COMPARISON THEOREM FOR THIRD-ORDER DIFFERENTIAL EQUATIONS

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Recently, Ohriska has shown ([7]) that using the *v-transformation* of an equation, we can deduce oscillatory and asymptotic behavior of the solutions of the equation

$$\Big(r(t)\big(r(t)u'(t)\big)'\Big)' + p(t)u\big(g(t)\big) = 0 \tag{1}$$

from that of the equation

$$y'''(t) + p(t)y\big(g(t)\big) = 0. \tag{2}$$

In this paper we have been motivated by the observation that there are very few effective criteria for transfering some asymptotic properties of the equation (1) to the equation

$$\Big(r_2(t)\big(r_1(t)u'(t)\big)'\Big)' + p(t)u\big(g(t)\big) = 0. \tag{3}$$

It is assumed that functions r_1, r_2, $r\colon [t_0,\infty) \to (0,\infty)$, $g\colon [t_0,\infty) \to \mathbb{R}$, and $p\colon [t_0,\infty) \to (-\infty,0)$ are continuous and $g(t) \to \infty$ as $t \to \infty$. We always suppose that

$$R_i(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{r_i(s)} \to \infty \quad \text{as } t \to \infty \text{ for } i = 1,2, \tag{4}$$

$$R(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{r(s)} \to \infty \quad \text{as } t \to \infty. \tag{5}$$

For convenience we formally set $r_0(t) = r_3(t) = 1$, $t \in [t_0,\infty)$ and make use of the following notation:

$$L_0(u;r_0)(t) = u(t),$$
$$L_i(u;r_0,\dots,r_i)(t) = r_i(t)\big[L_{i-1}(u;r_0,\dots,r_{i-1})(t)\big]' \text{ for } i = 1,2,3.$$

We consider only nontrivial solutions of (3). Such a solution is said to be oscillatory if the set of its zeros is unbounded, and nonoscillatory otherwise. In the examination of nonoscillatory solutions of (3) we may restrict our attention to positive solutions of (3). If u is a positive solution of (3) then $L_3(u; r_0, r_1, r_2, r_3)(t) > 0$ and so according to a generalization of a lemma by Kiguradze (see e.g. [6]) we have either

$$L_0(u; r_0)(t) > 0, \quad L_1(u; r_0, r_1)(t) > 0 \quad \text{and} \quad L_2(u; r_0, r_1, r_2)(t) < 0 \tag{6}$$

or

$$L_0(u; r_0)(t) > 0, \quad L_1(u; r_0, r_1)(t) > 0 \quad \text{and} \quad L_2(u; r_0, r_1, r_2)(t) > 0 \tag{7}$$

for all sufficiently large t. A function u satisfying (6) $\big((7)\big)$ is said to be of degree $\ell = 1$ ($\ell = 3$). Denote by N_ℓ the set of all positive solutions of (3) which are of degree ℓ. Then the set N of all positive solutions of (3) has the decomposition $N = N_1 \cup N_3$. We are interested in the extreme situation in which $N = N_3$. When this situation occurs, following Kiguradze [4], we say that equation (3) enjoys property (B).

It is well known that equation (3) has property (B) if

$$\int^\infty |p(s)|\,\mathrm{d}s = \infty,$$

therefore in the sequel we may assume that

$$\int^\infty |p(s)|\,\mathrm{d}s < \infty.$$

To the best of the author's knowledge it is usually the condition

$$r(t) \geqslant \max\{r_1(t), r_2(t)\}, \quad t \geqslant t_0 \tag{8}$$

that is imposed on the functions r, r_1 and r_2 in the comparison theorems which are applicable to equations (1) and (3). The objective of this paper is to establish a comparison result between equations (1) and (3) without imposing the condition (8) on the function r, and on the basis of the desired comparison theorem (Cf. Theorem 1) to obtain sufficient conditions for equation (3) to have property (B).

For other related results the reader is referred to the papers [2], [3] and [8]. In passing, the term "property (A)" refers to equation (3) for which $p(t) > 0$ and it is used to express the situation in which $N = N_0$.

We begin by formulating some preparatory results which are needed in the sequel.

Theorem A. (See Corollary 1 in [6].) *Let* (4) *hold. Equation* (3) *has property* (B) *if and only if so does the differential inequality*

$$\Big\{\Big(r_2(t)\big(r_1(t)u'(t)\big)'\Big)' + p(t)u\big(g(t)\big)\Big\} \operatorname{sgn} u\big(g(t)\big) \geqslant 0.$$

This theorem exhibits an important relationship between the differential equation (3) and the relevant differential inequality.

Theorem B. *Let* (5) *hold. Equation* (1) *has a solution* v *satisfying*

$$\lim_{t\to\infty} v(t) = a_0 \in \mathbb{R} - \{0\}$$

for some a_0 *if and only if*

$$\int^{\infty} |p(s)| R^2(s)\,\mathrm{d}s < \infty.$$

This theorem is a special case of Theorem 1 of Kitamura and Kusano (see [5]).

Lemma 1. *Suppose that* (4) *and* (5) *are satisfied. Let* u *be a positive solution of* (3) *such that* $u \in N_1$. *Assume that*

$$\frac{R_1(t)}{r_2(t)} \geqslant \frac{R(t)}{r(t)} \quad \text{for} \quad t \in [t_0, \infty) \tag{9}$$

and

$$\frac{R}{R_1} \text{ is a nondecreasing function.} \tag{10}$$

Then there exists a constant $c \geqslant 0$ *such that*

$$\begin{aligned} &\int_{t_1}^{t} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \int_{s_2}^{\infty} |p(s_3)| u\big(g(s_3)\big)\,\mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1 \\ &\qquad \geqslant -c + \int_{t_1}^{t} \frac{1}{r(s_1)} \int_{s_1}^{\infty} \frac{1}{r(s_2)} \int_{s_2}^{\infty} |p(s_3)| u\big(g(s_3)\big)\,\mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1, \end{aligned} \tag{11}$$

$t \geqslant t_1$, *provided* t_1 *is large enough.*

Proof. Suppose that u is a positive solution of equation (3) satisfying inequalities (6) for all $t \geqslant t_1$ ($\geqslant t_0$).

Set

$$P(t) = \int_t^{\infty} |p(s)| u\big(g(s)\big)\,\mathrm{d}s,$$

$$P_1(t) = \int_t^{\infty} \frac{1}{r(s)} P(s)\,\mathrm{d}s \quad \text{and} \quad P_2(t) = \int_t^{\infty} \frac{1}{r_2(s)} P(s)\,\mathrm{d}s$$

for $t \geqslant t_1$. Integration by parts yields

$$\int_{t_1}^{t} \frac{P_2(s)}{r_1(s)} \, ds = R_1(t)P_2(t) - R_1(t_1)P_2(t_1) + \int_{t_1}^{t} \frac{R_1(s)}{r_2(s)} P(s) \, ds, \quad t \geqslant t_1.$$

From (9) and (10) it follows that

$$\begin{aligned} R_1(t)P_2(t) &= R_1(t) \int_t^{\infty} \frac{1}{r_2(s)} P(s) \, ds \\ &\geqslant R_1(t) \int_t^{\infty} \frac{R(s)}{R_1(s)} \frac{P(s)}{r(s)} \, ds \geqslant R(t) \int_t^{\infty} \frac{1}{r(s)} P(s) \, ds, \quad t \geqslant t_1. \end{aligned} \tag{12}$$

On the other hand, using (9) we arrive at

$$\int_{t_1}^{t} \frac{R_1(s)}{r_2(s)} P(s) \, ds \geqslant \int_{t_1}^{t} \frac{R(s)}{r(s)} P(s) \, ds, \tag{13}$$

and furthermore there exists a constant $c \geqslant 0$ such that

$$-R_1(t_1)P_2(t_1) \geqslant -c - R(t_1)P_1(t_1). \tag{14}$$

Combining (12), (13) and (14) we get (11). The proof is complete. □

Now, we are prepared to compare equation (3) with equation (1).

Theorem 1. *Suppose that* (4), (5), (9) *and* (10) *are satisfied. Then equation* (3) *has property* (B) *if so does equation* (1).

P r o o f. Let u be a positive solution of (3). Suppose that u satisfies inequalities (6) on $[t_1, \infty)$ (i.e. $u \in N_1$). By integrating (3), with the aid of (6), we may write

$$u(t) \geqslant u(t_1) + \int_{t_1}^{t} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \int_{s_2}^{\infty} |p(s_3)| u\big(g(s_3)\big) \, ds_3 \, ds_2 \, ds_1, \quad t \geqslant t_1.$$

According to Lemma 1, there exists a $c \geqslant 0$ such that

$$u(t) \geqslant u(t_1) - c + \int_{t_1}^{t} \frac{1}{r(s_1)} \int_{s_1}^{\infty} \frac{1}{r(s_2)} \int_{s_2}^{\infty} |p(s_3)| u\big(g(s_3)\big) \, ds_3 \, ds_2 \, ds_1, \quad t \geqslant t_1. \tag{15}$$

We have supposed that equation (1) has property (B) and so (1) cannot have a solution v with the property $\lim_{t \to \infty} v(t) = a_0 \in R - \{0\}$ for any a_0, therefore Theorem B yields that

$$\int^{\infty} |p(s)| R^2(s) \, ds = \infty,$$

which implies that

$$(16)\qquad \int_{t_0}^{t} \frac{1}{r(s_1)} \int_{s_1}^{\infty} \frac{1}{r(s_2)} \int_{s_2}^{\infty} |p(s_3)|\,\mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1 \to \infty \quad \text{as} \quad t \to \infty.$$

Noting that u is increasing ($\ell = 1$), we may assume that $u(g(t)) \geqslant c_1 > 0$ for $t \geqslant t_1$, and hence, taking (16) into account, we see that the right hand side of (15) is unbounded as $t \to \infty$. Consequently, there exists a $t_2 \geqslant t_1$ such that

$$(17)\quad u(t) \geqslant u(t_1) + \int_{t_2}^{t} \frac{1}{r(s_1)} \int_{s_1}^{\infty} \frac{1}{r(s_2)} \int_{s_2}^{\infty} |p(s_3)| u\big(g(s_3)\big)\,\mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1, \quad t \geqslant t_2.$$

Let us denote the right hand side of (17) by $y(t)$. Repeated differentiation of y shows that $L_0(y; r_0)(t) > 0$, $L_1(y; r_0, r)(t) > 0$, $L_2(y; r_0, r, r)(t) < 0$ and $L_3(y; r_0, r, r, r_3)(t) > 0$ for $t \geqslant t_2$ and

$$\Big(r(t)\big(r(t)y'(t)\big)'\Big)' + p(t)u\big(g(t)\big) = 0, \quad t \geqslant t_2.$$

Since $u\big(g(t)\big) \geqslant y\big(g(t)\big)$ for all large t, say $t \geqslant t_3$, we obtain

$$\Big(r(t)\big(r(t)y'(t)\big)'\Big)' + p(t)y\big(g(t)\big) \geqslant 0, \quad t \geqslant t_3.$$

As y is a function of degree $\ell = 1$, using Theorem A applied to equation (1) we obtain from the last inequality that equation (1) cannot enjoy property (B). This is a contradiction, and the proof is complete. □

Now we illustrate an application of the above-mentioned comparison principle.

Theorem C. (Theorem 11 in [1].) *Let* (5) *hold. Assume that g satisfies*

$$(18)\qquad g \in C^1([t_0, \infty)), \quad g(t) \leqslant t \quad \text{and} \quad g'(t) > 0.$$

Then equation (1) *has property* (B) *provided*

$$\limsup_{t\to\infty} R^2\big(g(t)\big) \int_t^{\infty} p(s)\,\mathrm{d}s < -\frac{1}{3\sqrt{3}}.$$

Theorem 2. *Suppose that* (4) *and* (18) *are satisfied. Let*

$$(19)\qquad \limsup_{t\to\infty} \int_{t_0}^{g(t)} \frac{R_1(s)}{r_2(s)}\,\mathrm{d}s \int_t^{\infty} p(s)\,\mathrm{d}s < -\frac{1}{6\sqrt{3}},$$

and let

$$(20)\qquad \big(R_1(t)\big)^{-2}\int_{t_0}^{t}\frac{R_1(s)}{r_2(s)}\,\mathrm{d}s \quad \textit{be nondecreasing.}$$

Then equation (3) *has property* (B).

P r o o f. We consider equation (1) with the function r defined by the relation:

$$(21)\qquad \frac{R(t)}{r(t)}=\frac{R_1(t)}{r_2(t)},\quad t\geqslant t_0.$$

Integrating (21) and extracting the square root of the resulting equality, we arrive at

$$(22)\qquad R(t)=\sqrt{2}\Big(\int_{t_0}^{t}\frac{R_1(s)}{r_2(s)}\,\mathrm{d}s\Big)^{\frac{1}{2}}\to\infty \text{ as } t\to\infty,$$

where we have used (4). Hence, function R/R_1 is nondecreasing if and only if (20) holds. From (22) we conclude that condition (19) is equivalent to the condition

$$(23)\qquad \limsup_{t\to\infty} R^2\big(g(t)\big)\int_t^{\infty}p(s)\,\mathrm{d}s<-\frac{1}{3\sqrt{3}},$$

which, as we see from Theorem C, guarantees together with (18) that equation (1) has property (B). Consequently, Theorem 1 applied to equations (1) and (3) ensures that equation (3) has property (B). The proof is complete. □

Corollary 1. *Assume that the hypotheses of Theorem* 2 *hold except that the relation* (20) *is replaced by one of the following conditions:*

$$(24)\qquad \frac{2}{r_1(t)}\int_{t_0}^{t}\frac{R_1(s)}{r_2(s)}\,\mathrm{d}s\leqslant\frac{\big(R_1(t)\big)^2}{r_2(t)},\quad t\geqslant t_0,$$

or

$$(25)\qquad \frac{r_1}{r_2} \textit{ is nondecreasing.}$$

Then equation (3) *has property* (B).

P r o o f. The function $\big(R_1(t)\big)^{-2}\int_{t_0}^{t}\frac{R_1(s)}{r_2(s)}\,\mathrm{d}s$ is nondecreasing if its first derivative is nonnegative, which occurs if (24) holds. Using (25), it is not hard to see that

$$\int_{t_0}^{t}\frac{R_1(s)}{r_2(s)}\,\mathrm{d}s\leqslant\frac{r_1(t)}{r_2(t)}\int_{t_0}^{t}\frac{R_1(s)}{r_1(s)}\,\mathrm{d}s=\frac{r_1(t)}{r_2(t)}\frac{\big(R_1(t)\big)^2}{2},\quad t\geqslant t_0,$$

which implies (24). The proof is complete. □

Example 1. Let us consider the equation

$$\left(t^{\frac{1}{3}}\left(t^{\frac{1}{2}}u'(t)\right)'\right)' - \frac{a}{t^{\frac{13}{6}}}u(bt) = 0, \quad t \geqslant 1, \quad b \in (0,1].$$

By Corollary 1, this equation has property (B) if $a > \frac{49}{432\sqrt{3}b^{\frac{7}{6}}}$. □

By the technique we have used in the proof of Theorem 2 we can "produce" many more sufficient conditions for equation (3) to have property (B). The relation (21) shows how to define the function r to obtain equation (1) for comparing with equation (3). We give another example of the application of Theorem 1.

For the special case of equation (3), namely, for the equation

$$y'''(t) + p(t)y(t) = 0, \tag{26}$$

Chanturia and Kiguradze [3] have obtained the following result.

Theorem D. *Assume that*

$$\liminf_{t\to\infty} t\int_t^\infty s|p(s)|\,\mathrm{d}s > \frac{2\sqrt{3}}{9}$$

or

$$\limsup_{t\to\infty} t\int_t^\infty s|p(s)|\,\mathrm{d}s > 2.$$

Then equation (26) *has property* (B).

We extend their result to equation (3).

Theorem 3. *Assume that* (4) *and* (20) *hold and* $g(t) = t$. *Further suppose that*

$$\liminf_{t\to\infty}\left(\int_{t_0}^t \frac{R_1(s)}{r_2(s)}\,\mathrm{d}s\right)^{\frac{1}{2}}\int_t^\infty |p(s)|\left(\int_{t_0}^s \frac{R_1(x)}{r_2(x)}\,\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}s > \frac{\sqrt{3}}{9} \tag{27}$$

or

$$\limsup_{t\to\infty}\left(\int_{t_0}^t \frac{R_1(s)}{r_2(s)}\,\mathrm{d}s\right)^{\frac{1}{2}}\int_t^\infty |p(s)|\left(\int_{t_0}^s \frac{R_1(x)}{r_2(x)}\,\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}s > 1. \tag{28}$$

Then equation (3) *has property* (B).

Proof. Let us consider the equation

$$\left(r(t)\left(r(t)u'(t)\right)'\right)' + p(t)u(t) = 0, \tag{29}$$

where the function r is given by the relation (21). The theory of the *v-transformation* of an equation (see [7]) provides the result that equation (29) has property (B) if and only if so does the equation

$$y'''(t) + r\left(R^{-1}(t)\right) p\left(R^{-1}(t)\right) y(t) = 0, \tag{30}$$

where R^{-1} is the inverse function to R, which is defined by (5). On the other hand, Theorem D guarantees that equation (30) has property B if

$$\liminf_{t\to\infty} t \int_t^\infty s\, r\left(R^{-1}(s)\right) \left|p\left(R^{-1}(s)\right)\right| \mathrm{d}s > \frac{2\sqrt{3}}{9}$$

or

$$\limsup_{t\to\infty} t \int_t^\infty s\, r\left(R^{-1}(s)\right) \left|p\left(R^{-1}(s)\right)\right| \mathrm{d}s > 2,$$

which in view of (22) is equivalent to (27) or (28), respectively. Hence, equation (30) as well as equation (29) have property (B). Applying Theorem 1 to equations (29) and (3) we see that the assertion of this theorem holds true. □

The following considerations are aimed at extending the previous result to the equations with deviating arguments.

Theorem E. *Assume that* (4) *and* (18) *are satisfied. Then equation* (3) *has property* (B) *if so does the equation*

$$\left(r_2(t)\left(r_1(t)u'(t)\right)'\right)' + \frac{p\left(g^{-1}(t)\right)}{g'\left(g^{-1}(t)\right)} u(t) = 0.$$

For the proof of Theorem E see e.g. [1] or [6]. From Theorem E and Theorem 3 we have

Theorem 4. *Assume that* (4), (18) *and* (20) *hold. Further suppose that*

$$\liminf_{t\to\infty} \left(\int_{t_0}^{g(t)} \frac{R_1(s)}{r_2(s)} \mathrm{d}s\right)^{\frac{1}{2}} \int_t^\infty |p(s)| \left(\int_{t_0}^{g(s)} \frac{R_1(x)}{r_2(x)} \mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}s > \frac{\sqrt{3}}{9}$$

or

$$\limsup_{t\to\infty} \left(\int_{t_0}^{g(t)} \frac{R_1(s)}{r_2(s)} \mathrm{d}s\right)^{\frac{1}{2}} \int_t^\infty |p(s)| \left(\int_{t_0}^{g(s)} \frac{R_1(x)}{r_2(x)} \mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}s > 1.$$

Then equation (3) *has property* (B).

Taking Corollary 1 into account we see that if we replace the condition (20) either by the condition (24) or by (25) then the conclusions of Theorem 3 and 4 remain valid.

Now we return to Theorem 1. As a matter of fact we are able to establish a comparison theorem more general than Theorem 1. For our next considerations we need continuous functions z and w such that $w\colon [t_0,\infty) \to \mathbb{R}$, $z\colon [t_0,\infty) \to (-\infty,0)$, and $w(t) \to \infty$ as $t \to \infty$.

Theorem 5. *Assume that* (4), (5), (9) *and* (10) *are satisfied. Further, suppose that*

$$\begin{aligned} g(t) &\geqslant w(t), \quad t \geqslant t_0, \\ |p(t)| &\geqslant |z(t)|, \quad t \geqslant t_0. \end{aligned}$$

If the equation

$$\Big(r(t)\big(r(t)u'(t)\big)'\Big)' + z(t)u\big(w(t)\big) = 0$$

has property (B) *then so does equation* (3).

P r o o f. From Theorem 1 one gets that the equation

$$\Big(r_2(t)\big(r_1(t)u'(t)\big)'\Big)' + z(t)u\big(w(t)\big) = 0 \tag{32}$$

has property (B). An application of Theorem 1 from [6] to equations (32) and (3) then yields that (3) has property (B). The proof is complete. □

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