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BILINEAR FORMS AND NUCLEARITY

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INTRODUCTION

Back in 1965, A. Pietsch asked if a locally convex Hausdorff space (lcs) E must be nuclear whenever it has the property that every continuous bilinear form on $E \times E$ is nuclear (cf. [10], 7.4.5). The question remained open, even within the framework of Banach spaces where it translates to what is known as the "bounded non-nuclear operator problem": is a Banach space X necessarily finite-dimensional when all operators from X to its dual X^* are nuclear?

The related "compact non-nuclear operator problem" has a negative solution. In 1983, G. Pisier [12] constructed (separable, infinite-dimensional) Banach spaces P which, among others, have the property that every approximable operator $P \to P$ is nuclear. In 1990, K. John [8] observed that this is also true for approximable operators $P^{(m)} \to P^{(n)}$ for any choice of positive integers m and n; here $P^{(m)}$ is the m-th dual of P. He even proved that actually every compact operator $P \to P^*$ is nuclear.

It is open whether there are "Pisier spaces" which do not contain a copy of ℓ_1 (cf. [9]). In fact, for any such space P all operators $P \to P^*$ would be nuclear, and the answer to Pietsch's question would be negative even when restricted to Banach spaces.

Nevertheless, the Pisier spaces P can be used to give a negative answer to Pietsch's question within the class of Schwartz spaces; the clue is to change P's topology in such a way that compactness of the involved operators is automatic. An appropriate selection of a sequence of continuous seminorms on the resulting space makes it even possible to construct a non-nuclear Fréchet-Schwartz space on which all bounded bilinear forms are nuclear.

The topology in question is the compact-open topology on P. More generally, given any Banach space X, let us write

$$X_0$$

for X endowed with the topology of uniform convergence on compact subsets of X^*. This topology is known to be the finest Schwartz topology on X which is consistent with the duality $\langle X, X^* \rangle$; equivalently, it can be characterized as the coarsest locally convex topology on X which renders compact all continuous operators from X into any Banach space. See e.g. [2] for definitions and background. If X is infinite dimensional, then X_0 can never be nuclear. One way of seeing this is by using an immediate consequence of a result of S.Bellenot [1] on factorization properties of compact Hilbert space operators. It follows from this that regardless of how we choose the infinite dimensional Banach space X, every Hilbert-Schmidt operator $u\colon \ell_2 \to \ell_2$ admits a factorization $u\colon \ell_2 \overset{w}{\to} X \overset{v}{\to} \ell_2$; see also [4]. Clearly, v can be chosen compact, so that v is continuous from X_0 to ℓ_2. Nuclearity of X_0 would therefore entail that every Hilbert-Schmidt operator on ℓ_2 is nuclear—a plain contradiction.

RESULTS

In particular, if P is any Pisier space, then P_0 cannot be nuclear. However:

Theorem 1. *If P is a Pisier space, then every continuous bilinear form on $P_0 \times P_0$ is nuclear.*

A stronger result is the following:

Theorem 2. *If P is a Pisier space, then*

$$P_0 \otimes_\varepsilon P_0 = P_0 \otimes_\pi P_0.$$

Recall from Pisier's work [12] that $P \otimes_\varepsilon P = P \otimes_\pi P$. So the above can be looked at as a "quadratic" counterexample within the class of Schwartz spaces to Grothendieck's conjecture [2] that if two lcs E and F are such that $E \otimes_\varepsilon F = E \otimes_\pi F$, then one of them must be nuclear. Recall that there are "non-quadratic" such counterexamples, even within the class of all Fréchet-Schwartz spaces having a basis whose topologies are generated by hilbertian seminorms (cf. [7]); however, the hilbertian nature of such spaces prevents the existence of "quadratic" counterexamples of this kind (cf. [6]). Nevertheless, using Theorem 2 we are able to construct "quadratic" counterexamples within the class of all Fréchet-Schwartz spaces:

Theorem 3. *There exists a non-nuclear Fréchet-Schwartz space F such that*
(a) $F \otimes_\varepsilon F = F \otimes_\pi F$ *and*
(b) *every continuous bilinear form on* $F \times F$ *is nuclear.*

PRELIMINARIES

We are going to use standard terminology and results on Banach spaces, operator ideals, and locally convex spaces; our main references are [11] and [3]. Let us just recall some basic notions.

Let E be any lcs (all lcs will be over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$). The system of all closed, absolutely convex neighbourhoods of zero in E will be denoted by

$$\mathcal{U}(E).$$

Given $U \in \mathcal{U}(E)$, let p_U be its gauge functional, let

$$E_U$$

be the Banach space obtained from completing the associated normed space $E/\ker(p_U)$, and let

$$\Phi_U \colon E \to E_U$$

be the corresponding canonical map. If $V \in \mathcal{U}(E)$ is contained in U, then there is a unique $\Phi_{UV} \in \mathcal{L}(E_V, E_U)$ such that $\Phi_U = \Phi_{UV} \circ \Phi_V$.

We write

$$\mathcal{B}(E, E)$$

for the space of all continuous bilinear forms $E \times E \to \mathbf{K}$. Given $\beta \in \mathcal{B}(E, E)$, we can find $U \in \mathcal{U}(E)$ such that

$$|\beta(x, y)| \leqslant p_U(x) \cdot p_U(y)$$

for all $x, y \in E$. It follows that β admits a factorization $\beta = \beta_U \circ (\Phi_U \times \Phi_U)$ with $\beta_U \in \mathcal{B}(E_U, E_U)$. Since

$$\mathcal{L}(E_U, E_U{}^*) \to \mathcal{B}(E, E) \colon u \mapsto \langle \Phi_U^* \circ u \circ \Phi_U(\cdot), \cdot \rangle$$

is clearly a linear injection, we arrive at the identification

$$\mathcal{B}(E, E) = \bigcup_{U \in \mathcal{U}(E)} \mathcal{L}(E_U, E_U{}^*).$$

Let $\mathcal{N}$ denote the ideal of all nuclear operators between Banach spaces. The members of

$$\mathcal{B}_{\mathcal{N}}(E,E) := \bigcup_{U \in \mathcal{U}(E)} \mathcal{N}(E_U, E_U{}^*)$$

are called the *nuclear bilinear forms* on $E \times E$. To say that a bilinear form β: $E \times E \to \mathbf{K}$ is nuclear thus amounts to requiring the existence of a $U \in \mathcal{U}(E)$ and of sequences $(x_n^*), (y_n^*)$ in $E_U{}^*$ such that

$$\sum_n \|x_n^*\|_{E_U^*} \cdot \|y_n^*\|_{E_U^*} < \infty$$

and

$$\beta(x,y) = \sum_n \langle x_n^*, x\rangle \cdot \langle y_n^*, y\rangle$$

for all $x, y \in E$. Here we have used that the adjoint of Φ_U identifies E_U^* with a linear subspace of E^*, the continuous dual of E.

PROOFS

Though Theorem 2 implies Theorem 1, we start by a simple direct proof of the latter result.

Proof of Theorem 1. Let $\beta \in \mathcal{B}(P_0, P_0)$ be given. By what we have just explained, there is a $U \in \mathcal{U}(P_0)$ together with an operator $u \in \mathcal{L}(P_U, P_U{}^*)$ such that

$$\beta(x,y) = \langle(\Phi_U^* u \Phi_U)x, y\rangle$$

for all $x, y \in P$. By P_0's nature, $v := \Phi_U^* u \Phi_U \colon P \to P^*$ is compact; it was shown in [8] that it is even nuclear. Therefore it factors $v\colon P \overset{a}{\to} c_0 \overset{\Delta}{\to} \ell_1 \overset{b^*}{\to} P^*$ with Δ a diagonal operator and $a, b \in \mathcal{L}(P, c_0)$. Clearly, we may even chose a and b to be compact, so that $a = \tilde{a}\Phi_V$ and $b = \tilde{b}\Phi_V$ for some $V \in \mathcal{U}(P_C)$ and suitable operators $\tilde{a}, \tilde{b} \in \mathcal{L}(P_V, c_0)$. Of course, we may suppose $V \subset U$ so that, if we define $v \in \mathcal{N}(P_V, P_V{}^*)$ by $v := \tilde{b}^*\Delta\tilde{a}$, then $v = \Phi_{UV}^* u \Phi_{UV}$. It follows that $\beta_V \colon P_V \times P_V \to \mathbf{K}\colon (x,y) \mapsto \langle vx, y\rangle$ is a nuclear bilinear form, and since $\beta = \beta_V \circ (\Phi_V \times \Phi_V)$, we are done. □

In order to prove Theorem 2, we must look closer at the map

$$(*) \qquad \mathcal{L}(P_U, P_U^*) \to \mathcal{N}(P, P^*)\colon u \mapsto \Phi_U{}^* u \Phi_U$$

established in the preceding proof. We have already seen that for each $u \in \mathcal{L}(P_U, P_U{}^*)$ there is a $V \subset U$ in $\mathcal{U}(P_0)$ such that $\Phi_{UV}^* u \Phi_{UV}$ belongs to $\mathcal{N}(P_V, P_V{}^*)$;

so the range of the map $(*)$ is actually the union of all $\mathcal{N}(P_V, P_V{}^*), V \in \mathcal{U}(P_0)$. We are going to show that all of $\mathcal{L}(P_U, P_U{}^*)$ is actually mapped into $\mathcal{N}(P_V, P_V{}^*)$ for a single V. More precisely:

Proposition. *No matter how we select a neighbourhood $U \in \mathcal{U}(P_0)$, it contains a neighbourhood $V \in \mathcal{U}(P_0)$ such that $u \mapsto \Phi_{UV}{}^* u \Phi_{UV}$ defines a bounded operator of $\mathcal{L}(P_U, P_U{}^*)$ to $\mathcal{N}(P_V, P_V{}^*)$.*

Proof. We shall now use that, by [12], $\mathcal{N}(P, P^*)$ is a closed subspace of $\mathcal{L}(P, P^*)$!

The adjoint of $\Phi_U \otimes \Phi_U \colon P \tilde{\otimes}_\pi P \to P_U \tilde{\otimes}_\pi P_U$ is given by $u \mapsto \Phi_U{}^* u \Phi_U$, i.e. the map appearing in $(*)$.

Since U belongs to $\mathcal{U}(P_0)$, Φ_U is compact, and so the operator in $(*)$ is compact as well. There is thus a null sequence (v_n) in $\mathcal{N}(P, P^*)$ such that $\{\Phi_U{}^* u \Phi_U \colon u \in B_{\mathcal{L}(P_U, P_U{}^*)}\}$ is contained in $\overline{\text{conv}}\{v_n \colon n \in \mathbf{N}\}$. We may even assume that each v_n is of the form $v_n = \Phi_U{}^* u_n \Phi_U$ where $u_n \in B_{\mathcal{L}(P_U, P_U^*)}$, see e.g. 9.4.2 in [3].

Each v_n has a decompsition $v_n = b_n{}^* \Delta_n a_n$ where a_n and b_n are compact operators $P \to c_0$ and $\Delta \colon c_0 \to \ell_1$ is a diagonal operator. Let $\nu(\cdot)$ denote the nuclear norm. Clearly, we may suppose that $\|a_n\| \cdot \|\Delta_n\| \cdot \|b_n\| \leqslant 2 \cdot \nu(v_n)$, and we may arrange for $\|\Delta_n\| \leqslant 2$ and $\max\{\|a_n\|, \|b_n\|\} \leqslant \nu(v_n)^{\frac{1}{2}}$ for each n. In particular, $\lim\limits_{n \to \infty} \|a_n\| = \lim\limits_{n \to \infty} \|b_n\| = 0$.

Introduce the Banach space $X = c_0(c_0)$ of all norm null sequences in c_0, and let $p_n \colon X \to c_0$ be the projection onto the n-th coordinate, $n \in \mathbf{N}$. Then $a \colon P \to X \colon x \mapsto (a_n x)_n$ and $b \colon P \to X \colon x \mapsto (b_n x)_n$ are well-defined operators, and we may write $v_n = b^* p_n{}^* \Delta_n p_n a$ since $a_n = p_n a$ and $b_n = p_n b$ for each n. A standard diagonalization argument reveals that a and b are even compact. Therefore we can find $V \in \mathcal{U}(P_0)$ satisfying $V \subset U$, together with operators $\tilde{a}, \tilde{b} \in \mathcal{L}(P_V, X)$ such that $a = \tilde{a}\Phi_V$ and $b = \tilde{b}\Phi_V$. Write $v_n = \Phi_V{}^* w_n \Phi_V$ where $w_n := \tilde{b}^* p_n{}^* \Delta_n p_n \tilde{a}$ belongs to $\mathcal{N}(P_V, P_V{}^*)$. Since the involved operators are continuous and since Φ_V has dense range we may conclude that $\Phi_{UV}{}^* u_n \Phi_{UV} = w_n \in \mathcal{N}(P_V, P_V{}^*)$. Note that

$$\begin{aligned} \nu(w_n) &= \nu(\Phi_{UV}{}^* u_n \Phi_{UV}) = \nu(\tilde{b}^* p_n{}^* \Delta_n p_n \tilde{a}) \\ &\leqslant \|\tilde{a}\| \cdot \|\tilde{b}\| \cdot \nu(\Delta_n) \leqslant 2 \cdot \|\tilde{a}\| \cdot \|\tilde{b}\|. \end{aligned}$$

Let now $u \in B_{\mathcal{L}(P_U, P_U^*)}$ be arbitrary. There are $\lambda_n \geqslant 0$ such that $v := \Phi_U{}^* u \Phi_U$ has the representation $v = \sum_n \lambda_n v_n$ in $\mathcal{N}(P, P^*)$. To complete the proof, just observe that $w := \sum_n \lambda_n w_n$ exists in $\mathcal{N}(P_V, P_V{}^*)$ and equals $\Phi_{UV}{}^* u \Phi_{UV}$. □

The proof of Theorem 2 is now immediate. Given $U \in \mathcal{U}(P_0)$, let $V \in \mathcal{U}(P_0)$ be such that $V \subset U$ and $u \mapsto \Phi_{UV}{}^* u \Phi_{UV}$ maps $\mathcal{L}(P_U, P_U{}^*) = (P_U \otimes_\pi P_U)^*$

continuously into $\mathcal{N}(P_V, P_V{}^*)$ and hence continuously into $(P_V \otimes_\varepsilon P_V)^*$. Our map is the adjoint of $\Phi_{UV} \otimes \Phi_{UV}$ which therefore is continuous from $P_V \otimes_\varepsilon P_V$ to $P_U \otimes_\pi P_U$. But since $P_0 \tilde{\otimes}_\varepsilon P_0$ and $P_0 \tilde{\otimes}_\pi P_0$ have natural representations as projective limits of the $P_U \tilde{\otimes}_\varepsilon P_U$ and the $P_U \tilde{\otimes}_\pi P_U$, respectively ($U \in \mathcal{U}(P_0)$; cf. [3], 16.3.3 and 15.4.3), we may conclude that the identity $P_0 \otimes_\varepsilon P_0 \to P_0 \otimes_\pi P_0$ is continuous. □

Theorem 1 follows from Theorem 2: in fact, the latter implies that every continuous bilinear form on $P_0 \times P_0$ is integral. It must be nuclear since each $U \in \mathcal{U}(P_0)$ contains a $V \in \mathcal{U}(P_0)$ such that $\Phi_{UV}: P_V \to P_U$ is compact; cf. [11], 24.6.3.

Let us now proceed to our final goal.

Proof of Theorem 3. As before, (b) follows from (a); indeed, (a) and (b) are equivalent since we are dealing with metrizable lcs.

Given an lcs E, we call a neighbourhood $U \in \mathcal{U}(E)$ *non-nuclear* if there is no $V \in \mathcal{U}(E)$ such that the operator $\Phi_{UV}: E_V \to E_U$ is nuclear. Clearly, if $U \in \mathcal{U}(E)$ is non-nuclear, then any $V \in \mathcal{U}(E)$ which is contained in U is non-nuclear as well. Certainly, E is a non-nuclear lcs if and only if $\mathcal{U}(E)$ contains non-nuclear members.

Let P be a *separable* Pisier space. Then $[P^*, \sigma(P^*.P)]$ is separable; let $\{x_n^*: n \in \mathbf{N}\}$ be a countable dense subset of this space. By Theorem 2, each $U \in \mathcal{U}(P_0)$ contains a $V_U \in \mathcal{U}(P_0)$ such that $\Phi_{UV_U}: P_{V_U} \to P_U$ is compact and $\Phi_{UV_U} \otimes \Phi_{UV_U}: P_{V_U} \otimes_\varepsilon P_{V_U} \to P_U \otimes_\pi P_U$ is continuous. Since P_0 is a non-nuclear Schwartz space, we can construct a decreasing sequence $(U_n)_n$ of non-nuclear members of $\mathcal{U}(P_0)$ by fixing a non-nuclear $U_1 \in \mathcal{U}(P_0)$ and then setting $U_{n+1} = V_{U_n}$ for each $n \in \mathbf{N}$; moreover, we may certainly assume that $x_n^* \in U_n^\circ$ for each n. Then $\bigcup_n U_n^\circ$ is weak * dense in P^*, and so $\bigcap_n U_n = \{0\}$. Consequently, the seminorms p_{U_n} generate a metrizable lc topology $\mathcal{T}_m$ on P. The completion of $[P, \mathcal{T}_m]$ is a non-nuclear Fréchet-Schwartz space which has the property (a). □

Remarks. If E is P, or $[P, \sigma(P, P^*)]$, then $E \otimes_\varepsilon E = E \otimes_\pi E$ and so one might conjecture that this also holds when E is P endowed with any lc topology $\mathcal{T}$ which is compatible with $\langle P, P^* \rangle$. Such a conjecture, however, turns out be too optimistic.

(a) For a first counterexample, take $\mathcal{T}$ to be the lc topology $\mathcal{T}_2$ generated by all hilbertian seminorms on P, that is, by all seminorms of the form $\|u(\cdot)\|$, u any operator from P into any Hilbert space. It was shown in [5] that $P_2 := [P, \mathcal{T}_2]$ cannot be nuclear. This implies that $P_2 \otimes_\varepsilon P_2 \neq P_2 \otimes_\pi P_2$. In fact, we may either invoke [6] or argue that otherwise each $U \in \mathcal{U}(P_2)$ would contain a $V \in \mathcal{U}(P_2)$ such that $\Phi_{UV}: P_V \tilde{\otimes}_\varepsilon P_V \to P_U \tilde{\otimes}_\pi P_U$ is continuous, equivalently, that $\Phi_{UV}: P_V \to P_U$ would be a Hilbert-Schmidt operator between Hilbert spaces [4], contradicting the non-nuclearity of P_2.

(b) Another example can be obtained by essentially repeating the argument given at the end of the introduction.

Let $P_{2,0}$ be the space P with the lc topology generated by all seminorms $\|v(\cdot)\|$, $v\colon P \to \ell_2$ any compact operator. It follows from Bellenot's result [1] mentioned in the introduction that every Hilbert-Schmidt operator admits a factorization through P and hence through $P_{2,0}$ since we may assume the factors to be compact. As before, $P_{2,0} \otimes_\varepsilon P_{2,0} \neq P_{2,0} \otimes_\pi P_{2,0}$: otherwise $P_{2,0}$ would be nuclear and this would force all Hilbert-Schmidt operators to be nuclear.—$P_{2,0}$ and P_2 are different whenever P contains a copy of ℓ_1; see [9] for more on this.

We conclude by posing a more restricted version of the problem we started with and which has its origins of course in the main result of [7]: Is there a non-nuclear Fréchet-Schwartz space F with a basis such that all continuous bilinear forms $F \times F \to \mathbf{K}$ are nuclear or, equivalently, such that $F \otimes_\varepsilon F = F \otimes_\pi F$?

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