM - qL = sa^2 p, \quad pN - qM = sa^2 q \tag{6}$$

where s is the (intrinsical) proportionality factor. A nontrivial solution p, q of (6) exists if and only if $s^2 = (M^2 - LN)/a^4$. Assuming $E \neq 0$ from now on, it follows that

$$I = s^2/t^4 = (M^2 - LN)f^4/E^4$$

is an invariant function to the original integral (1). It determines the transformation rule for the $\mathcal{EL}$ operator (and for the $\mathcal{EL}$ *equation* $E = 0$).

2. The hyperbolic subcase. Supposing $M^2 > LN$, we may introduce the intrinsical requirements $s = 1$ and $s = -1$. They both provide the common specification $\bar{a} = (M^2 - LN)^{1/4}$ of the coefficient a, hence the specification $\bar{\omega} = \bar{a}\omega$ of the form ω. Moreover, we obtain two (not yet ultimate) specifications Z^+, Z^- of the family Z if the coefficients p, q are chosen to satisfy (6) with $s = 1$ or $s = -1$, respectively. Assuming $L \neq 0$ (for certainty), one can see that

$$Z^+ = w^+\Big(\partial_x + \frac{M - \bar{a}^2}{L}\partial_y\Big),\; Z^- = w^-\Big(\partial_x + \frac{M + \bar{a}^2}{L}\partial_y\Big) \tag{7}$$

where w^+, w^- are arbitrary functions. In geometrical terms, Z^+ and Z^- are intrinsical fields of directions. Our next aim is to specify w^+, w^- to obtain even intrinsical vector fields. We may use the relation $Z^+ \wedge Z^- = w\bar{\partial}$ where the coefficient

$$w = w^+ w^- 2\bar{a}^2 f/L \tag{8}$$

is of intrinsical nature. It would be possible to introduce the requirement $w = 1$. But this measure is not efficient enough and we abstain from it for a moment.

In order to employ a more efficient tool, let us look at the families of forms

$$\zeta^+ = Z^+ \rfloor \bar{\xi} = w^+ \Big(f\Big(\mathrm{d}y - \frac{M - \bar{a}^2}{L}\,\mathrm{d}x \Big) + \Big(Q - \frac{M - \bar{a}^2}{L} P \Big) \Big) \vartheta, \tag{9}$$

$$\omega^+ = Z^+ \rfloor \mathrm{d}\bar{\omega} = w^+ \Big(\Big(\partial_x + \frac{M - \bar{a}^2}{L} \partial_y \Big) \ln \bar{a} \cdot \bar{\omega} + \bar{a} \Big(\vartheta_x + \frac{M - \bar{a}^2}{L} \vartheta_y \Big) \Big), \tag{10}$$

and $\zeta^- = Z^- \rfloor \bar{\xi}$, $\omega^- = Z^- \rfloor \mathrm{d}\bar{\omega}$ not explicitly stated here. (At this place, it may be interesting to note the formula $\zeta^+ \wedge \zeta^- = w\bar{\xi}$ which means that the study of $\bar{\xi}$, hence of the integral (1), can be replaced by the study of ζ^+, ζ^- without loss of information. One can also observe that the intrinsical system $\bar{\omega} = \zeta^- = \zeta^+ = 0$ is equivalent to the system $\mathrm{d}x = \mathrm{d}y = \mathrm{d}u = 0$. So the family of variables x, y, u is of intrinsical nature and the equivalences are a mere *prolonged point transformations.*) This construction can be continued by setting

$$\omega^{++} = Z^+ \rfloor \mathrm{d}\omega^+ \,, \; \omega^{+-} = Z^+ \rfloor \mathrm{d}\omega^- \,, \; \omega^{--} = Z^- \rfloor \mathrm{d}\omega^- \,, \; \ldots, \tag{11}$$

and this will provide us the sought Frenet coframe later on.

Let us return to the problem of ultimate specification of Z^+, Z^-. We shall use the common method of moving frames reformulated in elementary terms as follows. First, the forms $\mathrm{d}x$ and $\mathrm{d}y$ can be expressed as linear combinations of ζ^+, ζ^-, $\bar{\omega}$. Second, the forms ϑ_x and ϑ_y can be expressed as linear combinations of ω^+, ω^-, $\bar{\omega}$. Third, ϑ is a multiple of $\bar{\omega}$. Altogether taken, the forms $\mathrm{d}\bar{\omega}$, $\mathrm{d}\zeta^+ \wedge \zeta^+$, $\mathrm{d}\zeta^- \wedge \zeta^-$ can be expressed (in terms of $\mathrm{d}x$, $\mathrm{d}y$, ϑ, ϑ_x, ϑ_y and hence) as linear combinations of exterior products of the forms ζ^+, ζ^-, $\bar{\omega}$, ω^+, ω^-. The coefficients are of intrinsical nature, of course. We shall not state intermediate calculations but only the most important part of the final result. Denoting

$$A^+ = \frac{M + \bar{a}^2}{L}, \; B^+ = Q - A^+ P, \; C^+ = f\Big(A^+ \frac{\partial}{\partial u_x} - \frac{\partial}{\partial u_y} \Big),$$

and analogously A^-, B^-, C^- with $-\bar{a}^2$ at the place of $\bar{a}^2$, one can derive the formulae

$$\begin{aligned}
\mathrm{d}\bar{\omega} &= \frac{1}{w}(\zeta^+ \wedge \omega^- - \zeta^- \wedge \omega^+) \\
&\quad + \frac{1}{w\bar{a}}(w^-(C^+ \ln \bar{a} - B^+)\omega^+ - w^+(C^- \ln \bar{a} - B^-) \wedge \bar{\omega}, \\
\mathrm{d}\zeta^+ \wedge \zeta^+ &= \ldots + \frac{w^+}{w\bar{a}}\Big(\Big(\frac{L}{2\bar{a}^2}C^+A^- + B^-\Big)\omega^+ + (w^+)^2\frac{f}{w}C^-A^-\omega^-\Big) \wedge \zeta^- \wedge \zeta^+, \\
\mathrm{d}\zeta^- \wedge \zeta^- &= \ldots + \frac{w^-}{w\bar{a}}\Big(\Big(\frac{L}{2\bar{a}^2}C^-A^+ + B^+\Big)\omega^- - (w^-)^2C^+A^+\omega^+\Big) \wedge \zeta^+ \wedge \zeta^-.
\end{aligned}$$

(It is to be noted that analogous development of the form $\mathrm{d}\xi$ does not give any useful result: the most interesting coefficients either vanish or are expressible in terms of I and w. A geometrical interpretation of this failure would be desirable.) The coefficients on the right hand side can be employed for specification of w^+, w^-.

For instance, if $C^+ \ln \bar{a} \neq B^+$, then the coefficient of $\omega^+ \wedge \bar{\omega}$ in the development of $\mathrm{d}\bar{\omega}$ can be equated to 1. Moreover, owing to (8), we obtain

$$w^+ = L(C^+ \ln \bar{a} - B^+)/2\bar{a}^3 f \tag{12}$$

which yields the ultimate specification of Z^+. If $C^- \ln \bar{a} \neq B^-$, then w^- (hence Z^-) can be specified in the analogous manner. (Then w given by (8) with these w^+, w^- substituted turns into an invariant. A lot of other invariants can be obtained by using the coefficients of the developments of the forms $\mathrm{d}\zeta^+ \wedge \zeta^+$ and $\mathrm{d}\zeta^- \wedge \zeta^-$.) Besides the already known form $\bar{\omega}$, we have intrinsical forms ζ^+ (cf. (9)), ζ^-, ω^+ (cf. (10)), ω^-, and the series (11). The only advantage of this choice of specification is that Z^+, Z^- play a symmetric role. But this is no longer true if we look for a Frenet coframe. Indeed, owing to $[Z^+, Z^-] \in \Omega^+$ (a consequence of $[\partial_x, \partial_y] = 0$), we have $[Z^+, Z^-] = AZ^+ + BZ^-$ with certain (in general nonvanishing) invariants A, B. As follows from the formula

$$\begin{aligned}
Z^+\rfloor dZ^-\rfloor \mathrm{d}\varphi &= \mathcal{L}_{Z^+}\mathcal{L}_{Z^-}\varphi = (\mathcal{L}_{Z^-}\mathcal{L}_{Z^+} + \mathcal{L}_{[Z^+,Z^-]})\varphi \\
&= Z^-\rfloor dZ^+\rfloor \mathrm{d}\varphi - (AZ^+ + BZ^-)\rfloor \mathrm{d}\varphi \quad (\varphi \in \Omega)
\end{aligned}$$

applied to $\varphi = \bar{\omega}$, the forms ω^+, ω^-, ω^{+-}, ω^{-+} are linearly dependent. A little generalized argument gives that the sought Frenet coframe may consist of the forms ζ^+, ζ^-, $\bar{\omega}$, ω^+, ω^- and only those forms of (11) for which the indices "+" precede all indices "−." (One can easily find that we indeed obtain a coframe by looking at the higher order summands ϑ_{ij} of these forms.) So the symmetry is lost in the final result. It seems that it is better to give up the symmetry from the very beginning and to choose, e.g., the specification (12) together with the requirement $w = 1$ which determines w^- (cf. (8)). Then, owing to the familiar formula

$$\mathrm{d}\varphi\,(X, Y) = X\varphi(Y) - Y\varphi(X) - \varphi([X, Y])$$

applied to $\varphi = \zeta^+, \zeta^-$ and $X = Z^+, Z^-$, and easily verifiable equations $\zeta^+(Z^+) = \zeta^-(Z^-) = 0$, $\zeta^+(Z^-) = -\zeta^-(Z^+) = w = 1$, one can derive explicit expressions $A = \mathrm{d}\zeta^-\,(Z^+, Z^-)$, $B = -\,\mathrm{d}\zeta^+\,(Z^+, Z^-)$. This facilitates the dependences between the terms of the sequence (11). Moreover, owing to the formula $\bar{\xi} = w\zeta^+ \wedge \zeta^- = \zeta^+ \wedge \zeta^-$ and hence $\mathrm{d}\bar{\xi} = \mathrm{d}\zeta^+ \wedge \zeta^- - \zeta^+ \wedge \mathrm{d}\zeta^-$, the Bianchi identities between the invariants arising from $\mathrm{d}^2\bar{\xi} = 0$ substantially simplify.

The above calculations fail if and only if $C^+ \ln \bar{a} = B^+$, $C^- \ln \bar{a} = B^-$. (One can verify that this happens if and only if we deal with a rather peculiar integrals (1) where f satisfies a Monge-Ampère equation of the kind $M^2 - LN = gf^{-4}$ with a positive function $g = g(x, y, u)$.) In this case, the development of the form $\bar{\omega}$ is useless but instead any one of the forms $\mathrm{d}\zeta^+ \wedge \zeta^+$, $\mathrm{d}\zeta^- \wedge \zeta^-$ can be employed to determine the sought specification of w^+, w^-. We shall omit more details since except for complicated formulae no new ideas appear.

3. The elliptical subcase $M^2 < LN$ can be settled by means of tedious complexification of the preceding results and separation of real and imaginary parts. Instead of this method, we shall adopt another approach.

Recall the intrinsical objects $Z = p\partial_x + q\partial_y \in \Omega^\perp$, $\omega = a\vartheta \in \Omega^0$ $(a \neq 0)$, and introduce the intrinsical family $\zeta \cong \hat{p}\,\mathrm{d}x + \hat{q}\,\mathrm{d}y \pmod{\Omega^0}$ with $\hat{p}$, $\hat{q}$ variable functions. (As this family is concerned, it is identical with the family $Z \rfloor \mathrm{d}\bar{\xi} \cong f(p\,\mathrm{d}y - q\,\mathrm{d}x)$ within the change of notation $\hat{p} = -qf$, $\hat{q} = pf$.) Then the mappings

$$\begin{aligned} &Z \to Z \rfloor \mathrm{d}\omega \cong a(p\vartheta_x + q\vartheta_y) \pmod{\Omega^0}, \\ &\zeta \to \zeta \wedge \mathrm{d}\bar{\xi} = \bar{\xi} \wedge \omega \wedge \left\{ \frac{1}{fa}((\hat{p}L + \hat{q}M)\vartheta_x + (\hat{p}M + \hat{q}N)\vartheta_y \right\} \to \{\ldots\} \end{aligned}$$

into Ω^1/Ω^0 make good sense. The former can be inverted and composed with the latter to the result

$$\zeta \cong \hat{p}\,\mathrm{d}x + \hat{q}\,\mathrm{d}y \to \{\ldots\} = a(p\vartheta_x + q\vartheta_y) \to p\partial_x + q\partial_y = Z \tag{13}$$

which is a polarity with respect to the quadratic form

$$\mathbf{Q}(\zeta) = \mathbf{Q}(\hat{p}\,\mathrm{d}x + \hat{q}\,\mathrm{d}y) = \frac{1}{fa^2}(L(\hat{p})^2 + 2M\hat{p}\hat{q} + N(\hat{q})^2).$$

In this polarity, the intrinsical 2-form $\bar{\xi} \cong f\,\mathrm{d}x \wedge \mathrm{d}y \pmod{\Omega^0}$ is transformed into the family of bivectors

$$f \cdot \frac{1}{fa^2}(L\partial_x + M\partial_y) \wedge \frac{1}{fa^2}(M\partial_x + N\partial_y) = \frac{LN - M^2}{a^4}\bar{\partial}.$$

It follows that the coefficient $(LN - M^2)/a^4$ is of intrinsical nature. In the elliptical case, it may be equated to 1. This yields the specifications $\bar{a} = (LN - M^2)^{1/4}$, $\bar{\omega} = \bar{a}\vartheta$.

Note moreover that the mapping (13) can be inverted. As a result, the polarity with respect to the quadratic form

$$\mathbf{Q}^{-1}(Z) = \mathbf{Q}^{-1}(p\,\partial_x + q\,\partial_y) = \frac{f}{\bar{a}^2}(N(p)^2 - 2Mpq + L(q)^2)$$

arises. (We use the specified values of quadratic forms without change of notation.)

We are passing to more advanced tools, the moving frames. But instead of introducing orthonormal frames with respect to $\mathbf{Q}$, $\mathbf{Q}^{-1}$, we shall deal with a quite general basis Z^1, Z^2 of $\Omega^{\perp}$:

$$Z^i \equiv p^i\partial_x + q^i\partial_y \qquad (i = 1, 2;\ p^1q^2 - q^1p^2 \neq 0) \tag{14}$$

and the families of forms

$$\zeta^i \equiv Z^i \rfloor \bar{\xi},\ \omega^i \equiv Z^i \rfloor \mathrm{d}\bar{\omega} \qquad (i = 1, 2). \tag{15}$$

Clearly $Z^1 \wedge Z^2 = w\bar{\partial}$ where the coefficient

$$w = (p^1q^2 - q^1p^2)f \tag{16}$$

is of intrinsical nature. One can then establish the existence of a development

$$\begin{aligned} \mathrm{d}\bar{\omega} &= \frac{1}{w}(\zeta^1 \wedge \omega^2 - \zeta^2 \wedge \omega^1) + (A^2\omega^1 - A^1\omega^2) \wedge \bar{\omega}, \\ A^i &\equiv \frac{1}{\bar{a}w}\Big(\Big(\frac{\partial \ln \bar{a}}{\partial u_x} + \frac{P}{f}\Big)q^i - \Big(\frac{\partial \ln \bar{a}}{\partial u_y} + \frac{Q}{f}\Big)p^i\Big). \end{aligned}$$

Assume that A^i are not identically vanishing. Then the requirement $A^1 = 0$ determines p^1, q^1 up to a nonvanishing factor which can be ultimately specified by the normalization $\mathbf{Q}^{-1}(Z^1) = 1$. With Z^1 already known, Z^2 can be determined (up to a sign) from the orthogonality $\mathbf{Q}^{-1}(Z^1, Z^2) = 0$ and normalization $\mathbf{Q}^{-1}(Z^2) = 1$. (With this specification, A^2 turns into an invariant but w is a constant, the area of an orthogonal frame.) Altogether taken, we may introduce a Frenet coframe consisting of the above forms ζ^1, ζ^2, $\bar{\omega}$, ω^1, ω^2 and those forms of the recurrently defined family

$$\omega^{iI} \equiv Z^i \rfloor \mathrm{d}\omega^I \tag{17}$$

($i = 1, 2$ and I is a sequence with terms 1 or 2) which have nondecreasing upper indices.

We shall not discuss the exceptional subcase when A^i are identically vanishing. (One can verify that it may happen if and only if $M^2 - LN = gf^4$ with an appropriate negative $g = g(x, y, u)$.) It seems that then the simultaneous investigation of the forms ζ^1, ζ^2 and the use of moving frames orthogonal with respect to the forms $\mathbf{Q}$, $\mathbf{Q}^{-1}$ is necessary but no serious difficulties arise.

4. The parabolical subcase $M^2 = LN$ leads to quite other results. As before, the module Ω, the $\mathcal{PC}$ form $\bar{\xi}$ (cf. (3)), the generator $\omega = a\vartheta$ ($a \neq 0$) of Ω^0, and the moving frame (14) are intrinsical objects. We shall assume $L \neq 0$, for certainty. Then, denoting $C = M/L$, we have $N = CM = C^2L$ and the formula (4) simplifies to

$$\mathrm{d}\bar{\xi} = \{E\,\mathrm{d}x \wedge \mathrm{d}y + L(\vartheta_x + C\vartheta_y) \wedge (C\,\mathrm{d}x - \mathrm{d}y)\} \wedge \vartheta.$$

Since $\{\ldots\}/a - t\bar{\xi} \cong L(\vartheta_x + C\vartheta_y) \wedge (C\,\mathrm{d}x - \mathrm{d}y)$ (mod Ω^0) is an intrinsical family (as before) and $t = E/fa$ is uniquely determined, we may introduce the intrinsical requirement $t = 1$ ($E \neq 0$ is tacitly assumed here). So we obtain the specification $\bar{a} = E/f$, $\bar{\omega} = \bar{a}\vartheta$ of higher order than in the preceding sections.

Let us pass to the frame (14). First of all, clearly

$$Z^1 \rfloor \mathrm{d}(\{\ldots\}/\bar{a} - \bar{\xi}) \cong -L(p^1C - q^1)(\vartheta_x + C\vartheta_y)/\bar{a} \quad (\mathrm{mod}\ \Omega^0)$$

and we may introduce the requirement $p^1C = q^1$. We obtain the family $Z^1 = p^1(\partial_x + C\partial_y)$ with a variable factor $p^1 \neq 0$. Secondly, by owing to the relation $Z^1 \wedge Z^2 = p^1(q^2 - Cp^2)f \cdot \bar{\partial}$, we may require

$$p^1(q^2 - Cp^2)f = 1. \tag{18}$$

Thirdly, by the formulae

$$\begin{aligned} &Z^2 \rfloor (\{\ldots\}/\bar{a} - \bar{\xi}) \cong -L(p^2C - q^2)(\vartheta_x + C\vartheta_y)/\bar{a}, \\ &Z^1 \rfloor \mathrm{d}\bar{\omega} \cong p^1\bar{a}(\vartheta_x + C\vartheta_y) \end{aligned}$$

permit to require $L(p^2C - q^2)/\bar{a} = \pm p^1\bar{a}$ ($\mp = \operatorname{sign} fL$, see below), that is, $-L/\bar{a}fp^1 = \pm p^1\bar{a}$ (use (18)). So we have the specification

$$p^1 = (\mp L/f)^{1/2}/\bar{a}^2 = f(\mp fL)^{1/2}E^{-2},$$

and thus the vector field Z^1 is ultimately determined.

Passing to the determination of the second term Z^2 of the frame (14), we shall employ the differential of the well-known form

$$\zeta^1 = Z^1 \rfloor \bar{\xi} = p^1(f(\mathrm{d}y - C\,\mathrm{d}x) + (Q - CP)\vartheta).$$

One can see that the form

$$\mathrm{d}\zeta^1 \wedge \zeta^1 = p^1 f(-\mathrm{d}C \wedge \mathrm{d}x + \mathrm{d}(A\vartheta)) \wedge \vartheta^1 \qquad \left(A = \frac{Q - PC}{f}\right)$$

can be expressed in terms of $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}u$, $\mathrm{d}u_x$, $\mathrm{d}u_y$ (no differentials of the second order derivatives appear), that is, in terms of ζ^1, ζ^2, $\bar{\omega}$, ω^1, ω^2. Coefficients of the terms $\zeta^2 \wedge \bar{\omega} \wedge \zeta^1$, $\omega^2 \wedge \bar{\omega} \wedge \zeta^1$, $\omega^2 \wedge \zeta^2 \wedge \zeta^1$ of this development are invariants (they depend on the already specified functions) and we shall omit then. Coefficients of $\omega^1 \wedge \bar{\omega} \wedge \zeta^1$, $\omega^1 \wedge \zeta^2 \wedge \zeta^1$ are as follows:

$$\text{(19)} \qquad \frac{p^1 f}{\bar{a}^2}\left(q^2\left(P\frac{\partial C}{\partial u_x} + f\frac{\partial A}{\partial u_x} + AP\right) - p^2\left(P\frac{\partial C}{\partial u_y} + f\frac{\partial A}{\partial u_y} - AQ\right)\right),$$

$$\frac{p^1 f}{\bar{a}}\left(p^1 f\left(q^2\frac{\partial C}{\partial u_x} - p^2\frac{\partial C}{\partial u_y}\right) + A\right).$$

Together with (18), the last coefficient equated to zero gives the system

$$p^1 f(q^2 - p^2 C) - 1 = p^1 f\left(q^2\frac{\partial C}{\partial u_x} - p^2\frac{\partial C}{\partial u_y}\right) + A = 0$$

for the specification of the coefficients p^2, q^2. If this system is not uniquely solvable (i.e., if $\partial C/\partial u_y = C\partial C/\partial u_x$), the coefficient (19) can be used in analogous manner with better final effect.

If both vector fields Z^1, Z^2 are specified then the determination of a Frenet coframe is easy (at least in principle) and need not be discussed.

5. Automorphisms of Ω. Let us return to the statements used but not proved in Section 1. First of all, we should like to prove that all invertible transformations which preserve the module Ω are the prolonged contact transformations. In other terms, *every automorphism of Ω preserves the submodules $\Omega^\ell = \{\vartheta_{ij};\ i+j \leqslant \ell\} \subset \Omega$ for $\ell = 1, \ldots$* In still other terms, these submodules Ω^ℓ are intrinsically related to Ω. It is however sufficient to deal only with the submodule Ω^0 since the other ones Ω^1, Ω^2, ... are determined from it by the recurrence $\Omega^{\ell+1} = \Omega^\ell + \mathcal{L}_Z\Omega^\ell$ (Z varies through $\Omega^\perp$). But Ω^0 consists of all multiples $a\vartheta$, so the forms of the kind $a\vartheta$

should be distinguished from the other forms of the module Ω. Roughly speaking, the distinctive property of the forms $a\vartheta$ ($a \neq 0$) is that they generate a basis of Ω after a certain application of operators $\mathcal{L}_Z$ ($Z \in \Omega^{\perp}$). In more detail, the idea is realized as follows.

Given two linearly independent vector fields $X, Y \in \Omega^{\perp}$ and $\omega \in \Omega$, we introduce forms

$$\omega_{rs} = (\mathcal{L}_X)^r (\mathcal{L}_Y)^s \omega = (X \rfloor d)^s (Y \rfloor d)^r \omega \in \Omega.$$

We may suppose $X = \partial_x$, $Y = \partial_y$ at a fixed point after an appropriate change of variables. Then, assuming $\omega = \sum a^{ij} \vartheta_{ij}$ and looking for the higher order identically nonvanishing summands $\omega = \ldots + \sum a^{ij} \vartheta_{ij}$ ($i + j = \ell$), one can see that

$$\omega_{rs} = \ldots + \sum a^{ij} \vartheta_{i+r,j+s} \qquad (i + j = \ell)$$

at the given point. It follows that the forms ω_{rs} ($r, s = 0, 1, \ldots$) are linearly independent. On the other hand, they constitute a basis of Ω if and only if $\ell = 0$, hence $\omega = a\vartheta$ ($a = a^{00} \neq 0$). This is the sought distinctive property of the forms $a\vartheta$ and we are done.

6. Identification of the $\mathcal{PC}$ form. Now, we should like to prove that the form (3) occupies a special (intrinsical) position in the family of all forms (2). This will be realized by successive reduction of the (rather wide) family (2).

One can observe that the module $\Xi = \{\mathrm{d}x\,, \mathrm{d}y\,, \mathrm{d}u\,, \mathrm{d}u_x\,, \mathrm{d}u_y\,\}$ plays a certain special role: this is the minimal module of 1-forms such that there exists a nonvanishing form in Ω^0 (e.g., the form ϑ) with the property that both this form and its differential can be algebraically expressed in terms of elements of the mentioned module. (Less formally: x, y, u, u_x, u_y is the minimal family of coordinates such that a nonvanishing form in Ω^0 can be expressed in terms of them.) So the reduced family involving such forms (2) which are expressible in terms of elements of Ξ is of intrinsical nature. The mentioned reduced family consists of forms (2) with $b^{ij} = c^{ij} = d^{ij}_{k\ell} = 0$ if at least one of the indices i, j, k, ℓ is greater than 1. (We may moreover assume $d^{ij}_{00} = d^{10}_{01} = 0$ without loss of generality.)

For this reduced family of forms (2) we introduce an additional requirement $\mathrm{d}\xi \cong 0$ (mod Ω^0, $\Omega \wedge \Omega$). One can verify that this is equivalent to the system

$$b^{10} = c^{01} = b^{01} + c^{10} = c^{00} + Q + \partial_x b^{01} = b^{00} + P + \partial_y c^{10} = 0.$$

So we deal with (a still more) reduced family of forms of the kind

$$\xi = \lambda - (P\,\mathrm{d}y - Q\,\mathrm{d}x + u\vartheta_x + v\vartheta_y) \wedge \vartheta + w\vartheta_x \wedge \vartheta_y - \mathrm{d}\,(r\vartheta)$$

where r, u, v, w are varying functions expressible in terms of the remaining variables c^{10}, d^{00}_{10}, d^{00}_{01}, d^{10}_{10} (for instance $r = c^{10}$).

The next intrinsical requirement $\mathrm{d}\xi \cong 0 \pmod{\Omega^0}$ immediately gives $w = 0$ and $u = v = 0$. So we have a rather narrow family of forms of the kind $\xi = \lambda - (P\,\mathrm{d}x - Q\,\mathrm{d}y) \wedge \vartheta - \mathrm{d}\,(r\vartheta)$ with variable r.

Finally, recall the frame (14) and denote $\xi^i \equiv Z^i \rfloor \xi$ $(i = 1, 2)$ where the family ξ is inserted on the right. One can find that ξ is a multiple of the product $\xi^1 \wedge \xi^2$ (the last intrinsical requirement) if and only if $r = 0$. So the ultimate reduction is achieved, the family (2) is intrinsically reduced to the $\mathcal{PC}$ form (3).

Second order double integrals

7. Preparatory results. Retaining the previous underlying space and the same module Ω of contact forms, we shall be interested in the second order variational integral

$$\iint f(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})\,\mathrm{d}x \wedge \mathrm{d}y \to \text{ extremum.} \tag{20}$$

Then the family (2) with $\lambda = f\,\mathrm{d}x \wedge \mathrm{d}y$ is taken for the primary intrinsical object, but recalling the module Ξ and the relevant reasonings of Section 6, the family (2) can be again reduced by assuming $b^{ij} = c^{ij} = d^{ij}_{k\ell}$ if at least one of the indices exceeds 1. For this narrower family of forms, the additional requirement $\mathrm{d}\xi \cong 0$ (mod Ω^0, $\Omega \wedge \Omega$) gives the system

$$b^{01} + \frac{\partial f}{\partial u_{xx}} = c^{01} + \frac{\partial f}{\partial u_{yy}} = b^{01} + c^{10} + \frac{\partial f}{\partial u_{xy}} =$$

$$= c^{00} + \frac{\partial f}{\partial u_y} - \partial_y \frac{\partial f}{\partial u_{yy}} + \partial_x b^{01} = b^{00} + \frac{\partial f}{\partial u_x} - \partial_y c^{10} + \partial_x \frac{\partial f}{\partial u_{xx}} = 0 \tag{21}$$

for the remaining coefficients appearing in (2). Simulating the approach of Section 6 rather closely, we recall the frame (14) and the families $\xi^i \equiv Z^i \rfloor \xi$, $\omega^i \equiv Z^i \rfloor \mathrm{d}\omega$ $(i = 1, 2;\ \omega = a\vartheta,\ a \neq 0)$. Then the congruence

$$\mathrm{d}\omega \cong \frac{1}{w}(\xi^1 \wedge \omega^2 - \xi^2 \wedge \omega^1) + \frac{a}{f}(b^{01} - c^{10})\omega^1 \wedge \omega^2 \pmod{\Omega^0}$$

can be verified by direct calculation and the intrinsical requirement $\mathrm{d}\omega \cong 0$ (mod Ω^0, ξ^1, ξ^2) implies $b^{01} = c^{10}$. Together with the third equation (21), this determines the coefficients

$$b^{01} = c^{10} = \frac{1}{2}\partial f / \partial u_{xy} \tag{22}$$

(and thus all b^{ij}, c^{ij} are ultimately specified). The forms ξ^i are still depending on the coefficients p^i, q^i. But the product

$$\xi^1 \wedge \xi^2 = w(\lambda + \sum (b^{ij}\,\mathrm{d}y \; - c^{ij}\,\mathrm{d}x\,) \wedge \vartheta_{ij} + \frac{1}{f}\sum b^{ij}\vartheta_{ij} \wedge \sum c^{ij}\vartheta_{ij})$$

involves only the intrinsical coefficient w (cf. (16)). (It would be possible to require $w = 1$ but we abstain from this measure for a moment.) Then the final requirement $\xi^1 \wedge \xi^2 = w\xi$ can be introduced. This yields the values of the remaining coefficients

$$d^{10}_{00} = \frac{1}{f}\begin{vmatrix} b^{00} & c^{00} \\ b^{10} & c^{10} \end{vmatrix}, \; d^{01}_{00} = \frac{1}{f}\begin{vmatrix} b^{00} & c^{00} \\ b^{01} & c^{01} \end{vmatrix}, \; d^{01}_{10} = \frac{1}{f}\begin{vmatrix} b^{01} & b^{10} \\ c^{01} & c^{10} \end{vmatrix}$$

and thus the sought intrinsical $\mathcal{PC}$ form $\bar{\xi}$.

By virtue of the properties of the form $\bar{\xi}$, a congruence of the kind $\mathrm{d}\bar{\xi} \cong E\vartheta \wedge \mathrm{d}x \wedge \mathrm{d}y \pmod{\Omega \wedge \Omega}$ is valid. Here $E = \sum(-\partial_x)^i(-\partial_y)^j \partial f/\partial u_{ij}$ is the familiar $\mathcal{EL}$ operator, of course. In terms of $\omega = a\vartheta$ and $\bar{\xi}$, the last congruence reads

$$\mathrm{d}\bar{\xi} \cong \frac{E}{fa}\omega \wedge \bar{\xi} \pmod{\Omega \wedge \Omega}.$$

Assuming $E \neq 0$, we may introduce the requirement $E/fa = 1$, that is, the intrinsical specifications $\bar{a} = E/f$, $\bar{\omega} = \bar{a}\vartheta$.

With this result, it remains to specify p^i, q^i. Indeed, if Z^1, Z^2 turn into intrinsical vector fields, then ξ^1, ξ^2, $\bar{\omega}$, (15) together with the appropriate forms selected from (17) will provide the sought Frenet coframe.

8. A digression to algebra. We shall deal with a two-dimensional vector space $\mathcal{V}$ equipped with a basis μ, ν and its automorphism $\mathcal{A}$ determined by

$$\mathcal{A}\mu = p^1\mu + q^1\nu,\;\; \mathcal{A}\nu = p^2\mu + q^2\nu \quad (\Delta = p^1q^2 - q^1p^2 \neq 0).$$

There are induced automorphisms denoted $S^2\mathcal{A}$, $S^3\mathcal{A}$, ... of the symmetrical tensor product spaces $S^2\mathcal{V}$, $S^3\mathcal{V}$, ..., respectively. They can be explicitly written down, e.g.,

$$S^2\mathcal{A}\mu \odot \mu = (p^1)^2\mu \odot \mu + 2p^1q^1\mu \odot \nu + (q^1)^2\nu \odot \nu.$$

The inversion $\mathcal{A}^{-1}$ arises after the substitution

$$p^1 \to q^2/\Delta,\;\; q^1 \to -q^1/\Delta,\;\; p^2 \to -p^2/\Delta,\;\; q^2 \to p^1/\Delta \tag{23}$$

in the above expression of $\mathcal{A}$. It follows that the same substitution (23) in the formula for $S^2\mathcal{A}, S^3\mathcal{A}, \ldots$ (not explicitly written here) would yield the inversions $(S^2\mathcal{A})^{-1} = S^2(\mathcal{A}^{-1})$, $(S^3\mathcal{A})^{-1} = S^3(\mathcal{A}^{-1})$, ...

9. Characteristic directions. The families Z^i can be ultimately specified by employing either $\bar{\xi}$ or $\mathrm{d}\bar{\omega}$. Both the ways, though technically different, lead to the same final result. (Such coincidences appear again and again. They indicate the existence of certain hidden geometrical structures.) For brevity, we shall consider only the congruence

$$\mathrm{d}\bar{\omega} = \mathrm{d}\bar{a} \wedge \vartheta + \bar{a}d\vartheta \cong \mathrm{d}\ln \bar{a} \wedge \bar{\omega} = \frac{1}{Ef} dE \wedge \bar{\omega} \quad (\mathrm{mod}\ \Omega^3).$$

This is equivalent to the investigation of the intrinsical form dE/Ef (mod Ω^3) but we shall consider only the congruence

$$dE/Ef \cong \sum A^{ij}\vartheta_{ij} \qquad (\mathrm{mod}\ \Omega^3, \zeta^1, \zeta^2) \tag{24}$$

where the sum is with $i + j = 4$ and the abbreviations $A^{40} = \frac{1}{Ef}\partial^2 f/\partial u_{xx}^2$, $A^{31} = \frac{2}{Ef}\partial^2 f/\partial u_{xx}\partial u_{xy}$, $A^{22} = \frac{1}{Ef}(2\partial^2 f/\partial u_{xx}\partial u_{yy} + \partial^2 f/\partial u_{xy}^2)$, $A^{13} = \frac{2}{Ef}\partial^2 f/\partial u_{yy}\partial u_{yx}$, $A^{04} = \frac{1}{Ef}\partial^2 f/\partial u_{yy}^2$ are used. The forms ϑ_{ij} in (24) can be expressed by intrinsical forms of the 4th order of the sequence (17), namely the forms

$$\omega_{40} = \omega^{1111},\ \omega_{31} = \omega^{1112},\ \ldots,\ \omega_{04} = \omega^{2222}$$

(with nondecreasing upper multiindices) are quite enough. The forms depend on the as yet unspecified coefficients p^i, q^i, of course. After the substitution, (24) turns into

$$dE/Ef \cong \sum B^{ij}\omega_{ij} \quad (\mathrm{mod}\ \Omega^3, \zeta^1, \zeta^2)$$

where B^{ij} $(i + j = 4)$ are coefficients of intrinsical nature.

The calculation of the coefficients is easy if one uses the hint of Section 8. Indeed, the lower order formulae

$$\omega^i = Z^i \rfloor \mathrm{d}\bar{\omega} \cong \bar{a}(p^i\vartheta_x + q^i\vartheta_y) \quad (\mathrm{mod}\ \Omega^0)$$

clearly induce analogous expressions of the higher order terms of the sequence (17), e.g.,

$$\omega^{11} = Z^1 \rfloor \mathrm{d}\omega^1 \cong \bar{a}((p^1)^2\vartheta_{xx} + 2p^1q^1\vartheta_{xy} + (q^1)^2\vartheta_{yy}) \ (\mathrm{mod}\ \Omega^1).$$

It is obvious that the multiples $\bar{a}S^2\mathcal{A}$, $\bar{a}S^3\mathcal{A}$, ... of the symmetrical tensor products appear. Consequently, the inversion of these formulae (i.e., the expression of the forms ϑ_{ij} by means of the forms ω_{ij}) easily appear after the substitution (23). For our aim, it is sufficient to deal only with 4th order forms.

As usual, we shall state only the most important part of the final results, namely the coefficients

$$B^{40} = \frac{1}{Ef}\frac{1}{a\Delta^4}\left(\frac{\partial^2 f}{\partial u_{xx}^2}(q^2)^4 - 2\frac{\partial^2 f}{\partial u_{xx}\partial u_{xy}}(q^2)^3 p^2 + \right.$$
$$\left. + \left(2\frac{\partial^2 f}{\partial u_{xx}\partial u_{yy}} + \frac{\partial^2 f}{\partial u_{xy}^2}\right)(q^2 p^2)^2 - 2\frac{\partial^2 f}{\partial u_{yy}\partial u_{xy}} q^3 (p^2)^2 + \frac{\partial^2 f}{\partial u_{yy}^2}(p^2)^4\right)$$

and B^{04} resulting from B^{40} after the substitution $q^2 \to q^1$, $p^2 \to p^1$. In rough terms, the next performances can be outlined as follows. If B^{40} is identically nonvanishing (i.e., if f is a nonlinear function of second derivatives), then the requirements $B^{04} = 0$, $B^{40} = 0$ provide (the same) 4th order algebraic equation (e.g.) for the quotients

$$w^1 = q^1/p^1, \quad w^2 = q^2/p^2.$$

We need $\Delta \neq 0$, hence $w^1 \neq w^2$, so that the algebraic equation should have at least two different real roots which are to be chosen for w^1, w^2. (The case of a quadruple root is similar to Section 4 so that quite other methods should be applied. The case of imaginary roots can be in principle resolved by complexification.) So we obtain two distinct intrinsical fields of directions

$$Z^i = p^i(\partial_x + w^i \partial_y) \quad (i = 1, 2; w^1 \neq w^2) \tag{25}$$

where p^i are not yet specified. (One can observe that they are nothing else than the common characteristic directions to the $\mathcal{EL}$ equation.) Then we may employ any two of the requirements

$$B^{31} = 1, \ B^{22} = 1, \ B^{13} = 1, \ w = 1.$$

They can be respectively expressed as

$$p^1(p^2)^3 B_1 = 1, \ (p^1 p^2)^2 B_2 = 1, \ (p^1)^3 p^2 B_3 = 1, \ (w^2 - w^1) p^1 p^2 f = 1$$

(cf. (16) for the last one) where B_1, B_2, B_3 are certain (in general nonvanishing) polynomials in the well-known values w^1, w^2.

After the ultimate specification of p^1, p^2, the families (25) turn into a certain intrinsical vector fields and we are done.

10. Point equivalences. If we deal with automorphisms of the space of variables x, y, u, then the module $\Theta = \{dx, dy, du\}$ is to be taken for an additional intrinsical object. Using $Z = p\partial_x + q\partial_y \in \Omega^\perp$, $\omega = a\vartheta \in \Omega^0$ ($\neq 0$) and the previous $\mathcal{PC}$ form $\bar{\xi}$, we look at the congruences modulo Θ:

$$
\begin{aligned}
Z\rfloor\bar{\xi} &\cong (pc^{10} - qb^{10})\vartheta_x + (pc^{01} - qb^{01})\vartheta_y,\\
Z\rfloor\,\mathrm{d}\omega &\cong a(p\vartheta_x + q\vartheta_y).
\end{aligned}
$$

The right hand sides are proportional if and only if

$$pc^{10} - qb^{10} = sap, \quad pc^{01} - qb^{01} = saq.$$

A solution $Z \neq 0$ exists if and only if the proportionality factor s satisfies $s^2 = (M^2 - LN)/a^2$ where $L = \partial f/\partial u_{xx}$, $M = \partial f/\partial u_{xy}$, $N = \partial f/\partial y_{yy}$. So we occur in a situation quite analogous to that in Section 1. For instance, if $M^2 > LN$ then the intrinsical requirement $s = \pm 1$ can be introduced and yields the common specifications $\bar{a} = (M^2 - LN)^{1/2}$, $\bar{\omega} = \bar{a}\vartheta$, Z^+, Z^- formally the same as in Section 2. (It is to be noted that these Z^+, Z^- have nothing in common with the characteristics of the $\mathcal{EL}$ equation!)

11. Fiber equivalences. Here the module $\{\mathrm{d}x\,, \mathrm{d}y\,\}$ and thus also the previous $\Theta = \{\mathrm{d}x\,, \mathrm{d}y\,\} \cup \Omega^0$ are taken for intrinsical objects. Consequently, the subfamily of (2) consisting of all forms

$$\xi = \lambda\,\mathrm{d}x\, + \sum(b^{ij}\,\mathrm{d}y\, - c^{ij}\,\mathrm{d}x\,) \wedge \vartheta_{ij}$$

with the typical property $\xi \cong 0$ (mod $\mathrm{d}x\,, \mathrm{d}y\,$) is of intrinsical nature. Simplified arguments of Section 7 applied to this subfamily lead to the familiar specification, the common $\mathcal{PC}$ form

$$\lambda\,\mathrm{d}x\, + \mathrm{d}y\, \wedge (b^{00}\vartheta + b^{10}\vartheta_x + b^{01}\vartheta_y) + \mathrm{d}x\, \wedge (c^{00}\vartheta + c^{10}\vartheta_x + c^{01}\vartheta_y),$$

where the coefficients b^{ij}, c^{ij} are given by (21, 22). Also the following calculations and results greatly simplify.

12. Preparatory results. Retaining the previous underlying space endowed with the same module Ω as before, we shall be interested in the divergence equivalences for the variational integral (1). The primary intrinsical objects are Ω (consequently all submodules $\Omega^\ell \subset \Omega$) and the family of all 2-forms $\xi\, \mathrm{d}\varphi$ with φ an arbitrary 1-form, that is, the family of all 3-forms $\mathrm{d}\xi$ (where ξ is expressed as (2)). The arguments of Section 5 (except the last one) imply that the differential $\mathrm{d}\bar\xi$ of the form (3) is of intrinsical nature. So, instead of intrinsical 2-form $\bar\xi$, we have the closed intrinsical 3-form $\mathrm{d}\bar\xi$ if we deal with the divergence equivalence.

Recalling the generator $\omega = a\vartheta$ ($a \neq 0$ is variable) of Ω^0 and formula (4), we conclude that $\{\ldots\}/a$ is intrinsical modulo Ω^0 analogously to Section 1. But the argument cannot be continued since the form $\bar\xi$ is not available here. Instead we introduce the intrinsical family

$$\chi \cong \{\ldots\}/a + (b\,\mathrm{d}y - c\,\mathrm{d}x) \wedge \vartheta \quad (\mathrm{mod}\ \Omega \wedge \Omega),$$

where $a \neq 0, b, c$ are variable functions. Using the frame (14), let

$$\chi^i = Z^i \rfloor \chi = \frac{p^i}{a}(E\,\mathrm{d}y - M\vartheta_x - N\vartheta_y - c\omega) - \frac{q^i}{a}(E\,\mathrm{d}x - L\vartheta_x - M\vartheta_y - b\omega),$$

(26) $$\omega^i = Z^i \rfloor \mathrm{d}\omega = Z^i \ln a \cdot \omega + a(p^i\vartheta_x + q^i\vartheta_y).$$

Assuming $E \neq 0$, one can then obtain

$$\chi^1 \wedge \chi^2 \cong w\frac{LN - M^2}{aE}\vartheta_x \wedge \vartheta_y, \quad \omega^1 \wedge \omega^2 \cong w\frac{a^3}{E}\vartheta_x \wedge \vartheta_y \ (\mathrm{mod}\ \Omega^0, \chi)$$

with the factor $w = (p^1q^2 - q^1p^2)E/a$ (other than the previous one (16) which does not make any sense here). Assume $M^2 \neq LN$ and let $\pm = \mathrm{sign}\,(M^2 - LN)$. Coefficients of $\vartheta_x \wedge \vartheta_y$ differ by a sign (an intrinsical requirement) if and only if we specify $\bar a = (\pm(M^2 - LN))^{1/4}$. We may also introduce $\bar\omega = \bar a\vartheta$, $\bar\chi \cong E\,\mathrm{d}x \wedge \mathrm{d}y\,/\bar a$ (mod Ω), $\bar w = (p^1q^2 - q^1p^2)E/\bar a$. In particular, it follows that the second order variational integral

$$\iint \frac{E}{\bar a}\,\mathrm{d}x \wedge \mathrm{d}y \to \text{ extremum}$$

is intrinsically related to the divergence problem. However, Sections 7–11 cannot be directly referred to since the function $E/\bar a$ is of a rather special kind, linear in the second order derivatives. Although the methods of the previous Sections could be easily adapted, we shall draw a slightly different way.

13. Concluding results. Instead of looking for the $\mathcal{PC}$ form of the above variational integral, we shall specify the coefficients b, c appearing in $\bar{\chi}$ by means of the requirement $\mathrm{d}\bar{\chi} \cong 0 \pmod{\Omega^0, \Omega \wedge \Omega}$. After some direct calculations, this requirement is expressed by

$$b = \bar{a}\left(\frac{\partial E/\bar{a}}{\partial u_x} + \partial_x \frac{L}{\bar{a}} + \partial_y \frac{M}{\bar{a}}\right), \quad c = \bar{a}\left(\frac{\partial E/\bar{a}}{\partial u_y} + \partial_x \frac{M}{\bar{a}} + \partial_y \frac{N}{\bar{a}}\right),$$

and then

$$\mathrm{d}\bar{\chi} \cong I\omega \wedge \chi \pmod{\Omega \wedge \Omega}, \; I = \frac{1}{E}\left(\frac{\partial E/\bar{a}}{\partial u} + \partial_x b + \partial_y c\right)$$

where I is clearly an invariant for the diverence equivalences.

The only remaining problem consists in the specification of p^i, q^i (since then the forms χ^1, χ^2, $\bar{\omega}$, ω^1, ω^2 together with the appropriate forms (17) provide the sought Frenet coframe). To this goal, we may employ $\bar{\omega}$, $\bar{w}$, I.

In more detail, ϑ_x and ϑ_y can be expressed as linear combinations of $\bar{\omega}$, ω^1, ω^2 (cf. (26)). Analogously $\mathrm{d}x$, $\mathrm{d}y$ can be expressed in terms of χ^1, χ^2, $\bar{\omega}$, ω^1, ω^2. One can then derive the development

$$\begin{aligned} \mathrm{d}\bar{\omega} &= \mathrm{d}\ln\bar{a} \wedge \bar{\omega} + \bar{a}(\mathrm{d}x \wedge \vartheta_x + \mathrm{d}y \wedge \vartheta_y) \\ &= \frac{1}{\bar{w}}(\chi^1 \wedge \omega^2 - \chi^2 \wedge \omega^1) + (A^2\omega^1 - A^1\omega^2) \wedge \bar{\omega} \end{aligned}$$

where the coefficients

$$A^i \equiv \frac{1}{\bar{a}^3\bar{w}}\left(q^i\left(L\partial_x + M\partial_y + E\left(\frac{\partial}{\partial u_x} - c\bar{a}\right)\right) - p^i\left(M\partial_x + N\partial_y + E\left(\frac{\partial}{\partial u_y} - b\bar{a}\right)\right)\bar{a}\right)$$

are of intrinsical nature. Even more interesting is the development

$$dI \cong \sum A^{ij}\vartheta_{ij} \qquad \pmod{\Omega^2, \chi^1, \chi^2}$$

quite analogous to (24) but with the summation over $i + j = 3$. Using the hint of Section 8, it may be adapted to

$$dI \cong \sum B^{ij}\omega_{ij} \qquad \pmod{\Omega^2, \chi^1, \chi^2}$$

where B^{ij} $(i + j = 3)$ are coefficients of intrinsical nature. As a result, we have a sufficient supply of means to reach the specification of Z^1, Z^2. The calculations are in a certain sense of intermediate nature between the previous cases of the first- and second-order variational integrals and do not bring any new ideas.

14. Preliminaries. Changing the underlying space and the module Ω, we shall be interested in the equivalence problem for the variational integrals

$$\iiint f(x,y,z,u,u_x,u_y,u_z)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \to \text{ extremum.}$$

The new underlying space with coordinates x, y, z, u_{ijk} $(i,j,k=0,1,\ldots)$ is endowed with the module $\Omega = \{\vartheta_{ijk}\,;\, i,j,k=0,1,\ldots\}$ where $\vartheta_{ijk} = \mathrm{d}u_{ijk} - u_{i+1,j,k}\,\mathrm{d}x - u_{i,j+1,k}\,\mathrm{d}y - u_{i,j,k+1}\,\mathrm{d}z$ are the contact forms, and with the Lagrange density $\lambda = f\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$. (Abbreviations like $u = u_{000}, \ldots$ analogous to the previous ones will be currently used.) The module Ω together with the family of all 3-forms ξ satisfying the congruence $\xi \cong \lambda \pmod{\Omega}$ are taken for primary intrinsical objects.

First of all, it can be verified that all automorphisms of the underlying space which preserve Ω are mere prolonged contact transformations. In other terms, they preserve all submodules $\Omega^\ell = \{\vartheta_{ijk}; i+j+k \leqslant \ell;\, i,j,k=0,1,\ldots\} \subset \Omega$. We omit the proof.

Secondly, the family of all forms ξ is very wide and we should like to pick out a certain special form $\bar{\xi}$ from it. This goal can be achieved by a successive reduction procedure quite analogously as in Section 6. (The reduction may run as follows. Since $\Omega^0 = \{\vartheta\} \subset \Omega$ is an intrinsical submodule, the group of coordinates x, y, z, u, u_x. u_y, u_z is intrinsical, too. Consequently, we may deal with such forms ξ which are expressible in terms of forms from the module $\Xi = \{\mathrm{d}x\,, \mathrm{d}y\,, \mathrm{d}z\,, \vartheta, \vartheta_x, \vartheta_y, \vartheta_z\}$. For this subfamily, the additional requirement $\mathrm{d}\xi \cong 0 \pmod{\Omega^0, \Omega \wedge \Omega}$ leads to a still narrower class of forms ξ of the kind

$$\begin{aligned}\xi = \lambda &+ (P\,\mathrm{d}y \wedge \mathrm{d}z + Q\,\mathrm{d}z \wedge \mathrm{d}x + R\,\mathrm{d}x \wedge \mathrm{d}y) \wedge \vartheta \\ &- \mathrm{d}\,(\vartheta \wedge (r\,\mathrm{d}x + s\,\mathrm{d}y + t\,\mathrm{d}z)) + \varphi\end{aligned}$$

where $P = \partial f/\partial u_x$, $Q = \partial f/\partial u_y$, $R = \partial f/\partial u_z$ are determined, r, s, t are variable functions, and $\varphi \cong 0 \pmod{\Omega^1 \wedge \Omega^1}$. Then, closely simulating Section 6, we introduce the requirement $\mathrm{d}\xi \cong 0 \pmod{\Omega^0}$. This implies the vanishing $\varphi = 0$. Finally, the form ξ can be represented as a product $\xi^1 \wedge \xi^2 \wedge \xi^3$ of three linear forms if and only if $r = s = t = 0$.) As a final result, the familiar $\mathcal{PC}$ form

$$\bar{\xi} = \lambda + (P\,\mathrm{d}y \wedge \mathrm{d}z + Q\,\mathrm{d}z \wedge \mathrm{d}x + R\,\mathrm{d}x \wedge \mathrm{d}y) \wedge \vartheta \tag{27}$$

appears. In principle, it is of intrinsical nature with respect to prolonged contact transformations (the automorphisms of Ω) but we shall now see that in reality only the prolonged point equivalences may be taken into account.

In more precise terms, we shall see that the module $\Theta = \{\mathrm{d}x\,,\mathrm{d}y\,,\mathrm{d}z\,,\mathrm{d}u\,\}$ is of intrinsical nature. To this aim, let

$$\partial_x = \partial/\partial x + \sum u_{i+1,j,k}\partial/\partial u_{ijk}, \quad \partial_y = \partial/\partial y + \ldots, \partial_z = \partial/\partial z + \ldots$$

be the familiar basis of $\Omega^\perp$. Then the moving frame

$$Z^i = p^i\partial_x + q^i\partial_y + r^1\partial_z \ \left(i = 1,2,3; \Delta = \det\begin{pmatrix} p^1 & \cdots \\ \cdots & r^3\end{pmatrix} \neq 0\right) \tag{28}$$

(a general basis of $\Omega^\perp$) is of intrinsical nature. It follows that the families of forms

$$\zeta^1 = Z^2\rfloor Z^3\rfloor\bar\xi, \quad \zeta^2 = Z^3\rfloor Z^1\rfloor\bar\xi, \quad \zeta^3 = Z^1\rfloor Z^2\rfloor\bar\xi \tag{29}$$

are intrinsical and consequently the module

$$\Theta = \{\mathrm{d}x\,,\mathrm{d}y\,,\mathrm{d}z\,,\vartheta\} = \{\zeta^1,\zeta^2,\zeta^3,\Omega^0\}$$

is of intrinsical nature, too. This is the desired result.

On this occasion, let us mention the intrinsical trivector $\bar\partial = \partial_x \wedge \partial_y \wedge \partial_z/f$ (with the property $\xi(\bar\partial) = 1$) and the relevant identity $Z^1 \wedge Z^2 \wedge Z^3 = f\Delta\bar\partial$ which means that the coefficient $f\Delta$ is of intrinsical nature. One can also verify the formula $\zeta^1 \wedge \zeta^2 \wedge \zeta^3 = (f\Delta)^3\bar\xi$. (Hint: use (29) on the left hand side and apply the operators $Z^i\rfloor Z^j\rfloor$ and $Z^1\rfloor Z^2\rfloor Z^3$ to the arising equation.)

15. Intrinsical polarity. Take a general vector field $Z = p\partial_x + q\partial_y + r\partial_z \in \Omega^\perp$ and recall the intrinsical family $\omega = a\vartheta$ ($a \neq 0$ is variable). Then the mapping

$$Z \to Z\rfloor\,\mathrm{d}\omega \cong a(p\vartheta_x + q\vartheta_y + r\vartheta_z) \quad (\mathrm{mod}\ \Omega^0)$$

between $\Omega^\perp$ and Ω^1/Ω^0 is bijective. On the other hand, take a general linear form $\zeta \in \Theta$ with the class $\zeta \cong \hat p\,\mathrm{d}x + \hat q\,\mathrm{d}y + \hat r\,\mathrm{d}z$ (mod Ω^0), and recall the $\mathcal{PC}$ form (27). Then the extension product

$$\begin{aligned}\zeta \wedge \mathrm{d}\bar\xi = \Big\{\frac{1}{fa}\Big(\Big(&\frac{\partial P}{\partial u_x}\vartheta_x + \frac{\partial P}{\partial u_y}\vartheta_y + \frac{\partial P}{\partial z}\vartheta_z\Big)\hat p \\ &+ \Big(\frac{\partial Q}{\partial u_x}\vartheta_x + \ldots\Big)\hat q + \Big(\frac{\partial R}{\partial u_x}\vartheta_x + \ldots\Big)\hat r\Big)\Big\} \wedge \omega \wedge \bar\xi\end{aligned}$$

makes a good sense and determines a mapping $\zeta \to \{\ldots\}$ of the class of ζ into Ω^1/Ω^0. Altogether taken, the composition

$$\zeta \to \{\ldots\} = a(p\vartheta_x + q\vartheta_y + r\vartheta_z) \to Z \tag{30}$$

is a polarity with respect to the quadratic form

$$\mathbf{Q}(\zeta) = \frac{1}{fa^2}\left(\frac{\partial^2 f}{\partial u_x^2}(\hat{p})^2 + 2\frac{\partial^2 f}{\partial u_x \partial u_y}\hat{p}\hat{q} + \dots\right)$$

depending on parameter a (which will be soon specified).

In more explicit terms, the polarity (30) is determined by the formulae

$$\mathrm{d}x \to \frac{1}{2}\frac{\partial \mathbf{Q}}{\partial \hat{p}}\partial_x, \mathrm{d}y \to \frac{1}{2}\frac{\partial \mathbf{Q}}{\partial \hat{q}}\partial_y, \mathrm{d}z \to \frac{1}{2}\frac{\partial \mathbf{Q}}{\partial \hat{r}}\partial_z.$$

It follows that the intrinsical class $\bar{\xi} \cong f\,\mathrm{d}x \wedge dy \wedge \mathrm{d}z \pmod{\Omega^0}$ is transformed into the trivector:

$$f\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \to f\cdot(\dots)\wedge(\dots)\wedge(\dots) = \frac{\mathrm{Hess}\, f}{fa^6}\bar{\partial}$$

(where Hess $f = \det\begin{pmatrix}\partial^2 f/\partial u_x^2 & \dots\\ \dots & \dots\end{pmatrix}$ is the Hessian). The coefficient of $\bar{\partial}$ is intrinsical. Assuming the *regularity* Hess $f \neq 0$, it may be equated to ± 1 (where $\pm$ is the sign of f Hess f). This yields the specification $\bar{a} = (\pm\,\mathrm{Hess}\, f/f)^{1/6}$ of the parameter a and thus the relevant specifications of all objects (as, e.g., $\omega, \mathbf{Q}$) depending on a.

16. An intrinsical form. First of all, let us recall the families (29) depending on $p^1, \dots, r^3$. In explicit terms, e.g.,

$$\begin{aligned}\zeta^1 &= f\left(\begin{vmatrix} r^2 & q^2\\ r^3 & q^3\end{vmatrix}\mathrm{d}x + \dots\right) + \left(P\begin{vmatrix} r^2 & q^2\\ r^3 & q^3\end{vmatrix} + \dots\right)\vartheta\\ &= p_1\,\mathrm{d}x + q_1\,\mathrm{d}y + r_1\,\mathrm{d}z + \frac{1}{f}(Pp_1 + Qq_1 + Rr_1)\vartheta.\end{aligned} \tag{31}$$

Secondly, following the common method, let us introduce the families $\omega^i \equiv Z^i \rfloor \mathrm{d}\bar{\omega}$ and (17) (where $i = 1, 2, 3$). One can then represent $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ as linear combinations of ζ^1, ζ^2, ζ^3, $\bar{\omega}$. Analogously ϑ_x, ϑ_y, ϑ_z can be expressed in terms of $\bar{\omega}$, ω^1, ω^2, ω^3 and (using a slight adaptation of Section 8) analogous formulae can be derived for higher order contact forms.

In virtue of these formulae (not explicitly stated here), a development of the kind

$$\begin{aligned}\mathrm{d}\bar{\omega} &= \mathrm{d}\ln\bar{a}\wedge\bar{\omega} + \bar{a}(\mathrm{d}x \wedge \vartheta_x + \mathrm{d}y \wedge \vartheta_y + \mathrm{d}z \wedge \vartheta_z)\\ &= \frac{1}{f\Delta}\sum \zeta^i \wedge \omega^i + \sum A^i \omega^i \wedge \bar{\omega}\end{aligned}$$

can be derived. (As the first summand of the last term is concerned, use $\zeta^i(Z^i) \equiv f\Delta$, $\zeta^i(Z^j) \equiv 0$ for $i \neq j$.) The coefficients

$$A^i \equiv \frac{1}{\bar{a}f\Delta}\left(\left(\frac{\partial \ln\bar{a}}{\partial u_x} + \frac{P}{f}\right)p^i + \left(\frac{\partial \ln\bar{a}}{\partial u_y} + \frac{Q}{f}\right)q^i + \left(\frac{\partial \ln\bar{a}}{\partial u_z} + \frac{R}{f}\right)r^i\right)$$

are intrinsical. So we obtain the intrinsical vector field

$$A = \frac{1}{af\Delta}\left(\left(\frac{\partial \ln \bar{a}}{\partial u_x} + \frac{P}{f}\right)\partial_x + (\ldots)\partial_y + (\ldots)\partial_z\right) \in \Omega^{\perp}$$

uniquely determined from the property $\zeta^i(A) \equiv A^i$. In our regular case, the polarity (30) can be inverted and transforms A into an intrinsical linear form α, more precisely, into the class $\alpha \cong u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z \in \Theta$ (mod Ω^0). (The functions u, v, w can be explicitly calculated but we omit the result.) Then the formula (31) for ζ^1 together with the analogous expressions of ζ^2, ζ^3 immediately implies that

$$\alpha = u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z + \frac{1}{f}(Pu + Qv + Rw)\vartheta \in \Theta$$

necessarily is an intrinsical form. (This is the unique element of the class of α modulo Ω^0 which is linearly depending on $\zeta^1, \zeta^2, \zeta^3$.)

17. Concluding note. One can also derive an invariant depending on the $\mathcal{EL}$ operator $E = \sum(-\partial_x)^i(-\partial_y)^j(-\partial_z)^k \partial f/\partial u_{ijk}$. Namely, the formula

$$\mathrm{d}\bar{\xi} \cong E\vartheta \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \frac{E}{\bar{a}f}\bar{\omega} \wedge \bar{\xi} \quad (\mathrm{mod}\ \Omega \wedge \Omega)$$

implies that the function $I = E/\bar{a}f$ is of this kind.

In general several strategies can be chosen to ultimately specify Z^1, Z^2, Z^3 and thus to reach the Frenet coframe. Assuming $A \neq 0$, one may choose $Z^1 = A$ and then consider the invariant function $\mathbf{Q}(\alpha)$ and the development of $d\mathbf{Q}(\alpha)$ in terms of ζ^1, ζ^2, ζ^3, $\bar{\omega}$, ω^1, ω^2, ω^3 which (in principle) yields a lot of other invariants permitting the determination of Z^2, Z^3. The case $A = 0$ is a rather peculiar one but then the higher order invariant I and the development of dI can be employed. On this occasion, it is to be noted that for every invariant B, the class $\beta = \partial_x B \cdot \mathrm{d}x + \partial_y B \cdot \mathrm{d}y + \partial_z B \cdot \mathrm{d}z \in \Theta/\Omega^0$ is of intrinsical nature and turns into an intrinsical vector field from $\Omega^{\perp}$ by the use of polarity. This vector field might be included into the sought family Z^1, Z^2, Z^3.

Because of a large number of particular subcases which may occur, we shall not pass to more detail. It would be desirable to discuss some particular examples, the case Hess $f = 0$, and the divergence problem.

18. Preparatory remarks. In the case of several variable functions, the true contact transformations do not exist, they are mere prolonged point transformations (see below for a simple proof). But this fact does not made the equivalence problem easier since, on the other hand, the order increasing transformations may appear (even between variational integrals of a fixed order!) in the infinite jet space (cf. [3]). It is highly probable that this trouble cannot appear if we deal with mere *regular* variational problems but we shall not go into this domain at this place. Our present aim is a modest one: to deal with order preserving equivalences for the variational integral

$$\iint f(x,y,u,v,u_x,v_x,u_y,v_y)\,\mathrm{d}x \wedge \mathrm{d}y \rightarrow \text{ extremum.} \tag{32}$$

Using the alternative notation

$$w^1 = u,\ w^2 = v,\ w^i_{jk} \equiv \partial^{j+k} w^i / \partial x^j \partial y^k$$

and abbreviations like $w^i \equiv w^i_{00}$, $w^i_x \equiv w^i_{10}$, $w^i_y \equiv w^i_{01}$, ..., we can make our task more precise as follows.

Our reasonings are carried out in the space of variables x, y, w^i_{jk} $(i = 1, 2;\ j, k = 0, 1, \ldots)$. This space is equipped with the module $\Omega = \{\vartheta^i_{jk}\,;\ i = 1, 2\,;\ j, k = 0, 1, \ldots\}$ generated by the contact forms $\vartheta^i_{jk} \equiv dw^i_{jk} - w^i_{j+1,k}\,\mathrm{d}x - w^i_{j,k+1}\,\mathrm{d}y$, and with the Lagrange density $\lambda = f\,\mathrm{d}x \wedge \mathrm{d}y$. We are interested in *order preserving* equivalences, i.e., the submodules $\Omega^\ell = \{\vartheta^i_{jk}\,;\ j + k \leqslant \ell\}$ are apriori taken for intrinsical objects. As the integral (32) is concerned, the family of forms ξ satisfying the congruence $\xi \cong \lambda \pmod{\Omega}$ is taken for additional intrinsical object, too.

The aim of the present Section is to determine two other (and crucial) intrinsical objects.

First of all, since $\Omega^0 = \{\vartheta^1, \vartheta^2\}$ is intrinsical (abbreviation $\vartheta^i \equiv \vartheta^i_{00}$), the module

$$\Xi = \{\mathrm{d}x\,, \mathrm{d}y\,, \mathrm{d}u\,, \mathrm{d}v\,, \mathrm{d}u_x\,, \mathrm{d}v_x\,, \mathrm{d}u_y\,, \mathrm{d}v_y\,\}$$

is intrinsical, too. (Hint: observe that x, ..., v_y is the minimal group of coordinates such that there exists a basis of Ω^0 expressible in terms of them.) Let us consider a form $\omega = a\vartheta^1 + b\vartheta^2 \in \Omega$ where a, b are fixed but arbitrary functions. Then in the cotangent space of variables $\mathrm{d}x$, ..., $\mathrm{d}v_y$, the exterior systems $\mathrm{d}\omega =$

$$da \wedge \vartheta^1 + db \wedge \vartheta^2 + a(\mathrm{d}x \wedge \mathrm{d}u_x + \mathrm{d}y \wedge \mathrm{d}u_y) + b(\mathrm{d}x \wedge \mathrm{d}v_y + \mathrm{d}y \wedge \mathrm{d}v_y) = 0$$

have the common solution $\vartheta^1 = \vartheta^2 = \mathrm{d}x = \mathrm{d}y = 0$ for all choices of a, b. One can then see that every common solution involves both the conditions $\vartheta^1 = \vartheta^2 = 0$ and also $\mathrm{d}x = \mathrm{d}y = 0$. It follows that the above solution is the minimal one (and thus unique), i.e., *the module* $\Theta = \{\mathrm{d}x\,, \mathrm{d}y\,, \mathrm{d}u\,, \mathrm{d}v\}$ *is of intrinsical nature.* (We have derived a version of the *Lie-Bäcklund theorem*: an automorphism of Ω^0 is a prolonged point transformation.)

Now we shall look for the $\mathcal{PC}$ form. In virtue of the previous result, the subfamily of such forms ξ which may be expressed only in terms of differentials $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}u$, $\mathrm{d}v$ is of intrinsical nature. These are just the forms of the kind

$$\xi = \lambda - \sum (P^i\,\mathrm{d}y - Q^i\,\mathrm{d}x\,) \wedge \vartheta^i + r\vartheta^1 \wedge \vartheta^2$$

where P^1, P^2, Q^1, Q^2, r are varying functions. Then the intrinsical requirement $\mathrm{d}\xi \cong 0 \ (\mathrm{mod}\ \Omega^0, \Omega \wedge \Omega)$ determines the coefficients

$$P^i \equiv \partial f / \partial w_x^i, \quad Q^i \equiv \partial f / \partial w_y^i.$$

Moreover, one can see that ξ can be represented as an exterior product of appropriate 1-forms if and only if $r = (p^1 q^2 - q^1 p^2)/f$. So we have the $\mathcal{PC}$ form

$$\bar{\xi} = \lambda - \sum (P^i\,\mathrm{d}y - Q^i\,\mathrm{d}x\,) \wedge \vartheta^i + \bar{r}\vartheta^1 \wedge \vartheta^2 \quad \left(\bar{r} = \frac{1}{f}\begin{vmatrix} P^1 & Q^2 \\ Q^1 & P^2 \end{vmatrix}\right).$$

It is intrinsical with respect to point equivalences (and if the last summand of $\bar{\xi}$ is omitted, the well-known form intrinsical to mere fiber-preserving equivalences appears).

19. Principal results. We should like to simulate Section 1. However, already the counterpart to (4), the formula

$$\begin{aligned} \mathrm{d}\bar{\xi} = {} & \sum \{E^i\,\mathrm{d}x \wedge \mathrm{d}y - (L^{ij}\vartheta_x^i + M^{ij}\vartheta_y^i) \wedge \mathrm{d}y \\ & + (M^{ij}\vartheta_x^i + N^{ij}\vartheta_y^i) \wedge \mathrm{d}x\,\} \wedge \vartheta^j + \bar{r}\,\mathrm{d}\,(\vartheta^1 \wedge \vartheta^2) + \alpha \end{aligned} \tag{33}$$

(where $E^i \equiv \partial f/\partial w^i - \partial_x(\partial f/\partial w_x^i) - \partial_y(\partial f/\partial w_y^i)$ are $\mathcal{EL}$ operators, $L^{ij} \equiv \partial^2 f/\partial w_x^i \partial w_x^j$, $M^{ij} \equiv \partial^2 f/\partial w_x^i \partial w_y^i$, $N^{ij} \equiv \partial^2 f/\partial w_y^i \partial w_y^j$, and the last summand $\alpha = (A\,\mathrm{d}x + B\,\mathrm{d}y\,) \wedge \vartheta^1 \wedge \vartheta^2$ need not be explicitly stated here) seems to be much more complex so that we are not able to follow it too closely.

Let us introduce the common family of vectors $Z = p\partial_x + q\partial_y \in \Omega^\perp$ where p, q are varying functions and

$$\partial_x = \partial/\partial x + \sum w_{j+1,k}^i \partial w_{jk}^i, \ \partial_y = \partial/\partial y + \sum w_{j,k+1}^i \partial w_{jk}^i$$

are the total derivatives. Let moreover $\omega^i \equiv a^i\vartheta^1 + b^i\vartheta^2$ $(i = 1,2)$ be a general coframe of Ω^0, i.e., a^i and b^i are varying functions with $a = a^1b^2 - a^2b^1 \neq 0$. Using (33), one can derive a formula of the kind $Z\rfloor \mathrm{d}\bar{\xi} = \beta \wedge \omega^1 + \gamma \wedge \omega^2$ where β, γ are certain differential forms expressible in terms of $\mathrm{d}x$, $\mathrm{d}y$, ϑ^i, ϑ^i_x, ϑ^i_y. They are unique modulo Ω^0 (hence modulo Θ) and thus the requirement

$$Z\rfloor \mathrm{d}\omega^1 \quad \text{is proportional to} \quad \beta \quad (\mathrm{mod}\ \Theta) \tag{34}$$

makes good sense. Assuming $pq \neq 0$ (the subcase $pq = 0$ is easier), (34) leads to the condition

$$\begin{gathered}(L^{11}q^2 - 2M^{11}pq + N^{11}p^2)(L^{22}q^2 - 2M^{22}pq + N^{22}p^2)\\ -(L^{12}q^2 - 2M^{12}pq + N^{12}p^2)(L^{21}q^2 - 2M^{21}pq + N^{21}p^2) = 0\end{gathered} \tag{35}$$

for the coefficients p, q, and to a homogeneous linear system (with coefficients depending on p, q) for a^1, a^2, b^1, b^2. In general, (35) has four mutually distinct roots $\lambda_{(j)} = (p/q)_{(j)}$; $j = 1, \ldots, 4$, and for each of them, the linear system mentioned has a unique nonvanishing solution $a^i_{(j)}, B^i_{(j)}$ determined up to a factor. The corresponding families

$$Z_{(j)} = w_j(\partial_x + \lambda_{(j)}\partial_y),\ \omega^i_{(j)} \equiv w^i_j(a^i_{(j)}\vartheta^1 + b^i_{(j)}\vartheta^2),$$

where w_j, w^i_j $(i = 1,2;\ j = 1, \ldots, 4)$ are variable functions, are of intrinsical nature. (We shall not state more details here for an obvious reason: they are mere routine. It is to be noted that the above uniqueness of $a^i_{(j)}, b^i_{(j)}$ is ensured if a certain determinant is nonvanishing which may be regarded as a generalized *regularity* condition. We also tacitly omit the discussion of complex and multiple roots, in particular the latter ones cause much troubles.)

Specification of coefficients can be obtained by the use of the intrinsical requirements

$$Z_{(j)}\rfloor Z_{(k)}\rfloor \xi = w_j w_k(\lambda_{(j)} - \lambda_{(k)})f = \pm 1 \quad (j,k = 1,2,3)$$

where the sign is appropriately chosen. (In the case of multiple roots, only a part of these requirements can be set up.) They yield, e.g., the specification of w_1:

$$(w_1)^2 = \frac{w_1w_2 \cdot w_1w_3}{w_2w_3} = \pm\frac{1}{f}\frac{\lambda_{(2)} - \lambda_{(3)}}{(\lambda_{(1)} - \lambda_{(2)})(\lambda_{(1)} - \lambda_{(3)})}.$$

Analogous equations determine w_2, w_3 (and even w_4). Then we may introduce the intrinsical forms $\zeta_{(j)} = Z_{(j)}\rfloor\bar{\xi}$; $j = 1, \ldots, 4$. (For the construction of a Frenet coframe, only two of them are enough.) Specification of coefficients c^i_j can be comfortably derived from the developments of the forms $d\omega^i_{(j)}$ modulo Ω^0. (Alternatively, appropriate linear combinations of the forms $\zeta_{(j)}$ belong to Ω^0 and at the same time are of intrinsical nature.)

We conclude that there are many ways (not stated above, e.g., the use of $d\zeta_{(j)}$, $d\lambda_{(j)}$, etc.) to the sought Frenet coframe, however, it seems that it is not reasonable to continue in this generality.

20. Three remarks. (i) There exist unique functions t^1, t^2 satisfying the congruence $d\bar{\xi} \cong \bar{\xi} \wedge \sum t^i\omega^i$ (mod Ω), namely

$$t^1 = \frac{1}{af}(E^1b^2 - E^2a^2), \quad t^2 = \frac{1}{af}(E^2a^1 - E^1b^1). \tag{36}$$

Since t^i are intrinsical for the equivalence transformations between variational integrals, and transformation rules for the values f, $a = a^1b^2 - b^1a^2$, a^1, a^2, b^1, b^2 are clear (f and a are multiplied by a determinant, a^i and b^i are changed in such a manner that the forms $\omega^i \equiv a^1\vartheta^1 + b^i\vartheta^2$ are invariant, i.e., contragradiently to ϑ^i), the formulae (36) determine the transformation rule for the $\mathcal{EL}$ operators E^1, E^2. If moreover a^i, b^i are replaced by specified values, t^1 and t^2 turn into invariants of the integral (32).

(ii) A biquadratic form intrinsically related to the integral (32) can be determined by simulating the method of Section 3,15. The above mentioned form $\mathbf{Q}(Z \otimes \varphi)$ will be defined on the tensor product $\Omega^\perp \otimes \Omega^0$ and will be quadratic in the factors $Z \in \Omega^\perp$ and $\varphi \in \Omega^0$ separately. In explicit terms, let $Z = p\partial_x + q\partial_y$, $\omega = a\vartheta^1 + b\vartheta^2$. We introduce the mapping

$$Z \otimes \omega \to Z\rfloor\, d\omega \cong a(p\vartheta^1_x + q\vartheta^1_y) + b(p\vartheta^2_x + q\vartheta^2_y) \in \Omega^1/\Omega^0. \tag{37}$$

It determines a bijection between the spaces $\Omega^\perp \otimes \Omega^0$ and Ω^1/Ω^0. Now, in order to define the "bipolarity" relevant to the sought form $\mathbf{Q}$, we have to introduce the dual space

$$(\Omega^\perp \otimes \Omega^0)\hat{} = (\Omega^\perp)\hat{}\otimes (\Omega^0)\hat{}.$$

Here $(\Omega^\perp)\hat{}$ consists of restrictions of linear forms ζ to the space $\Omega^\perp$; we denote $\zeta \cong \hat{p}\,dx + \hat{q}\,dy \in (\Omega^\perp)\hat{}$. If we introduce the basis ∂_x, ∂_y, $\partial/\partial\vartheta^i_{jk}$ ($i = 1,2; j,k = 0,1,\ldots$) of the module of all vector fields (the basis is *dual* to the basis dx, dy, ϑ^i_{jk}), then the restriction to Ω^0 of a vector field $C = c\partial_x + c'\partial_y + \sum c^i_{jk}\partial/\partial\vartheta^i_{jk}$ clearly is $C \cong c^1\partial/\partial\vartheta^1 + c^2\partial/\partial\vartheta^2 \in (\Omega^0)\hat{}$. With this notation and formula (33), we have the mapping

$$\zeta \otimes C \to C\rfloor(\zeta \wedge \bar{\xi}) \cong \{\ldots\} \to \{\ldots\} \in \Omega^1/\Omega^0, \tag{38}$$

where $\{\ldots\} =$

$$(39)\qquad \begin{aligned}&\frac{1}{f}\sum c^j(L^{ij}\hat{p}\vartheta^i_x + M^{ij}(\hat{q}\vartheta^i_x + \hat{p}\vartheta^i_y) + N^{ij}\hat{q}\vartheta^i_y)\\ &+\frac{\bar{r}}{f}(c^1(\hat{p}\vartheta^2_y - \hat{q}\vartheta^2_x) - c^2(\hat{p}\vartheta^1_y - \hat{a}\vartheta^1_x)) \in \Omega^1/\Omega^0.\end{aligned}$$

The inversion of (37) composed with (39) yields the sought "bipolarity" mapping $\mathbf{P} : \Omega^\perp \otimes \Omega^0 \to (\Omega^\perp \otimes \Omega^0)\hat{}$. The relevant biquadratic form is defined by the duality pairing $\mathbf{Q}(\zeta \otimes C) = \mathbf{B}(\zeta \otimes C)(\zeta \otimes C)$, where $\mathbf{B}(\zeta \otimes C) \in (\Omega^\perp \otimes \Omega^0)\hat{}$ is regarded as a linear function on $\Omega^\perp \otimes \Omega^0$.

(iii) If we deal with fiber equivalences, the module $\{\mathrm{d}x\,, \mathrm{d}y\,\}$ is of intrinsical nature. Then the last summand of (39) may be omitted and $\mathbf{Q}$ gets simplier. Analogous arguments applied to $\bar{\xi}$ implies that both the forms $\lambda = f\,\mathrm{d}x \wedge \mathrm{d}y$ and $\mu = \sum(P^i\,\mathrm{d}y - Q^i\,\mathrm{d}x\,) \wedge \vartheta^i$ are of intrinsical nature. We may consider the mappings $Z \to Z\rfloor\lambda$, $C \to C\rfloor\mu$ into the module $\{\mathrm{d}x\,, \mathrm{d}y\,\}$. The first can be inverted and so we obtain an intrinsical mapping $(\Omega^0)^\perp \to \Omega^\perp$ by composition, and thus a biquadratic form $\mathbf{Q}(\zeta \otimes Z)$ by substitution.

References

[1] *E. Cartan*: Les espaces métriques fondés sur la notion d'aire. Exposés de Géometrie I. Paris, 1933.

[2] *S.S. Chern*: Local equivalences and Euclidean connexes in Finsler spaces. Selected Papers II. Springer Verlag, 1989, pp. 194–212.

[3] *J. Chrastina*: On the equivalence of variational problems I. Journal of Differential Equations *98* (1992), 76–90; Part II. To appear.

[4] *J. Chrastina*: On formal theory of differential equations III. Mathematica Bohemica *116* (1991), 60–90.

[5] *R.B. Gardner*: The method of equivalence and its application. CBMS-NSF Regional Conference Series in Appl. Mathematics 58. Philadelphia, 1981.

[6] *N. Kamran*: On the equivalence problem of Élie Cartan. Academie Royale de Belgique, Mem. Cl. Sc., Collection in $8^0 - 2^e$ série, T. XLV-Fascicule 7 et dernier, 1989.

[7] *N. Kamran and P.L. Olver*: III. New invariant differential equations. Nonlinearity *5* (1992), 601–621.

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