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Garyfalos Papaschinopoulos; John Schinas

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SOME RESULTS CONCERNING SECOND AND THIRD ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT

GARYFALOS PAPASCHINOPOULOS and JOHN SCHINAS, Xanthi

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INTRODUCTION

Recently, there has been a lot of work concerning differential equations with piecewise constant argument (see Aftabizadeh, Wiener and Xu [1], Cooke and Wiener [4], [5], Huang [6], Ladas, Partheniadis and Schinas [7], [8], Papaschinopoulos [9], Papaschinopoulos and Schinas [10], Partheniadis [11], Wiener and Cooke [12] and the references cited therein). The current strong interest in these equations is motivated by the fact that they describe hybrid dynamical systems (a combination of continuous and discrete) and therefore combine properties of both differential and difference equations. We also note that these equations may have applications in certain biomedical models [2].

In this paper we study the second and the third order neutral delay differential equations with piecewise constant argument of the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(y(t)+py(t-1)) = -qy\Big(2\Big[\frac{t+1}{2}\Big]\Big), \tag{1}$$

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}(y(t)+py(t-1)) = -qy\Big(2\Big[\frac{t+1}{2}\Big]\Big) \tag{2}$$

where $t \in [-1,\infty)$, p, q are real constants and $[\cdot]$ denotes the greatest-integer function.

We note that some results concerning first-order equations of this form were investigated in [10] and some results concerning second order equations of the same form with $p=0$ are included in [7] and [8].

A function $y\colon [-1,\infty) \mapsto \mathbb{R}$ is a solution of (1) if the following conditions are satisfied:

(i) y is continuous on $[-1,\infty)$,

(ii) $\frac{d}{dt}(y(t)+py(t-1))=g(t)$ exists on $[0,\infty)$ and g is continuous on $[0,\infty)$,

(iii) $\frac{d^2}{dt^2}(y(t)+py(t-1))$ exists on $[0,\infty)$ except possibly at the points $2n-1$, $n\in\{1,2,\dots\}$ where one-sided second derivatives exist,

(iv) y satisfies (1) on the interval $[0,1)$ and on each interval $[2n-1,2n+1)$, $n\in\{1,2,\dots\}$.

A function $y\colon[-1,\infty)\mapsto\mathbb{R}$ is solution of (2) if the following conditions are satisfied:

(i) y is continuous on $[-1,\infty)$,

(ii) $\frac{d^2}{dt^2}(y(t)+py(t-1))=f(t)$ exists on $[0,\infty)$ and f is continuous on $[0,\infty)$,

(iii) $\frac{d^3}{dt^3}(y(t)+py(t-1))$ exists on $[0,\infty)$ except possibly at the points $2n-1$, $n\in\{1,2,\dots\}$ where one-sided third derivatives exist,

(iv) y satisfies (2) on the interval $[0,1)$ and on each interval $[2n-1,2n+1)$, $n\in\{1,2,\dots\}$.

As is customary a solution of (1) is called oscillatory if it has arbitrarily large zeros.

In Proposition 1 of this paper we prove that if $q\neq-2$ then for every initial function $y_0\colon[-1,0]\mapsto\mathbb{R}$ continuous on $[-1,0]$ and for every $a_1\in\mathbb{R}$ there exists a unique solution $y(t)$ of (1) such that $y(t)=y_0(t)$, $-1\leqslant t\leqslant 0$ and $y(1)=a_1$. We also find necessary and sufficient conditions in order equation (1) to be asymptotically stable (see Proposition 2 below) in contrast to second order equations studied in [11] which are not asymptotically stable. In Proposition 3 we give sufficient conditions for the oscillatory behavior of the solutions of (1). In Proposition 4 we prove that if $q\neq-6$ then for every initial function $y_0\colon[-1,0]\mapsto\mathbb{R}$ continuous on $[-1,0]$ and for every $a_1,a_2\in\mathbb{R}$ there exists a unique solution $y(t)$ of (2) such that $y(t)=y_0(t)$, $-1\leqslant t\leqslant 0$ and $y(1)=a_1$, $y(2)=a_2$. We also prove that the equation of the form (2) is not asymptotically stable (see Proposition 5 below). Finally in Proposition 6 we give two sufficient conditions for the oscillatory behavior of the solutions of (2).

Main results

We prove now our main results.

I. The second order equation

First we deal with the second order equation of the form (1).

Proposition 1. *Consider equation* (1) *where $q\neq-2$. Let $y_0\colon[-1,0]\mapsto\mathbb{R}$ be a continuous function on $[-1,0]$ and $a_0,a_1\in\mathbb{R}$. Then if $p\neq0$ (resp. $p=0$) equation*

(1) *has a unique solution* $y(t)$ *which satisfies*

$$y(t) = y_0(t), \quad t \in [-1,0] \quad (\text{resp.} \quad y(0) = a_0), \quad y(1) = a_1. \tag{3}$$

Moreover for $t = 2n - 1 + \theta$, $n \in \{0, 1, \dots\}$, $0 \leqslant \theta \leqslant 1$ *the function* $y(t)$ *is given by*

$$\begin{aligned} y(t) = (-p)^{2n}\Big(y_0(\theta - 1) + \Big(\frac{q}{2}(\theta^2 - \theta) - \theta\Big)\, a_0 + (\theta - 1)a_{-1}\Big) \\ + (1 - \theta)a_{2n-1} + \Big(\theta + \frac{q}{2}(\theta - \theta^2)\Big)\, a_{2n} \\ + (\theta^2 - \theta)\Big(\frac{qp}{2} - \frac{q}{2}\Big) \sum_{k=0}^{n-1} (-p)^{2n-2k-1} a_{2k} \end{aligned} \tag{4}$$

and if $t = 2n + \theta$, $n \in \{0, 1, \dots\}$, $0 \leqslant \theta \leqslant 1$

$$\begin{aligned} y(t) = (-p)^{2n+1}\Big(y_0(\theta - 1) + \Big(\frac{q}{2}(\theta^2 - \theta) - \theta\Big)\, a_0 + (\theta - 1)a_{-1}\Big) \\ + \Big(1 - \theta + \frac{q}{2}(\theta^2 - \theta)(p - 1)\Big)\, a_{2n} + \theta a_{2n+1} \\ + (\theta^2 - \theta)\Big(\frac{qp}{2} - \frac{q}{2}\Big) \sum_{k=0}^{n-1} (-p)^{2n-2k} a_{2k} \end{aligned} \tag{5}$$

where $a_{-1} = y_0(-1)$, $a_0 = y_0(0)$ *and* a_{2n+1}, a_{2n}, a_{2n-1} *are given by the difference equation*

$$\begin{pmatrix} a_{2n+3} \\ a_{2n+2} \\ a_{2n+1} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{2n+1} \\ a_{2n} \\ a_{2n-1} \end{pmatrix}, \tag{6}$$

$$\begin{aligned} b_{11} &= \frac{2p^2 - 4p - 5q + 4pq + 6}{2 + q}, \quad b_{12} = \frac{8p - 4p^2 + q^2 - 4pq - 4}{2 + q}, \\ b_{13} &= \frac{2p(p + q - 2)}{2 + q}, \quad b_{21} = \frac{2(2 - p)}{2 + q}, \quad b_{22} = \frac{4p - q - 2}{2 + q}, \\ b_{23} &= \frac{-2p}{2 + q}. \end{aligned}$$

Proof. To prove our proposition we consider a solution $y(t)$ of (1) such that (3) is satisfied. For each $t \in [-1, \infty)$ there exists a $n \in \{0, 1, \dots\}$ such that $n \leqslant \frac{t+1}{2} < n + 1$. Then $2n - 1 \leqslant t < 2n + 1$, $n \in \{0, 1, \dots\}$. We set

$$y(n) = a_n, \quad n \in \{-1, 0, 1, \dots\}. \tag{7}$$

Thus from (1) and (7) it follows

$$\frac{d^2}{dt^2}(y(t)+py(t-1))=-qa_{2n} \tag{8}$$

where $0\leqslant t<1$, $n=0$ or $2n-1\leqslant t<2n+1$, $n\in\{1,2,\ldots\}$.

If

$$\beta_n=\frac{d}{dt}(y(t)+py(t-1)) \quad \text{at} \quad t=n, \quad n\in\{0,1,\ldots\} \tag{9}$$

then by integrating (8) from $2n$ to t where $t\in[0,1)$, $n=0$ or $t\in[2n-1,2n+1)$, $n\in\{1,2,\ldots\}$ we take

$$\frac{d}{dt}(y(t)+py(t-1))=\beta_{2n}-q(t-2n)a_{2n}. \tag{10}$$

Hence, by integrating (10) from $2n$ to t where $t\in[0,1)$, $n=0$ or $t\in[2n-1,2n+1)$, $n\in\{1,2,\ldots\}$ we get

$$y(t)+py(t-1)=(t-2n)\beta_{2n}+a_{2n}+pa_{2n-1}-\frac{q}{2}(t-2n)^2a_{2n}. \tag{11}$$

Since $y(t)$ is continuous in $[-1,\infty)$ by taking the limits as $t\to 2n-1$, $t\to 2n+1$ in (11) and using (7) we obtain correspondingly

$$a_{2n-1}+pa_{2n-2}=a_{2n}+pa_{2n-1}-\beta_{2n}-\frac{q}{2}a_{2n}, \quad n\in\{1,2,\ldots\}. \tag{12}$$

$$a_{2n+1}+pa_{2n}=a_{2n}+pa_{2n-1}+\beta_{2n}-\frac{q}{2}a_{2n}, \quad n\in\{0,1,\ldots\}. \tag{13}$$

By the continuity of the function g given previously in the definition of the solution of (1) and if we take the limits as $t\to 2n-1$, $t\to 2n+1$ in (10) we get from (9) respectively

$$\beta_{2n-1}=\beta_{2n}+qa_{2n}, \quad n\in\{1,2,\ldots\}. \tag{14}$$

$$\beta_{2n+1}=\beta_{2n}-qa_{2n}, \quad n\in\{0,1,\ldots\}. \tag{15}$$

Using (12), (13), (14), (15) and performing some algebraic calculations we can prove that a_{2n+1}, a_{2n}, a_{2n-1} are given by the difference equation (6).

We prove now that $y(t)$ satisfies (4) and (5). Applying Lemma 3 [11, p. 463] to (11) and using (13) we take for $t=2n-1+\theta$, $n\in\{0,1,\ldots\}$, $0\leqslant\theta\leqslant 1$

$$\begin{aligned} y(t)&=(-p)^{2n}y_0(\theta-1)+\sum_{k=0}^{n-1}(-p)^{2n-2k-1}z(2k+\theta)\\ &\quad+\sum_{k=1}^{n}(-p)^{2n-2k}z(2k-1+\theta) \end{aligned} \tag{16}$$

and for $t = 2n + \theta$, $n \in \{0, 1, \ldots\}$, $0 \leqslant \theta \leqslant 1$

$$y(t) = (-p)^{2n+1} y_0(\theta - 1) + \sum_{k=0}^{n} (-p)^{2n-2k} z(2k + \theta) \\ + \sum_{k=1}^{n} (-p)^{2n-2k+1} z(2k - 1 + \theta) \tag{17}$$

where

$$z(2k+\theta) = \Big(1 - \frac{q}{2}\theta^2 + \theta\Big(p - 1 + \frac{q}{2}\Big)\Big) a_{2k} + \theta a_{2k+1} + p(1-\theta) a_{2k-1},$$
$$z(2k-1+\theta) = \Big(1 - \frac{q}{2}(\theta-1)^2 + (\theta - 1)\Big(p - 1 + \frac{q}{2}\Big)\Big) a_{2k} \\ + (\theta - 1) a_{2k+1} + p(2-\theta) a_{2k-1}.$$

By setting

$$c_1(n) = \sum_{k=0}^{n-1} (-p)^{2n-2k-1} a_{2k+1}, \quad c_2(n) = \sum_{k=0}^{n-1} (-p)^{2n-2k-1} a_{2k}, \\ c_3(n) = \sum_{k=0}^{n} (-p)^{2n-2k} a_{2k-1} \tag{18}$$

we can easily get

$$c_1(n) + p c_3(n) = p(-p)^{2n} a_{-1}. \tag{19}$$

Moreover from (16) and (18) we obtain for $t = 2n - 1 + \theta$, $n \in \{0, 1, \ldots\}$, $0 \leqslant \theta \leqslant 1$

$$y(t) = (-p)^{2n} y_0(\theta - 1) + (\theta + p - \theta p) c_1(n) + \Big(\lambda(\theta) - p\lambda(\theta - 1)\Big) c_2(n) \\ + (2p - p\theta + \theta - 1) c_3(n) + (\theta - 1) a_{2n+1} + \lambda(\theta - 1) a_{2n} + (1 - \theta) a_{2n-1} \\ - (-p)^{2n} \Big(\lambda(\theta - 1) a_0 + (\theta - 1) a_1 + p(2 - \theta) a_{-1}\Big) \tag{20}$$

where λ is a function defined by $\lambda(\theta) = 1 - \frac{q}{2}\theta^2 + \theta(p - 1 + \frac{q}{2})$.

Also from (17) and (18) we take for $t = 2n + \theta$, $n \in \{0, 1, \ldots\}$, $0 \leqslant \theta \leqslant 1$

$$y(t) = (-p)^{2n+1} y_0(\theta - 1) + (\theta p^2 - p^2 - \theta p) c_1(n) \\ + \Big(p^2 \lambda(\theta - 1) - p\lambda(\theta)\Big) c_2(n) + (p - p\theta - 2p^2 + \theta p^2) c_3(n) \\ + (p + \theta - p\theta) a_{2n+1} + \Big(\lambda(\theta) - p\lambda(\theta - 1)\Big) a_{2n} \\ - (-p)^{2n+1} \Big(\lambda(\theta - 1) a_0 + (\theta - 1) a_1 + p(2 - \theta) a_{-1}\Big). \tag{21}$$

Furthermore putting in (20) $\theta = 0$ and using (7) we get

$$(22)\qquad \begin{aligned} &pc_1(n) + \left((p-1)^2 + pq\right) c_2(n) + (2p-1)c_3(n) \\ &\quad = a_{2n+1} + (p+q-2)a_{2n} + (-p)^{2n}\Big((2-p-q)a_0 - a_1 + (2p-1)a_{-1}\Big). \end{aligned}$$

If $p \neq 1$, relations (19) and (22) imply that

$$(23)\qquad \begin{aligned} c_1(n) &= p\nu a_{2n+1} + p\mu a_{2n} + (-p)^{2n+1}(\mu a_0 + \nu a_1) - (\nu q p^2 + p)c_2(n), \\ c_3(n) &= -\nu a_{2n+1} - \mu a_{2n} + (-p)^{2n}(\mu a_0 + \nu a_1 + a_{-1}) + (\nu q p + 1)c_2(n) \end{aligned}$$

where $\nu = \frac{1}{(p-1)^2}$, $\mu = \frac{p+q-2}{(p-1)^2}$. Then from the relations (20) (resp. (21)), (23) we can show that $y(t)$ satisfies (4) for $t = 2n-1+\theta$, (resp. (5) for $t = 2n+\theta$), $n \in \{0, 1, \ldots\}$, $0 \leqslant \theta \leqslant 1$.

Suppose now $p = 1$. By adding (12) and (13) we can take

$$(24)\qquad a_{2n+1} - a_1 = \sum_{k=1}^{n} (a_{2k+1} - a_{2k-1}) = \sum_{k=1}^{n} \Big((1-q)a_{2k} - a_{2k-2}\Big).$$

So from (18) and (24) it follows that

$$(25).\qquad qc_2(n) = (1-q)a_0 - a_1 + a_{2n+1} + (q-1)a_{2n}$$

Then, using (20) (resp. (21)), (19) and (25) we can easily prove that $y(t)$ satisfies (4) for $t = 2n - 1 + \theta$ (resp. (5) for $t = 2n + \theta$), $n \in \{0, 1, \ldots\}$, $0 \leqslant \theta \leqslant 1$ in the case $p = 1$. Therefore we proved that if $y(t)$ is a solution of (1) which satisfies (3) then $y(t)$ is defined by (4) and (5).

Conversely let y be a function which satisfies (4) and (5). We can prove that y is a continuous function which satisfies (1) and (3). Therefore y is the unique solution of (1) which satisfies (3). This completes the proof of the proposition. □

In the following proposition of this paper we study the asymptotic stability of (1).

Proposition 2. *Consider equation* (1), *where* $q \neq -2$. *Then,* (1) *is asymptotically stable if and only if the condition*

$$(26)\qquad 0 < p < 1, \quad 0 < q < 2p^2 + 2$$

is satisfied.

P r o o f. Suppose first that (1) is asymptotically stable. Then it is obvious that the difference equation (6) is also asymtotically stable. We can find that the characteristic equation of the coefficient matrix of (6) is the equation

$$(27)\qquad x^3 + \kappa_1 x^2 + \kappa_2 x + \kappa_3 = 0$$

where

$$\kappa_1 = \frac{6q - 2p^2 - 4pq - 4}{2+q}, \quad \kappa_2 = \frac{4p^2 - 4pq + q + 2}{2+q}, \quad \kappa_3 = \frac{-2p^2}{2+q}.$$

Since (6) is asymptotically stable, we have that every root of (27) is of modulus less that 1. Then from Lemma 4 [11,p.467], the following conditions are satisfied

$$\begin{aligned} &\text{(i)}\ \ (q+2)q(1-p) > 0, \quad \text{(ii)}\ \ (q+2)(2p^2 - q + 2) > 0, \quad \text{(iii)}\ \ q(q+2) > 0, \\ &\text{(iv)}\ \ (q+2)(-2p^2 + 2pq + 2 - q) > 0, \quad \text{(v)}\ \ pq(-4p + q + 2 + 2p^2) > 0. \end{aligned} \tag{28}$$

From (i) and (iii) of (28) we take $p < 1$ and $q > 0$ or $q < -2$.

If $q > 0$ from (ii) of (28) we take $q < 2p^2 + 2$ and from (v) of (28) it holds $p > 0$. Thus (26) is satisfied.

If $q < -2$ from (ii) of (28) we have $2p^2 + 2 < q$ which is a contradiction. Hence we proved that if (1) is asymptotically stable then (26) is satisfied.

Conversely, suppose that (26) is satisfied. Then, we can easily prove that all the conditions of (28) hold. So, from Lemma 4 [11,p.467], we have that equation (6) is asymptotically stable. Hence there exist constants $K > 0$, $0 < p < 1$ such that

$$|a_{2n}| \leqslant K p^n, \quad n \in \{0, 1, \ldots\}. \tag{29}$$

Therefore, from (4), (5) and (29) we can easily prove that (1) is asymptotically stable. This completes the proof of the proposition. □

In the following proposition the oscillatory behavior of the solutions of (1) is obtained. We need the following definition:

Consider the difference equation

$$\chi_{n+1} = A\chi_n, \quad n \in \{0, 1, \ldots\}$$

where A is a $r \times r$ matrix. We say that a solution of the difference equation oscillates if and only if any component of the solution is not eventually of fixed sign.

Proposition 3. *Consider equation* (1), *where* $q \neq -2$. *Then, every solution of* (1) *oscillates if one of the following conditions is true*

$$p < \frac{1}{4}, \quad q < \min\left\{\frac{-4p^2 - 2}{1 - 4p}, -2\right\}, \tag{30}$$

$$p = 0 \quad \text{and} \quad q > 0, \tag{31}$$

$$p = 1 \quad \text{and} \quad q > 0. \tag{32}$$

P r o o f. From Lemma 1 [3, p. 52] every solution of the difference equation (6) oscillates if the characteristic equation (27) of the coefficient matrix of (6) has no positive roots. We can prove that if (30) is satisfied then $\kappa_i \geqslant 0$, $i = 1, 2, 3$, κ_i are defined in (27). This implies that equation (27) has no positive roots and so every solution of the difference equation (6) oscillates. Then it is obvious that every solution of (1) oscillates if the condition (30) is satisfied.

Suppose that the condition (31) holds. Then using Lemma 1 [3, p. 52] and Corrolary 1 [11, p. 462] we can show that every solution of the difference equation (6) oscillates. This implies that every solution of (1) oscillates.

Let now the condition (32) is satisfied. Then from (12), (13), (14), (15) we take

$$a_{2n+4} + \frac{3q-4}{2+q} a_{2n+2} + \frac{2}{2+q} a_{2n} = 0. \tag{33}$$

Using Corrolary 1 [11, p. 462], we can prove that every solution of (33) oscillates. Therefore every solution of (1) oscillates. Thus the proof of the proposition is completed.

II. The third order equation

We consider now the equation of the form (2).

In the following proposition we give a result concerning existence and uniqueness of the solutions of (2).

Proposition 4. *Consider equation* (2) *where* $q \neq -6$. *Let* $y_0 \colon [-1, 0] \mapsto \mathbb{R}$ *be a continuous function on* $[-1, 0]$ *and* $a_0, a_1, a_2 \in \mathbb{R}$. *Then, if* $p \neq 0$ *(resp.* $p = 0$*) equation* (2) *has a unique solution* $y(t)$ *which satisfies* (3) *and*

$$y(2) = a_2.$$

Moreover for $t = 2n - 1 + \theta$, $n \in \{0, 1, \ldots\}$, $0 \leqslant \theta \leqslant 1$ the function $y(t)$ is given by

$$\begin{aligned} y(t) &= (-p)^{2n} y_0(\theta - 1) + \frac{\theta(\theta-1)}{2} a_{2n+1} \\ &+ \left(\theta(2-\theta) + (\theta - \theta^2)\Big(\frac{pq}{12} - \frac{q}{3} + \frac{\theta q}{6}\Big)\right) a_{2n} + \frac{(\theta-1)(\theta-2)}{2} a_{2n-1} \\ &+ (-p)^{2n}\Big(\lambda_1(\theta) a_1 + \lambda_2(\theta) a_0 + \lambda_3(\theta) a_{-1}\Big) + \lambda_4(\theta) \sum_{k=0}^{n-1} (-p)^{2n-2k-1} a_{2k} \end{aligned}$$

and if $t = 2n + \theta$, $n \in \{0, 1, \ldots\}$, $0 \leqslant \theta \leqslant 1$

$$y(t) = (-p)^{2n+1} y_0(\theta - 1) + \frac{(\theta^2 - \theta)(q + 6)}{12} a_{2n+2} + \theta(2 - \theta) a_{2n+1}$$
$$+ \left(\frac{(\theta - 1)(\theta - 2)}{2} + (\theta^2 - \theta)(p - 1)\Big(\frac{\theta q}{6} - \frac{q}{4} + \frac{pq}{12}\Big) \right) a_{2n}$$
$$+ (-p)^{2n+1} \Big(\lambda_1(\theta) a_1 + \lambda_2(\theta) a_0 + \lambda_3(\theta) a_{-1} \Big) + \lambda_4(\theta) \sum_{k=0}^{n-1} (-p)^{2n-2k} a_{2k}$$

where

$$\lambda_1(\theta) = \frac{\theta(1 - \theta)}{2}, \quad \lambda_2(\theta) = \theta(\theta - 2) + (\theta^2 - \theta)\Big(\frac{pq}{12} - \frac{q}{3} + \frac{\theta q}{6}\Big),$$
$$\lambda_3(\theta) = \frac{(1 - \theta)(\theta - 2)}{2}, \quad \lambda_4(\theta) = \frac{q(\theta^2 - \theta)(p - 1)(2\theta + p - 3)}{12},$$

$a_{-1} = y_0(-1)$, $a_0 = y_0(0)$ and a_{2n+2}, a_{2n+1}, a_{2n}, a_{2n-1} are given by the difference equation

$$(34) \qquad \begin{pmatrix} a_{2n+4} \\ a_{2n+3} \\ a_{2n+2} \\ a_{2n+1} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{2n+2} \\ a_{2n+1} \\ a_{2n} \\ a_{2n-1} \end{pmatrix},$$

$$c_{11} = \frac{6p^2 - 18p - 20q + 5pq + 36}{q + 6}, \quad c_{12} = \frac{-18p^2 + 54p - 48}{q + 6},$$
$$c_{13} = \frac{18p^2 - 54p - 3q + pq + 18}{q + 6}, \quad c_{14} = \frac{p(18 - 6p)}{q + 6}.$$

Using the same argument to prove Proposition 1 we can prove the above proposition. □

In the following propsition of this paper we prove that equation (2) is not asymptotically stable.

Proposition 5. *Consider equation* (2) *where* $q \neq -6$. *Then* (2) *is not asymptotically stable.*

Proof. Suppose that (2) is asymptotically stable. Then it is obvious that the difference equation (34) is also asymptotically stable. We can easily find that the characteristic equation of the coefficient matrix of (34) is the equation

$$(35) \qquad x^4 + \mu_1 x^3 + \mu_2 x^2 + \mu_3 x + \mu_4 = 0$$

where

$$\mu_1 = \frac{-6p^2 + 23q - 8pq - 18}{q+6}, \quad \mu_2 = \frac{18p^2 + 23q - 32pq + 18}{q+6},$$
$$\mu_3 = \frac{-18p^2 - 8pq + q - 6}{q+6}, \quad \mu_4 = \frac{6p^2}{q+6}.$$

Equation (34) is asymptotically stable if and only if every root of (35) is of modulus less that 1. Therefore from Lemma 1 [9] the following conditions are satisfied

$$\text{(i) } \alpha_1 \neq 0, \quad \text{(ii) } \frac{\alpha_2}{\alpha_1} > 0, \quad \text{(iii) } \frac{\alpha_3}{\alpha_1} > 0, \quad \text{(iv) } \frac{\alpha_5}{\alpha_1} > 0,$$
$$\text{(v) } \frac{\alpha_4}{\alpha_1}\left(\frac{\alpha_2\alpha_3}{\alpha_1^2} - \frac{\alpha_4}{\alpha_1}\right) > \frac{\alpha_2^2\alpha_5}{\alpha_1^3} \tag{36}$$

where

$$\alpha_1 = \frac{48q(1-p)}{q+6}, \quad \alpha_2 = \frac{48q}{q+6}, \quad \alpha_3 = \frac{8q(8p-5)}{q+6},$$
$$\alpha_4 = \frac{8(-6p^2 - 5q + 6)}{q+6}, \quad \alpha_5 = \frac{16(3p^2 - pq + 3)}{q+6}.$$

Using conditions (i), (ii), (iii) of (36) we can easily get

$$\frac{5}{8} < p < 1. \tag{37}$$

Condition (iv) of (36) and relation (37) imply that

$$0 < q < \frac{3p^2 + 3}{p}. \tag{38}$$

Finally from the condition (v) of (36) we take

$$pq^2 + 2q\Big(6(p^2+1) + (p^2-1)(5-2p)\Big) + 12(p^2-1)^2(1-p) < 0. \tag{39}$$

Let

$$D = \Big(6(p^2+1) + (p^2-1)(5-2p)\Big)^2 - 12p(p^2-1)^2(1-p). \tag{40}$$

After some simple computations in (40) we obtain

$$D = 12(p^2+1)(8p^2-2) + 24p(p^2+1)(1-p^2) + (p^2-1)^2(16p^2-32p+25). \tag{41}$$

Relations (37) and (41) imply that $D > 0$. Therefore from (39) it holds

$$\frac{-c-\sqrt{D}}{p} < q < \frac{-c+\sqrt{D}}{p} \tag{42}$$

where $c = 6(p^2+1) + (p^2-1)(5-2p)$. Since from (37) $p < 1$ we can easily show that $c > 0$. Then using (40) we take $\sqrt{D} < c$. So from (37) and (42) it is obvious that $q < 0$ which contradicts (38). This implies that the difference equation (34) is not asymptotically stable and so equation (2) also is not asymptotically stable. This completes the proof of the proposition. □

In the last proposition we give two sufficient conditions for the oscillatory behavior of the solutions of (2).

Proposition 6. *Consider equation* (2) *with* $q \neq -6$. *Then every solution of* (2) *oscillates if one of the following conditions is satisfied:*

$$\text{(i)}\ \ p < \frac{1}{8}, \quad q > \frac{6+18p^2}{1-8p}, \qquad \text{(ii)}\ \ p = 0, \quad q < -6. \tag{43}$$

P r o o f. We can show that if one of the conditions (43) is satisfied then $\mu_i \geqslant 0$, $i = 1, 2, 3, 4$, μ_i are defined in (35). Therefore arguing as in Proposition 3 the proof of the proposition follows immediately. □

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Authors' address: Democritus University of Thrace, School of Engineering, 67100 Xanthi, Greece.