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ON THE EXISTENCE OF SOLUTIONS AND ONE-STEP METHOD FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH PARAMETERS

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INTRODUCTION

For an integer $i \geqslant 0$, let $C^i(I, \mathbb{R}^p)$ denote the collection of functions with continuous derivatives up to the order i on $I = [a,b]$, $a < b$, into $\mathbb{R}^p$ and put $C(I, \mathbb{R}^p) = C^0(I, \mathbb{R}^p)$. We are concerned with the question of solving boundary value problems for Volterra neutral functional differential equations of the form

$$x'(t) = f\big(t, x(\cdot), x'(\cdot), \lambda\big), \quad t \in I, \tag{1}$$

$$x(a) = x_p, \quad L\big(x(\cdot), \lambda\big) = \theta \in \mathbb{R}^q, \tag{2}$$

where $f\colon I \times C^1(I, \mathbb{R}^p) \times C(I, \mathbb{R}^p) \times \mathbb{R}^q \to \mathbb{R}^p$, $L\colon C^1(I, \mathbb{R}^p) \times \mathbb{R}^q \to \mathbb{R}^q$, $x_p \in \mathbb{R}^p$ are given. It is assumed that the mapping $t \to f\big(t, x(\cdot), x'(\cdot), \lambda\big)$ is continuous on I for any $x \in C^1(I, \mathbb{R}^p)$ and $\lambda \in \mathbb{R}^q$. We seek $x \in C^1(I, \mathbb{R}^p)$ and $\lambda \in \mathbb{R}^q$ such that (1)–(2) to be satisfied. We mean that the problem (1)–(2) is solved if such x and λ are found.

Indeed special cases of (1) are following equations:

$$x'(t) = f\big(t, x(t), x'(t), \lambda\big), \quad t \in I,$$

or

$$x'(t) = f\Big(t, x\big(\alpha_1(t)\big), \ldots, x\big(\alpha_r(t)\big), x'\big(\beta_1(t)\big), \ldots, x'\big(\beta_s(t)\big), \lambda\Big), \quad t \in I,$$

where α_i and β_j are continuous functions such that $a \leqslant \alpha_i(t) \leqslant t$, $a \leqslant \beta_j(t) \leqslant \beta_j t$, $0 \leqslant \beta_j \leqslant 1$ for $t \in I$ and $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, s$, or

$$x'(t) = f\Big(t, x(t), x'(t), \int_a^{\alpha(t)} g\big(t, \tau, x(\tau), x'(\tau)\big)\, \mathrm{d}\tau, \lambda\Big), \quad t \in I,$$

where $a \leqslant \alpha(t) \leqslant t$, or directly

$$x'(t) = f\big(t, x(\cdot), \lambda\big), \quad t \in I,$$

if the function f is independent on the derivative x' (for example see [6, 7, 10]).

By the substitution $y(t) = x'(t)$, $t \in I$, the problem (1)–(2) is equivalent to the following

$$y(t) = f\Big(t, x_p + \int_a^{\cdot} y(\tau)\,\mathrm{d}\tau, y(\cdot), \lambda\Big), \quad t \in I, \tag{3}$$

$$L\Big(x_p + \int_a^{\cdot} y(\tau)\,\mathrm{d}\tau, \lambda\Big) = \theta. \tag{4}$$

To show the problem (3)–(4) has a solution $(y, \lambda) \in C(I, \mathbb{R}^p) \times \mathbb{R}^q$ we introduce two sequences $\{y_n\}$ and $\{\lambda_n\}$ by formulas

$$\begin{cases} y_0(t) = x_p, \quad t \in I, \\ y_{n+1}(t) = f\Big(t, x_p + \displaystyle\int_a^{\cdot} y_n(\tau)\,\mathrm{d}\tau, y_n(\cdot), \lambda_n\Big) = F(t, y_n, \lambda_n), \quad n = 0, 1, \ldots, \end{cases} \tag{5}$$

and

$$\begin{cases} \lambda_0 \text{ is an arbitrary vector in } \mathbb{R}^q, \\ \lambda_{n+1} = \lambda_n - B^{-1} L\Big(x_p + \displaystyle\int_a^{\cdot} y_{n+1}(\tau)\,\mathrm{d}\tau, \lambda_n\Big), \quad n = 0, 1, \ldots, \end{cases} \tag{6}$$

where a nonsingular square matrix B of order q will be defined later. The general sufficient conditions by which the sequences $\{y_n\}$, $\{\lambda_n\}$ have the limits $\bar{y}$ and $\bar{\lambda}$, respectively, and that $(\bar{y}, \bar{\lambda})$ is a solution of (3)–(4), are given in the first part.

To solve the problem (1)–(2) numerically we apply the one-step methods for finding y combined with the Newton method for finding λ. Due to this fact we divide the interval I into N subintervals all of the same length $h = (b - a)/N$. The points t_{hi} of division are defined by $t_{hi} = a + ih$, $i = 0, 1, \ldots, N$. Now we can describe our method by

$$\begin{cases} y_h^j(t_{hn} + rh) = y_h^j(t_{hn}) + h\Phi\big(t_{hn}, y_h^j(\cdot), z_h^j(\cdot), \lambda_{hj}, r, h\big), \\ \hfill r \in (0, 1],\ n = 0, 1, \ldots, N - 1, \\ z_h^j(t_{hn} + rh) = \Psi(t_{hn}, y_h^j(\cdot), z_h^j(\cdot), \lambda_{hj}, r, h), \quad r \in (0, 1],\ n = 0, 1, \ldots, N - 1, \\ \lambda_{h,j+1} = \lambda_{hj} - B^{-1} L\big(y_h^j(\cdot), \lambda_{hj}\big), \end{cases} \tag{7}$$

for $j = 0, 1, \ldots$. Here usually $y_h^j(a) = x_p + \xi_{1j}(h)$, $\lambda_{h0} = \lambda_0 \in \mathbb{R}^q$ is given and $z_h^j(a) = \tilde{x}_{pj} + \xi_{2j}(h)$ where $\tilde{x}_{pj}$ is a solution of the equation

$$\tilde{x}_{pj} = f(a, x_p, \tilde{x}_{pj}, \lambda_{hj}),$$

and

$$\lim_{\substack{h\to 0\\ j\to\infty}} \xi_{1j}(h) = \lim_{\substack{h\to 0\\ j\to\infty}} \xi_{2j}(h) = 0.$$

In the second part of this paper we study the convergence of (y_h^j, λ_{hj}) to the solution (φ, λ) of (1)–(2).

This paper is an extension of some results obtained in [6, 7, 8].

PART 1

We introduce the following

Assumption $\mathbf{H_1}$. Suppose that

1° $f\colon I\times C^1(I,\mathbb{R}^p)\times C(I,\mathbb{R}^p)\times\mathbb{R}^q\to\mathbb{R}^p$, $L\colon C^1(I,\mathbb{R}^p)\times\mathbb{R}^q\to\mathbb{R}^q$, and for any $x\in C^1(I,\mathbb{R}^p)$, $\lambda\in\mathbb{R}^q$ the mapping $t\to f\big(t,x(\cdot),x'(\cdot),\lambda\big)$ is continuous on I,

2° there exist a constant $\beta\in[0,1]$ and nondecreasing functions $K_1,K_2,K_3\in C(I,\mathbb{R}_+)$, $\mathbb{R}_+=[0,\infty)$ such that

$$\begin{aligned}&\big\|f\big(t,x_1(\cdot),x_2(\cdot),\mu_1\big)-f\big(t,\bar x_1(\cdot),\bar x_2(\cdot),\mu_2\big)\big\|\\ &\leqslant K_1(t)\sup_{[a,t]}\|x_1(s)-\bar x_1(s)\|+K_2(t)\sup_{[a,\beta t]}\|x_2(s)-\bar x_2(s)\|+K_3(t)\|\mu_1-\mu_2\|,\end{aligned}$$

for $t\in I$, $x_1,\bar x_1\in C^1(I,\mathbb{R}^p)$, $x_2,\bar x_2\in C(I,\mathbb{R}^p)$ and $\mu_1,\mu_2\in\mathbb{R}^q$,

3° there exist a nonsingular square matrix B of order q and constants $m\geqslant 0$, $d>0$ such that $md<1$, $d>\|B^{-1}\|$ and

$$\begin{aligned}\Big\|L\Big(x_p+\int_a^{\cdot} F(t,x,\mu_1)\,\mathrm{d}t,\mu_1\Big)-L\Big(x_p+\int_a^{\cdot} F(t,x,\mu_2)\,\mathrm{d}t,\mu_2\Big)-B(\mu_1-\mu_2)\Big\|\\ \leqslant m\|\mu_1-\mu_2\|,\end{aligned}$$

for $x\in C(I,\mathbb{R}^p)$, $\mu_1,\mu_2\in\mathbb{R}^q$, where the matrix norm is consistent with the vector norm,

4° for any $x_1,x_2\in C^1(I,\mathbb{R}^p)$ and $\mu\in\mathbb{R}^q$ we have the inequality

$$\big\|L\big(x_1(\cdot),\mu\big)-L\big(x_2(\cdot),\mu\big)\big\|\leqslant L_1\sup_{s\in I}\|x_1(s)-x_2(s)\|,$$

where $L_1\geqslant 0$.

Assumption $\mathbf{H_2}$. Suppose that

1° there exists a nondecreasing solution $w^*\in C(I,\mathbb{R}_+)$ of the inequality

$$Gw(t)+K_3(t)\,\mathrm{d}L_1(1-md)^{-1}\int_a^b Gw(s)\,\mathrm{d}s+v(t)\leqslant w(t),\quad t\in I, \tag{8}$$

where

$$
\begin{aligned}
Gw(t) &= K_1(t)\int_a^t w(s)\,\mathrm{d}s + K_2(t)w(\beta t),\\
v_1(t) &= \sup_{s\in[a,t]}\Big\|f\Big(s, x_p + \int_a^{\cdot} y_0(\tau)\,\mathrm{d}\tau, y_0(\cdot), \lambda_0\Big) - y_0(s)\Big\|,\\
v^* &= \sup_{s\in I}\Big\|L\Big(x_p + \int_a^s F(t, y_0, \lambda_0)\,\mathrm{d}t, \lambda_0\Big)\Big\|,\\
v(t) &= v_1(t) + K_3(t)\,\mathrm{d}v^*(1-md)^{-1},
\end{aligned}
$$

2° in the class of functions $w \in M(I, \mathbb{R}_+)$ satisfying the condition $0 \leqslant w(t) \leqslant w^*(t)$, $t \in I$, the function $w(t) = 0$, $t \in I$ is the only solution of the equation

$$
Gw(t) + K_3(t)\,\mathrm{d}L_1(1-md)^{-1}\int_a^b Gw(s)\,\mathrm{d}s = w(t), \quad t \in I, \tag{9}
$$

where $M(I, \mathbb{R}_+)$ denotes the class of measurable and bounded functions defined in I with a range in $\mathbb{R}_+$.

Remark 1. Instead of (8) and (9) we can take (10) and (11), respectively, where

$$
Gw(t) + K(t)\int_a^b w(s)\,\mathrm{d}s + V(t) \leqslant w(t), \quad t \in I, \tag{10}
$$

$$
Gw(t) + K(t)\int_a^b w(s)\,\mathrm{d}s = w(t), \quad t \in I, \tag{11}
$$

and

$$
\begin{aligned}
r &= 1 - md + \mathrm{d}L_1\int_a^b K_3(s)\,\mathrm{d}s,\\
K(t) &= K_3(t)\,\mathrm{d}L_1/r,\\
V(t) &= v(t) - K(t)\int_a^b v(s)\,\mathrm{d}s\,.
\end{aligned}
$$

Now we are in a position to establish the existence of the solution of (3)–(4). We have

Theorem 1. *If Assumptions* H_1 *and* H_2 *are satisfied then there exists a solution* $(\bar{\lambda}, \bar{y}) \in \mathbb{R}^q \times C(I, \mathbb{R}^p)$ *of the problem* (3)–(4). *This solution is the limit of the* *sequences* $\{\lambda_n\}$, $\{y_n\}$ *and the following estimations*

$$\|\bar{\lambda} - \lambda_n\| \leqslant u_n, \quad n = 0, 1, \dots, \tag{12}$$

$$\sup_{[a,t]} \|\bar{y}(s) - y_n(s)\| \leqslant w_n(t), \quad t \in I, \ n = 0, 1, \dots, \tag{13}$$

hold with

$$\begin{cases} u_0 = u^* = d(1 - md)^{-1} \Big[L_1 \int_a^b G w^*(s) \, \mathrm{d}s + v^* \Big], \\ u_{n+1} = d \Big[m u_n + L_1 \int_a^b G w_n(s) \, \mathrm{d}s \Big], \quad n = 0, 1, \dots, \end{cases}$$

and

$$\begin{cases} w_0(t) = w^*(t), \quad t \in I, \\ w_{n+1}(t) = G w_n(t) + K_3(t) u_n, \quad t \in I, \ n = 0, 1, \dots . \end{cases}$$

Moreover, this solution $(\bar{\lambda}, \bar{y})$ *is unique in the class satisfying the conditions*

$$\|\bar{\lambda} - \lambda_0\| \leqslant u^*, \qquad \sup_{[a,t]} \|\bar{y}(s) - y_0(s)\| \leqslant w^*(t), \quad t \in I. \tag{14}$$

Proof. Using the following relations

$$\begin{aligned} \|y_{n+1}(t) - y_0(t)\| &\leqslant \|F(t, y_n, \lambda_n) - F(t, y_0, \lambda_0)\| + \|F(t, y_0, \lambda_0) - y_0(t)\| \\ &\leqslant K_1(t) \int_a^t \|y_n(\tau) - y_0(\tau)\| \, \mathrm{d}\tau + K_2(t) \sup_{[a,\beta t]} \|y_n(s) - y_0(s)\| \\ &\quad + K_3(t) \|\lambda_n - \lambda_0\| + v_1(t), \end{aligned}$$

and

$$\begin{aligned} &\|\lambda_{n+1} - \lambda_0\| \\ &= \Big\| B^{-1} \Big[L\Big(x_p + \int_a^{\cdot} F(t, y_n, \lambda_0) \, \mathrm{d}t, \lambda_0\Big) - L\Big(x_p + \int_a^{\cdot} F(t, y_n, \lambda_n) \, \mathrm{d}t, \lambda_n\Big) \\ &\quad - B(\lambda_0 - \lambda_n) - L\Big(x_p + \int_a^{\cdot} F(t, y_n, \lambda_0) \, \mathrm{d}t, \lambda_0\Big) \\ &\quad + L\Big(x_p + \int_a^{\cdot} F(t, y_0, \lambda_0) \, \mathrm{d}t, \lambda_0\Big) - L\Big(x_p + \int_a^{\cdot} F(t, y_0, \lambda_0) \, \mathrm{d}t, \lambda_0\Big) \Big] \Big\| \\ &\leqslant d \Big\{ m \|\lambda_n - \lambda_0\| + L_1 \int_a^b \Big[K_1(t) \int_a^t \|y_n(\tau) - y_0(\tau)\| \, \mathrm{d}\tau \\ &\quad + K_2(t) \sup_{[a,\beta t]} \|y_n(s) - y_0(s)\| \Big] \, \mathrm{d}t + v^* \Big\}, \end{aligned}$$

we can prove

$$\|\lambda_n - \lambda_0\| \leqslant u^*, \quad n = 0, 1, \dots,$$
$$\sup_{[a,t]} \|y_n(s) - y_0(s)\| \leqslant w^*(t), \quad t \in I, \ n = 0, 1, \dots,$$

by induction. Similarly we obtain

$$\|\lambda_{n+j} - \lambda_n\| \leqslant u_n, \quad n = 0, 1, \dots, \tag{15}$$
$$\sup_{[a,t]} \|y_{n+j}(s) - y_n(s)\| \leqslant w_n(t), \quad t \in I, \ n = 0, 1, \dots, \tag{16}$$

by induction.

Indeed the sequences $\{u_n\}$, $\{w_n\}$ are nondecreasing and bounded on I, so they are convergent. In view of Assumption H_2 we have

$$u_n \to 0, \qquad w_n(t) \rightrightarrows 0, \quad t \in I,$$

where the sign $\rightrightarrows$ denotes the uniform convergence on I. Hence $\lambda_n \to \bar{\lambda}$, $y_n(t) \rightrightarrows \bar{y}(t)$, $t \in I$, $\bar{y} \in C(I, \mathbb{R}^p)$ and $(\bar{\lambda}, \bar{y})$ is a solution of (3)–(4). There is no problem to prove that this solution is unique in the class satisfying the relations (14). The estimations (12)–(13) follow from (15)–(16) if $j \to \infty$. Now the proof of Theorem 1 is completed. □

We give the conditions by which the problem (3)–(4) has at most one solution. They do not guarantee the existence of the solution. We have

Theorem 2. *If Assumption* H_1 *is satisfied and in the class* $u \in M(I, \mathbb{R}_+)$, *the function* $u(t) = 0$, $t \in I$ *is the only solution of the inequality*

$$u(t) \leqslant \mathrm{d}L_1(1 - md)^{-1} K_3(t) \int_a^b Gu(s)\,\mathrm{d}s + Gu(t), \quad t \in I, \tag{17}$$

then the problem (3)–(4) *has at most one solution.*

Proof. Assuming that the problem (3)–(4) has two solutions $(\bar{\lambda}_i, \bar{y}_i) \in \mathbb{R}^q \times C(I, \mathbb{R}^p)$ we are able to get the estimations

$$\|\bar{y}_1(t) - \bar{y}_2(t)\| = \|F(t, \bar{y}_1, \bar{\lambda}_1) - F(t, \bar{y}_2, \bar{\lambda}_2)\|$$
$$\leqslant K_1(t) \int_a^t \|\bar{y}_1(\tau) - \bar{y}_2(\tau)\|\,\mathrm{d}\tau + K_2(t) \sup_{[a,\beta t]} \|\bar{y}_1(s) - \bar{y}_2(s)\| + K_3(t)\|\bar{\lambda}_1 - \bar{\lambda}_2\|$$

and

$$\|\bar{\lambda}_1 - \bar{\lambda}_2\| =$$

$$\Big\|B^{-1}\Big[B(\bar{\lambda}_1 - \bar{\lambda}_2) - L\Big(x_p + \int_a^{\cdot} F(t,\bar{y}_1,\bar{\lambda}_1)\,\mathrm{d}t, \bar{\lambda}_1\Big) + L\Big(x_p + \int_a^{\cdot} F(t,\bar{y}_1,\bar{\lambda}_2)\,\mathrm{d}t, \bar{\lambda}_2\Big)$$

$$-L\Big(x_p + \int_a^{\cdot} F(t,\bar{y}_1,\bar{\lambda}_2)\,\mathrm{d}t, \bar{\lambda}_2\Big) + L\Big(x_p + \int_a^{\cdot} F(t,\bar{y}_2,\bar{\lambda}_2)\,\mathrm{d}t, \bar{\lambda}_2\Big)\Big]\Big\|$$

$$\leqslant d\Big\{m\|\bar{\lambda}_1 - \bar{\lambda}_2\| + L_1\int_a^b \Big[K_1(t)\int_a^t \|\bar{y}_1(s) - \bar{y}_2(s)\|\,\mathrm{d}s$$

$$+K_2(t)\sup_{[a,\beta t]}\|\bar{y}_1(s) - \bar{y}_2(s)\|\Big]\,\mathrm{d}t\Big\}.$$

Now combining these inequalities we have the assertion of our theorem. □

Let for $t \in I$

$$K_1(t) = k_1, \qquad K_2(t) = k_2.$$

Now we can cite the following

Lemma 1 (see [7]). *If*

1° $v, K_3 \in C(I, \mathbb{R}_+)$ *and are nondecreasing,*

2° $0 \leqslant \beta \leqslant 1$,

3° $0 \leqslant k_2\beta < 1$,

4° $T_1(t) = \sum\limits_{n=0}^{\infty} k_2^n v(t\beta^n) < \infty$ *and* $T_1 \in C(I, \mathbb{R}_+)$,

5° $T_2(t) = \sum\limits_{n=0}^{\infty} k_2^n K_3(t\beta^n) < \infty$ *and* $T_2 \in C(I, \mathbb{R}_+)$,

6° *there exists a unique nondecreasing solution* $\tilde{u} \in C(I, \mathbb{R}_+)$ *of the equation*

$$u(t) = k_1\sum_{n=0}^{\infty} k_2^n \int_a^{t\beta^n} u(\tau)\,\mathrm{d}\tau + \mathrm{d}L_1(1 - md)^{-1}T_2(t)\int_a^b Gu(\tau)\,\mathrm{d}\tau + T_1(t), \quad t \in I, \tag{18}$$

where $k_1, m, L_1 \geqslant 0$, $d > 0$ *and* $md < 1$, *then*

i° *in the class of functions* $u \in M(I, \mathbb{R}_+)$ *satisfying the condition* $0 \leqslant u(t) \leqslant \tilde{u}(t)$, $t \in I$, *the function* $\tilde{u}$ *is the unique, continuous and nondecreasing solution of the equation* (8),

ii° *in the class of functions* $u \in M(I, \mathbb{R}_+)$ *satisfying the condition* $0 \leqslant u(t) \leqslant \tilde{u}(t)$, $t \in I$, *the function* $u(t) = 0$, $t \in I$ *is the unique solution of the inequality* (17).

Lemma 3 (see [7]). *If the assumptions (1)–(4) of Lemma 1 are satisfied, $a \geqslant 0$ and the function K_3 is such that for some constants $L_2 \geqslant 0$ and $\varrho > 0$ satisfying the condition*

$$\varrho > (1 - k_2\beta)^{-1}\{k_1 + L_2\,\mathrm{d}L_1(1 - md)^{-1}[k_1(b - a) + k_2]\},$$

the inequality

$$K_3(t) \leqslant L_2\big(\exp(\varrho t) - \exp(\varrho a)\big)\big(\exp(\varrho b) - \exp(\varrho a)\big)^{-1}, \quad t \in I,$$

holds then there exists a unique nondecreasing solution $\tilde{u} \in C(I, \mathbb{R}_+)$ of (18).

Part 2

Now we are concerned with the numerical solution of the problem (1)–(2). As it was mentioned earlier we apply the method defined by (7). At first we introduce the following

Assumption H$_3$. Suppose that

1° $\Phi, \Psi\colon I \times C(I, \mathbb{R}^p) \times \tilde{C}(I, \mathbb{R}^p) \times \mathbb{R}^q \times [0,1] \times H \to \mathbb{R}^p$, $L\colon C(I, \mathbb{R}^p) \times \mathbb{R}^q \to \mathbb{R}^q$, $H = [0, h_0]$, $h_0 > 0$ and $\tilde{C}(I, \mathbb{R}^p)$ denotes the space of piecewise continuous functions from I into $\mathbb{R}^p$, $\Phi\big(t, y(\cdot), z(\cdot), \lambda, \cdot, h\big)$ and $\Psi\big(t, y(\cdot), z(\cdot), \lambda, \cdot, h\big)$ are continuous for fixed t, y, z, λ, h, and $\Phi\big(t, y(\cdot), z(\cdot), \lambda, 0, h\big) \equiv 0$,

2° there exist constants $M_1, M_2, M_3 \geqslant 0$ and a function $\delta\colon I \times [0,1] \times H \to \mathbb{R}_+$ such that the conditions

$$\Big\|\Phi\big(t, y_1(\cdot), z_1(\cdot), \mu_1, r, h\big) - \Phi\big(t, y_2(\cdot), z_2(\cdot), \mu_2, r, h\big)\Big\|$$
$$\leqslant M_1 \sup_{[a,t]} \|y_1(s) - y_2(s)\| + M_2 \sup_{[a,t]} \|z_1(s) - z_2(s)\| + M_3\|\mu_1 - \mu_2\| + \delta_1(t, r, h),$$

$$\lim_{N\to\infty} h \sum_{i=0}^{N-1} \sup_{r\in[0,1]} \delta_1(t_{hi}, r, h) = 0,$$

hold for $t \in I$, $h \in H$, $y_1, y_2 \in C(I, \mathbb{R}^p)$, $z_1, z_2 \in \tilde{C}(I, \mathbb{R}^p)$, $\mu_1, \mu_2 \in \mathbb{R}^q$,

3° there exist constants $D_1, D_3 \geqslant 0$, $0 \leqslant D_2 < 1$ and a function $\delta_2\colon I \times [0,1] \times H \to \mathbb{R}_+$ such that for $t \in I$, $h \in H$, $r \in (0,1]$, $y_1, y_2 \in C(I, \mathbb{R}^p)$, $z_1, z_2 \in \tilde{C}(I, \mathbb{R}^p)$, $\mu_1, \mu_2 \in \mathbb{R}^q$ we have

$$\Big\|\Psi\big(t, y_1(\cdot), z_1(\cdot), \mu_1, r, h\big) - \Psi\big(t, y_2(\cdot), z_2(\cdot), \mu_2, r, h\big)\Big\|$$
$$\leqslant D_1 \sup_{[a,t+h]} \|y_1(s) - y_2(s)\| + D_2 \sup_{[a,t+h]} \|z_1(s) - z_2(s)\| + D_3\|\mu_1 - \mu_2\| + \delta_2(t, r, h),$$

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} \sup_{r\in[0,1]} \delta_2(t_{hi}, r, h) = 0,$$

4° there exist a nonsingular matrix $B_{q\times q}$ and constants $d > 0$, $m_1 \geqslant 0$ such that $d \geqslant \|B^{-1}\|$, $m_1 d < 1$ and

$$(19)\qquad \big\|L\big(y_1(\cdot),\mu_1\big) - L\big(y_1(\cdot),\mu_2\big) - B(\mu_1-\mu_2)\big\| \leqslant m_1\|\mu_1-\mu_2\|,$$

for $y_1 \in C(I,\mathbb{R}^p)$ and $\mu_1,\mu_2 \in \mathbb{R}^q$, where the matrix norm is consistent with the vector norm,

5° for $y_1, y_2 \in C(I,\mathbb{R}^p)$ and $\mu \in \mathbb{R}^q$ we have

$$\big\|L\big(y_1(\cdot),\mu\big) - L\big(y_2(\cdot),\mu\big)\big\| \leqslant m_2 \sup_{s\in I}\|y_1(s)-y_2(s)\|,$$

where $m_2 \geqslant 0$.

We introduce the standard definitions of convergence and consistency.

Definition 1. The method (7) is said to be convergent to the solution (φ,λ) of (1)–(2) if

$$\lim_{\substack{h\to 0\\ j\to\infty}} \sup_{t\in I}\|y_h^j(t)-\varphi(t)\| = 0, \qquad \lim_{\substack{h\to 0\\ j\to\infty}} \|\lambda_{hj}-\lambda\| = 0.$$

Definition 2. The method (7) is consistent with (1)–(2) on (φ,λ) if for $(t,r,h) \in I\times[0,1]\times H$ the following conditions

$$\begin{gathered}
\big\|\varphi(t+rh)-\varphi(t)-h\Phi\big(t,\varphi(\cdot),\varphi'(\cdot),\lambda,r,h\big)\big\| \leqslant \varepsilon_1(t,r,h),\\
\big\|\varphi'(t+rh)-\Psi\big(t,\varphi(\cdot),\varphi'(\cdot),\lambda,r,h\big)\big\| \leqslant \varepsilon_2(t,r,h),\\
\lim_{N\to\infty}\sum_{i=0}^{N-1}\sup_{r\in[0,1]}\varepsilon_1(t_{hi},r,h) = 0, \quad \varepsilon_1(t,0,h)\equiv 0,\\
\lim_{N\to\infty}\sum_{i=0}^{N-1}\sup_{r\in[0,1]}\varepsilon_2(t_{hi},r,h) = 0,
\end{gathered}$$

are satisfied.

Put

$$\tilde{\varepsilon}_2(t,r,h) = \varepsilon_2(t,r,h) + \delta_2(t,r,h),$$
$$\tilde{\tilde{\varepsilon}}_2(t,h) = \sup_{r\in[0,1]} \tilde{\varepsilon}_2(t,r,h),$$
$$\tilde{\varepsilon}_1(t,r,h) = \varepsilon_1(t,r,h) + h\delta_1(t,r,h),$$
$$\tilde{\tilde{\varepsilon}}_1(t,h) = \sup_{r\in[0,1]} \tilde{\varepsilon}_1(t,r,h),$$
$$\tilde{M}_i = M_i + M_2 D_i (1 - D_2)^{-1}, \quad i = 1,3,$$
$$\eta(h) = M_2\big[\|z_h^j(a) - \varphi'(a)\| + \sum_{i=0}^{N-1} \tilde{\tilde{\varepsilon}}_2(t_{hi},h)\big](1 - D_2)^{-1},$$
$$\beta(h) = (b-a)\big[\tilde{M}_3\|\lambda_{hj} - \lambda\| + \eta(h)\big] + \sum_{i=0}^{N-1} \tilde{\tilde{\varepsilon}}_1(t_{hi},h),$$
$$W(h) = dm_2\big[\|y_h^j(a) - \varphi(a)\| + (b-a)\eta(h) + \sum_{i=0}^{N-1} \tilde{\tilde{\varepsilon}}_1(t_{hi},h)\big] \exp\big((b-a)\tilde{M}_1\big).$$

Now we can formulate the main theorem of this part.

Theorem 3. *If Assumption* H_3 *is satisfied and if*

1° *there exists the solution* $(\varphi,\lambda) \in C^1(I,\mathbb{R}^p) \times \mathbb{R}^q$ *of* (1)–(2),

2° $A = d\big[m_1 + m_2\tilde{M}_3(b-a)\exp\big((b-a)\tilde{M}_1\big)\big] < 1$,

3° $\lim\limits_{\substack{h\to 0\\ j\to\infty}} \|y_h^j(a) - x_p\| = \lim\limits_{\substack{h\to 0\\ j\to\infty}} \|z_h^j(a) - \varphi'(a)\| = 0$, *then the method* (7) *is convergent to the solution* (φ,λ) *of* (1)–(2)*. Furthermore, the following estimations*

$$\|\lambda_{hj} - \lambda\| \leqslant u_j(h), \quad j = 0,1,\dots, \tag{20}$$
$$\sup_{t\in I} \|y_h^j(t) - \varphi(t)\| \leqslant w_j(h), \quad j = 0,1,\dots, \tag{21}$$
$$\sup_{t\in I} \|z_h^j(t) - \varphi'(t)\| \leqslant v_j(h), \quad j = 0,1,\dots, \tag{22}$$

hold, where

$$u_j(h) = A^j\|\lambda_{h0} - \lambda\| + W(h)\frac{1-A^j}{1-A},$$
$$w_j(h) = \Big[\|y_h^j(a) - x_p\| + (b-a)\tilde{M}_3 u_j(h) + (b-a)\eta(h)$$
$$+ \sum_{i=0}^{N-1} \tilde{\tilde{\varepsilon}}_1(t_{hi},h)\Big] \exp\big((b-a)\tilde{M}_1\big),$$
$$v_j(h) = (1-D_2)^{-1}\Big[\|z_h^j(a) - \varphi'(t)\| + D_1 w_j(h) + D_3 u_j(h) + \sum_{i=0}^{N-1} \tilde{\tilde{\varepsilon}}_2(t_{hi},h)\Big],$$

for $j = 0, 1, \ldots$.

Proof. Put

$$e_h^j(t) = \|y_h^j(t) - \varphi(t)\|, \qquad E_{hn}^j = \sup_{[a,t_{hn}]} e_h^j(t),$$
$$g_h^j(t) = \|z_h^j(t) - \varphi'(t)\|, \qquad G_{hn}^j = \sup_{[a,t_{hn}]} g_h^j(t),$$
$$z_{hj} = \|\lambda_{hj} - \lambda\|,$$

for $n = 0, 1, \ldots, N$, $j = 0, 1, \ldots$. Using the assumptions we get

$$\begin{aligned} g_h^j(t_{hn} + rh) &= \big\|\Psi\big(t_{hn}, y_h^j(\cdot), z_h^j(\cdot), \lambda_{hj}, r, h\big) - \Psi\big(t_{hn}, \varphi(\cdot), \varphi'(\cdot), \lambda, r, h\big) \\ &\quad + \Psi\big(t_{hn}, \varphi(\cdot), \varphi'(\cdot), \lambda, r, h\big) - \varphi'(t_{hn} + rh)\big\| \\ &\leqslant D_1 E_{h,n+1}^j + D_2 G_{h,n+1}^j + D_3 z_{hj} + \tilde{\varepsilon}_2(t_{hn}, r, h), \\ &\qquad n = 0, 1, \ldots, N-1 \end{aligned}$$

and hence by induction we see that

$$G_{hn}^j \leqslant G_{h0}^j + D_1 E_{hn}^j + D_2 G_{hn}^j + D_3 z_{hj} + \sum_{i=0}^{n-1} \tilde{\tilde{\varepsilon}}_2(t_{hi}, h), \quad n = 1, 2, \ldots, N-1,$$

or

$$(23) \quad G_{hn}^j \leqslant (1 - D_2)^{-1}\big[G_{h0}^j + D_1 E_{hn}^j + D_3 z_{hj} + \sum_{i=0}^{n-1} \tilde{\tilde{\varepsilon}}_2(t_{hi}, h)\big], \quad n = 0, 1, \ldots, N.$$

Similarly, for the error e_h^j we get

$$\begin{aligned} e_h^j(t_{hn} + rh) &= \big\|y_h^j(t_{hn}) + h\Phi\big(t_{hn}, y_h^j(\cdot), z_h^j(\cdot), \lambda_{hj}, r, h\big) - \varphi(t_{hn}) \\ &\quad - h\Phi\big(t_{hn}, \varphi(\cdot), \varphi'(\cdot), \lambda, r, h\big) + \varphi(t_{hn}) \\ &\quad + h\Phi\big(t_{hn}, \varphi(\cdot), \varphi'(\cdot), \lambda, r, h\big) - \varphi(t_{hn} + rh)\big\| \\ &\leqslant e_h^j(t_{hn}) + hM_1 E_{hn}^j + hM_2 G_{hn}^j + hM_3 z_{hj} + \tilde{\varepsilon}_1(t_{hn}, r, h), \\ &\qquad n = 0, 1, \ldots, N-1 \end{aligned}$$

and hence and (23)

$$\begin{aligned} e_h^j(t_{hn} + rh) &\leqslant e_h^j(t_{hn}) + h\tilde{M}_1 E_{hn}^j + h\tilde{M}_3 z_{hj} + h\eta(h) + \tilde{\varepsilon}_1(t_{hn}, r, h), \\ &\qquad r \in (0,1),\ n = 0, 1, \ldots, N-1. \end{aligned}$$

Now by induction we see

$$E_{hn}^j \leqslant E_{h0}^j + h\tilde{M}_1 \sum_{i=0}^{n-1} E_{hi}^j + nh\big[\tilde{M}_3 z_{hj} + \eta(h)\big] + \sum_{i=0}^{n-1} \tilde{\tilde{\varepsilon}}_1(t_{hi}, h), \quad n = 0, 1, \dots, N$$

or

$$E_{hn}^j \leqslant d_n = E_{h0}^j + h\tilde{M}_1 \sum_{i=0}^{n-1} E_{hi}^j + \beta(h), \quad n = 0, 1, \dots, N. \tag{24}$$

Indeed, we have

$$d_{n+1} \leqslant (1 + h\tilde{M}_1) d_n, \quad n = 0, 1, \dots, N-1,$$

so

$$E_{hn}^j \leqslant \big[E_{h0}^j + \beta(h)\big] \exp\big((b-a)\tilde{M}_1\big), \quad n = 0, 1, \dots, N.$$

We next note that

$$\begin{aligned} z_{h,j+1} &= \big\| B^{-1}\big[L\big(y_h^j(\cdot), \lambda\big) - L\big(y_h^j(\cdot), \lambda_{hj}\big) - B(\lambda - \lambda_{hj}) \\ &\quad + L\big(\varphi(\cdot), \lambda\big) - L\big(y_h^j(\cdot), \lambda\big)\big]\big\| \\ &\leqslant d[m_1 z_{hj} + m_2 E_{hN}^j] \leqslant A z_{hj} + W(h), \quad j = 0, 1, \dots . \end{aligned}$$

From this we obtain the estimation (20) and then (21)–(22). According to our assumptions we see that

$$\lim_{\substack{h\to 0\\ j\to\infty}} u_j(h) = \lim_{\substack{h\to 0\\ j\to\infty}} w_j(h) = \lim_{\substack{h\to 0\\ j\to\infty}} v_j(h) = 0.$$

It means that our method (y_h^j, λ_{hj}) is convergent to the solution (φ, λ) of (1)–(2) and the proof of this theorem is completed. □

Remark 2 (see [9]). The condition (19) is satisfied provided that

$$\big\| D_\mu L\big(y(\cdot), \mu\big) - B \big\| \leqslant m_1 \quad \text{for all } y \in C(I, \mathbb{R}^p),\ v \in \mathbb{R}^q, \tag{25}$$

where

$$D_\mu L\big(y(\cdot), \mu\big) = \left[\frac{\partial L_i(y(\cdot), \mu)}{\partial \mu_j}\right].$$

Now if

$$L\big(y(\cdot), \mu\big) = \tilde{M} y(b) + \tilde{N}\mu + \tilde{K}, \tag{26}$$

then the condition (25) takes the form

$$\|\tilde{N} - B\| \leqslant m_1, \tag{27}$$

where $\tilde{M}_{q\times p}$, $\tilde{N}_{q\times q}$, $\tilde{K}_{q\times 1}$. And if $p = q$ we may choose $B = \tilde{M} + \tilde{N}$ and (25) leads us to $\|\tilde{M}\| \leqslant m_1$, and

$$\|(\tilde{M} + \tilde{N})^{-1}\| m_1 < 1, \tag{28}$$

provided that $\tilde{M} + \tilde{N}$ is nonsingular.

Remark 3 (see [9]). Assume that there exist matrices $Q_{q\times q}$, $\mathbb{Z}_{q\times q}$ such that for all $y \in C(I, \mathbb{R}^p)$, $\mu \in \mathbb{R}^q$ the matrix

$$P(y(\cdot), \mu) = D_\mu L(y(\cdot), \mu) + Q(y(\cdot), \mu),$$

has a representation of the form

$$P(y(\cdot), \mu) = P_0(I + Z(y(\cdot), \mu))$$

with a constant nonsingular matrix P_0. Moreover, we assume that

$$\|P_0 Z(y(\cdot), \mu)\| \leqslant \nu_1, \quad \|Q(y(\cdot), \mu)\| \leqslant \nu_2, \qquad \text{for all } y \in C(I, \mathbb{R}^p),\ \mu \in \mathbb{R}^q.$$

Now taking $B = P_0$, the condition (19) is satisfied with $m_1 = \nu_1 + \nu_2$ and $\|P_0^{-1}\|(\nu_1 + \nu_2) < 1$.

Moreover if $p = q$ and the function L is linear of the form (26) than we may put

$$Q(y(\cdot), \mu) = D_y L(y(\cdot), \mu) = \tilde{M}.$$

Choosing $B = \tilde{M} + \tilde{N}$ we have

$$P(y(\cdot), \mu) = \tilde{M} + \tilde{N}, \qquad \nu_1 = 0, \qquad \|\tilde{M}\| \leqslant \nu_2 = m_1,$$

which lead to the condition (28).

Now we are interested in the construction of the method (7). The increment functions Φ and Ψ can be created in the analogous way as for ordinary differential equations. To adopt these methods we need a interpolation scheme to compute the numerical solution y_h^j (and z_h^j) on the interval I. It requires storing all previous values $y_h^j(t_{hi})$ as well as $z_h^j(t_{hi})$ for $i = 0, 1, \ldots, n$ becase they may be needed to compute y_h^j and z_h^j on the interval $(t_{hn}, t_{h,n+1}]$.

i° The Euler method is defined by

$$
\begin{aligned}
y_h^j(t_{hn}+rh) &= y_h^j(t_{hn}) + rhf\big(t_{hn}, y_h^j(\cdot), z_h^j(\cdot), \lambda_{hj}\big), \quad r \in [0,1],\\
z_h^j(t_{hn}+rh) &= \frac{\mathrm{d}}{\mathrm{d}r} y_h^j(t_{hn}+rh), \quad r \in (0,1),\\
z_h^j(t_{h,n+1}) &= f\big(t_{h,n+1}, y_h^j(\cdot), z_h^j(\cdot), \lambda_{hj}\big),
\end{aligned}
$$

for $n = 0, 1, \ldots, N-1$, $j = 0, 1, \ldots$.

ii° The improved Euler method is

$$
\begin{aligned}
y_h^j(t_{hn}+rh) = y_h^j(t_{hn}) &+ h\big((r - \tfrac{1}{2}r^2) f\big(t_{hn}, y_h^j(\cdot), z_h^j(\cdot), \lambda_{hj}\big)\\
&+ \tfrac{1}{2}r^2 f\big(t_{h,n+1}, \bar{y}_h^j(\cdot), \bar{z}_h^j(\cdot)\big)\big),\\
&\qquad r \in [0,1],\ n = 0, 1, \ldots, N-1,\ j = 0, 1, \ldots,
\end{aligned}
$$

where $z_h^j(t_{hn}+rh)$, $r \in (0,1)$ is defined as in (i) and

$$
\bar{y}_h^j(s) = \begin{cases} y_h^j(s), & a \leqslant s \leqslant t,\\ y_h^j(t) + (s-t) f\big(t, y_h^j(\cdot), z_h^j(\cdot), \lambda_{hj}\big), & t < s \leqslant t+h, \end{cases}
$$

$$
\bar{z}_h^j(s) = \begin{cases} z_h^j(s), & a \leqslant s \leqslant t,\\ f\big(t, y_h^j(\cdot), z_h^j(\cdot), \lambda_{hj}\big), & t < s \leqslant t+h. \end{cases}
$$

iii° The one-step method defined by

$$
\begin{aligned}
y_h^j(t_{hn}+rh) = y_h^j(t_{hn}) &+ h\big((r - \tfrac{3}{2}r^2 + \tfrac{2}{3}r^3) f\big(t_{hn}, y_h^j(\cdot), z_h^j(\cdot), \lambda_{hj}\big)\\
&+ (2r^2 - \tfrac{4}{3}r^3) f\big(t_{hn} + .5h, \bar{\bar{y}}_h^j(\cdot), \bar{\bar{z}}_h^j(\cdot), \lambda_{hj}\big)\\
&+ (-\tfrac{1}{2}r^2 + \tfrac{2}{3}r^3) f\big(t_{h,n+1}, \bar{\bar{y}}_h^j(\cdot), \bar{\bar{z}}_h^j(\cdot), \lambda_{hj}\big)\big), \quad r \in [0,1],
\end{aligned}
$$

where

$$
\bar{\bar{y}}_h^j(s) = \begin{cases} y_h^j(s), & a < s \leqslant t,\\ y_h^j(t) + (s-t) f\big(t, y_h^j(\cdot), z_h^j(\cdot), \lambda_{hj}\big) + (s-t)^2/(2h)\big(f\big(t+h, \bar{y}_h^j(\cdot),\\ \qquad \bar{z}_h^j(\cdot), \lambda_{hj}\big) - f\big(t, y_h^j(\cdot), z_h^j(\cdot), \lambda_{hj}\big)\big), & t < s \leqslant t+h, \end{cases}
$$

$$
\bar{\bar{z}}_h^j(s) = \begin{cases} z_h^j(s), & a \leqslant s \leqslant t,\\ f\big(t, y_h^j(\cdot), z_h^j(\cdot), \lambda_{hj}\big) + \big((s-t)/h\big)\big(f\big(t+h, \bar{y}_h^j(\cdot), \bar{z}_h^j(\cdot), \lambda_{hj}\big)\\ \qquad - f\big(t, y_h^j(\cdot), z_h^j(\cdot), \lambda_{hj}\big)\big), & t < s \leqslant t+h, \end{cases}
$$

where $\bar{y}_h^j$ and $\bar{z}_h^j$ are given in (ii) while $z_h^j(t_{hn}+rh)$, $r \in (0,1)$ is determined in (i). Such methods were described in [1, 4, 12].

Usually in the above mentioned methods $y_h^j(a) = x_p$, whereas $z_h^j(a)$ is determined from the equation

$$z_h^j(a) = f\big(a, x_p, z_h^j(a), \lambda_{hj}\big), \quad j = 0, 1, \dots .$$

Indeed $\lambda_{h0} = \lambda_0 \in \mathbb{R}^q$ is given and

$$\lambda_{h,j+1} = \lambda_{hj} - B^{-1}L\big(y_h^j(\cdot), \lambda_{hj}\big), \quad j = 0, 1, \dots .$$

This procedure works as follows: for $y_h^0(a)$ and λ_0 find $z_h^0(a)$ and determine $y_h^0(t)$, $z_h^0(t)$ for $t \in I$, then find the new value for λ_{h1} and $y_h^1(a)$, $z_h^1(a)$ to determine $y_h^1(t)$, $z_h^1(t)$ for $t \in I$ and so on.

We may also consider approximations y_h^j and z_h^j of φ and φ' only on the grid points t_{hn}. Indeed, we have then the sets of discrete values for y_h^j and z_h^j. We define them by

$$y_h^j(t_{h,n+1}) = y_h^j(t_{hn}) + h\Phi_0\big(t_{hn}, y_h^j(\cdot), z_h^j(\cdot), \lambda_{hj}, h\big), \quad n = 0, 1, \dots, N-1,$$

where

$$\begin{aligned} &y_h^j(\cdot) = y_h^j(t_{hs}) \quad \text{if} \quad t_{hs} \leqslant \cdot < t_{h,s+1}, \\ &z_h^j(\cdot) = z_h^j(t_{hs}) = \big(y_h^j(t_{hs}) - y_h^j(t_{h,s-1})\big)/h \quad \text{if} \quad t_{hs} \leqslant \cdot < t_{h,s+1} \text{ and } s \geqslant 1. \end{aligned}$$

For example, for the problem

$$y'(t) = f\big(t, y(\alpha_1(t)), \dots, y(\alpha_r(t)), y'(\beta_1(t)), \dots, y'(\beta_s(t)), \lambda\big),$$

we have now

$$\begin{aligned} y_h^j(t_{h,n+1}) = y_h^j(t_{hn}) + h\Phi_1\big(&t_{hn}, y_h^j(t_{h,c_1^\alpha(n)}), \dots, y_h^j(t_{h,c_r^\alpha(n)}), \\ &z_h^j(t_{h,c_1^\beta(n)}), \dots, z_h^j(t_{h,c_s^\beta(n)}), \lambda_{hj}, h\big), \quad n = 0, 1, \dots, N-1, \end{aligned}$$

where

$$\begin{aligned} c_i^\alpha(n) &= E\Big(\frac{\alpha_i(t_{hn}) - a}{h}\Big), \quad E \text{ denotes integer part}, \\ z_h^j(t_{hq}) &= \big(y_h^j(t_{hq}) - y_h^j(t_{h,q-1})\big)/h. \end{aligned}$$

Here Φ_0 and Φ_1 are increment functions. Such schemes were discussed in [2, 8] for special cases of our problems. Indeed, we may also consider more complicated algorithms to approximate $y(\alpha_i(t))$ and $y'(\beta_q(t))$.

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