Ladislav Bican

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LADISLAV BICAN, Praha

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1. INTRODUCTION

All groups in this paper are abelian. If x is an element of a torsionfree group G then $|x|_G$, or simply $|x|$ is the characteristic and $t_G(x) = t(x)$ is the type of x in G. The corank of a pure subgroup H of G is the rank of G/H. A subgroup H of G is said to be a regular subgroup of G if G/H is torsion and $t_H(x) = t_G(x)$ for each $x \in H$. The letter G will usually denote a general torsionfree group, while the letter B will be used for Butler groups. For unexplained terminology and notation see [F1]. As usual, CH denotes the *continuum hypothesis*, i.e. $2^{\aleph_0} = \aleph_1$. By a *smooth increasing union of a group* G we mean a collection of subgroups G_α indexed by an initial segment of ordinals with the property that $G_\beta \subseteq G_\alpha$ when $\beta < \alpha$ and $G_\alpha = \bigcup\limits_{\beta<\alpha} G_\beta$ whenever α is a limit ordinal.

An exact sequence $E\colon 0 \longrightarrow H \longrightarrow G \xrightarrow{\beta} K \longrightarrow 0$ with K torsionfree is *balanced* if the induced map $\beta_*\colon \operatorname{Hom}(J, G) \longrightarrow \operatorname{Hom}(J, K)$ is surjective for each rank one torsionfree group J. Equivalently, E is balanced if all rank one (completely decomposable) torsionfree groups are projective with respect to E. A torsionfree group B is said to be a B_1*-group* (*Butler group*) if $\operatorname{Bext}(B, T) = 0$ for all torsion groups T, where Bext is the subfunctor of Ext consisting of all balanced-exact extensions. It is known [BS] that this definition coincides with the familiar one if B has finite rank, i.e. if it is a pure subgroup of a completely decomposable group, or, equivalently [B], a torsionfree homomorphic image of a completely decomposable group of finite rank.

A torsionfree group B is called a B_2*-group* if it has a B*-filtration*, i.e. B is a smooth union of pure subgroups B_α, $\alpha < \lambda$ an ordinal and $B_{\alpha+1} = B_\alpha + H_\alpha$ for all $\alpha < \lambda$ where H_α is a Butler group of finite rank. It is well-known [BS] that the class of B_2-groups is contained in the class of B_1-groups and consequently it is natural to ask whether the converse is true or not. For groups of cardinality $\leqslant \aleph_1$

these classes coincide (see [DHR], [AH] and [BS] for the countable case). However, for groups of higher cardinality, the problem is undecidable in ZFC. In [DT] it is shown that the denial of CH leads to the negative answer already at the cardinality $\aleph_2$. On the other hand, assuming CH, Dugas, Hill and Rangaswamy [DHR] gave the affirmative answer for groups of cardinalities up to $\aleph_\omega$, while Fuchs and Magidor recently obtained positive solution in the constructive universe L.

Proving the equality of the classes mentioned above one of the important steps is to show that a B_1-group B of regular cardinality has a κ-filtration consisting of separative and TEP subgroups. In [BF2] we introduced the notion of a preseparative subgroup generalizing both the separative and prebalanced ones. Using this concept, our primary objective in this paper is to develop a technique which enables us to obtain a similar result in singular cardinalities (Th. 4.1). Doing this we get that a B_1-group B is a B_2-group if it has 'enough' balanced, preseparative chains and B_2-subfactorgroups of cardinalities less than $|B|$ (Th. 4.2). This yields a simplification of the proof of the result of [FMa], especially in the singular cardinality case. As applications we shall use the technique developed to the study of Butler groups with countable typesets. Generalizing some results of [B4], [N] and [DR] we show that within the class of torsionfree groups with countable typesets the classes of B_1-groups and B_2-groups coincide and are closed with respect to pure subgroups.

2. Some important subgroups

Recall that a subgroup H of a group G with a torsionfree quotient G/H is called *prebalanced* if for each rank one (pure) subgroup J of G/H there is a pair (X, φ) consisting of a finite rank completely decomposable group X and a homomorphism $\varphi\colon X \to G$ such that $\beta\varphi X = J$, β being the canonical projection $G \to G/H$. An exact sequence $0 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 0$ is prebalanced if αH is a prebalanced subgroup of G. This concept was introduced by Richman [R] under the name 'semi-balanced' subgroup and rediscovered by Fuchs-Viljoen [FV].

It has been noted in Fuchs-Metelli [FMe] that the prebalancedness of H in G can be rephrased by requiring that for each $g \in G$ there are a non-zero integer m and a finite subset $\{h_0, \ldots, h_n\} \subseteq H$ such that $|mg+H|_{G/H} = |mg+h_0|_G \cup \ldots \cup |mg+h_n|_G$. As in the case of balancedness, the prebalanced exact sequences form the subfunctor PBext of Ext.

2.1. Lemma. *Let λ be a limit ordinal and $H = \bigcup\limits_{\alpha<\lambda} H_\alpha$ a smooth increasing union such that $H_{\alpha+1}$ is balanced in G for all $\alpha < \lambda$. If H is prebalanced in G, then it is balanced.*

P r o o f. For any $g \in G \setminus H$ there is $0 \neq m < \omega$ and $h_0, \dots, h_k \in H$ such that $|mg + H| = \bigcup_{i=0}^{k} |mg + h_i|$. Consequently there is $\alpha < \lambda$ non-limit such that $\{h_0, h_1, \dots, h_k\} \subseteq H_\alpha$, showing that $|mg + H| = |mg + H_\alpha|$. Thus $t(g + H) = t(mg + H) = t(mg + H_\alpha) = t(g + H_\alpha) = t(g + h)$ for some $h \in H_\alpha \subseteq H$. □

2.2. Lemma. *Let λ be a limit ordinal and $H = \bigcup_{\alpha<\lambda} H_\alpha$ a smooth increasing union such that $H_{\alpha+1}$ is prebalanced (balanced) in G for all $\alpha < \lambda$. If $\operatorname{cof}\lambda \neq \omega$ then H is prebalanced (balanced) in G.*

P r o o f. Take $g \in G \setminus H$ arbitrarily. For each $0 \neq m < \omega$ there is a countable subset $S_m = \{h_0^m, h_1^m, \dots\} \subseteq H$ such that $|mg+H| = \bigcup_{n<\omega} |mg+h_n^m|$. Since $\operatorname{cof}\lambda > \omega$, there is $\alpha < \lambda$ non-limit such that $\{h_n^m \mid m, n < \omega\} \subseteq H_\alpha$. Further, there is $k < \omega$ and $h_0, h_1, \dots, h_l \in H_\alpha$ with $|kg + H_\alpha| = \bigcup_{i=0}^{l} |kg + h_i|$, H_α being prebalanced. Now $|kg + H| = \bigcup_{n<\omega} |kg + h_n^k| \leqslant |kg + H_\alpha| = \bigcup_{i=0}^{l} |kg + h_i| \leqslant |kg + H|$ and we are through.
The balanced case follows from the above one and Lemma 2.1. □

Note that a pure subgroup H of a torsionfree group G is called *separative* if for each $g \in G$ there is a countable subset $\{h_0, h_1, \dots\} \subseteq H$ such that, given any $h \in H$, there is an index $n < \omega$ with $|g + h| \leqslant |g + h_n|$. This concept, introduced by P. Hill under the name 'separable', is one of the most important tools in the study of Butler groups of infinite rank (c.f. [AH], [DHR], [FMa]). In [BF2] the class of separative subgroups has been extended to that of preseparative ones. The main advantage of this class lies in two facts. First, it contains all prebalanced subgroups and, second, it works with the types instead of the characteristics. Recall that a pure subgroup H of a torsionfree group G is said to be *preseparative*, if for each $g \in G \setminus H$ there is a countable subset $\{h_0, h_1, \dots\} \subseteq H$ such that for each $h \in H$ there are $m, n < \omega$, $m \neq 0$, with $t(g + h) \leqslant t(mg + h_0) \cup t(mg + h_1) \cup \dots \cup t(mg + h_n)$. In this case we shall also say that *$\{h_0, h_1, \dots\}$ is a preseparative set for g over H.*

2.3. Lemma. *Let $K \leqslant H$ be pure subgroups of a torsionfree group G. Then:*

(i) *if H is either separative or prebalanced in G then it is preseparative;*

(ii) *if K is prebalanced in G then H/K is preseparative in G/K if and only if H is preseparative in G;*

(iii) *if K is preseparative in G and H/K is countable, then H is preseparative in G;*

(iv) *if K is preseparative in H and H is preseparative in G, then K is preseparative in G;*

(v) *if* $\{H_n \mid n < \omega\}$ *is an increasing sequence of preseparative subgroups of* G *then* $H = \bigcup_{n<\omega} H_n$ *is preseparative in* G.

Proof. See [BF2; L.2.1]. $\square$

2.4. Lemma. *(CH) Let* λ *be a limit ordinal with* $\operatorname{cof}\lambda \geqslant \omega_2$ *and let* $H = \bigcup_{\alpha<\lambda} H_\alpha$ *be a smooth increasing union with* $H_{\alpha+1}$ *preseparative in a torsionfree group* G *for each* $\alpha < \lambda$. *Then* H *is preseparative in* G.

Proof. Let $g \in G \setminus H$ be arbitrary. Define an equivalence relation $\sim$ on H by $x \sim y$, $x, y \in H$, if and only if $|g + x| = |g + y|$, and select a set $S \subseteq H$ of representatives of equivalence classes. Obviously, $|S| \leqslant 2^{\aleph_0}$, since there is at most $2^{\aleph_0}$ characteristics. Thus, there is $\alpha < \lambda$ non-limit such that $S \subseteq H_\alpha$. Now if $\{h_n \mid n < \omega\}$ is a preseparative set for g over H_α then for an arbitrary $h \in H$ there is $\tilde{h} \in S$ with $\tilde{h} \sim h$ and consequently $t(g + h) = t(g + \tilde{h}) \leqslant \bigcup_{i=0}^{n} t(mg + h_i)$ for some $m, n < \omega, m \neq 0$. $\square$

Another relevant concept in the study of infinite rank Butler groups is the *torsion extension property* (TEP), due to Dugas and Rangaswamy [DR] (cf. also [B2]). A pure subgroup A of a torsionfree group B is said to have TEP in B, or briefly, A is TEP(-subgroup) in B, if every homomorphism $A \to T$ with T torsion extends to a homomorphism $B \to$T.

2.5. Lemma. *If* $G = H + B$ *where* H *is pure in* G *and* B *is a finite rank Butler group, then* H *is* TEP *in* G.

Proof. Let $f\colon H \to T$ be arbitrary. Since $H \cap B$ is pure in B, [DR; Th.2] yields that $f|H \cap B$ extends to $k\colon B \to T$. Now $g\colon G \to T$ given by $g(h + b) = f(h) + k(b), h \in H, b \in B$, is the desired extension of f. $\square$

2.6. Corollary. *If* $B = \bigcup_{\alpha<\lambda} B_\alpha$ *is a* B*-filtration of a torsionfree group* B, *then* B_α *is* TEP *in* B *for each* $\alpha < \lambda$.

2.7. Lemma. (i) *If* A *is a* TEP *subgroup of a* B_1*-group* B, *then* B/A *is a* B_1*-group;*

(ii) *if* A *is a prebalanced subgroup of a* B_1*-group* B *such that* B/A *is a* B_1*-group, then* A *is* TEP *in* B.

Proof. (i) The exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} B/A \to 0$ induces the exact sequence $\operatorname{Hom}(B,T) \to \operatorname{Hom}(A,T) \to \operatorname{Ext}(B/A,T) \xrightarrow{\beta^*} \operatorname{Ext}(B,T)$ where β^*

is monic, A being TEP in B. Now it is easy to see that β^* induces a monomorphism $\mathrm{Bext}(B/A,T) \to \mathrm{Bext}(B,T) = 0$.

(ii) The exact sequence given above induces $\mathrm{Hom}(B,T) \xrightarrow{\alpha^*} \mathrm{Hom}(A,T) \longrightarrow \mathrm{PBext}(B/A,T)$ for each torsion group T. Since $\mathrm{PBext}(B/A,T) = \mathrm{Bext}(B/A,T) = 0$ by [FM; Th. 1.5], α^* is an epimorphism. $\square$

The following result has been proved in [DHR]. The idea of the proof we borrowed from [F2].

2.8. Lemma. *Let κ be an uncountable regular cardinal and let $B = \bigcup_{\alpha<\kappa} B_\alpha$ be any κ-filtration of a B_1-group B of cardinality κ. If each B_α is a B_1-group then there is a cub $C \subseteq \kappa$ such that B_α has* TEP *in B for each $\alpha \in C$.*

Proof. Denote $\tilde{E} = \{\alpha < \kappa \mid B_\alpha$ is not TEP in $B\}$ and $E = \{\alpha < \kappa \mid B_\alpha$ is not TEP in B_β for some $\beta > \alpha\}$. Then $E \subseteq \tilde{E}$ and the equality holds provided E is not stationary.

Supposing E stationary, there is no loss of generality in assuming $\beta = \alpha + 1$ in the definition of E.

Given any $\alpha \in E$, select a torsion group T_α and a homomorphism $\varphi_\alpha : B_\alpha \to T_\alpha$ which has no extension to $B_{\alpha+1} \to T_\alpha$. For $\alpha < \kappa, \alpha \notin E$, we take $T_\alpha = 0$, $\varphi_\alpha = 0$. Setting $S_\alpha = \oplus_{\beta<\alpha} T_\beta$ and $\Pi_\alpha : S_{\alpha+1} \to T_\alpha$ the canonical projection, we can form the commutative diagram

$$\begin{array}{rccccccccc}
E_\alpha : 0 & \longrightarrow & S_\alpha & \longrightarrow & S_\alpha \oplus B_\alpha & \xrightarrow{\sigma_\alpha} & B_\alpha & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle \eta_\alpha} & & \downarrow & & \\
E_{\alpha+1} : 0 & \longrightarrow & S_{\alpha+1} & \longrightarrow & S_{\alpha+1} \oplus B_{\alpha+1} & \xrightarrow{\sigma_{\alpha+1}} & B_{\alpha+1} & \longrightarrow & 0
\end{array}$$

with exact rows, where $\eta_\alpha(s_\alpha, b_\alpha) = (s_\alpha + \varphi_\alpha b_\alpha, b_\alpha)$ and the remaining maps are natural embeddings and projections. Take the direct limit $E\colon 0 \to S \to G \to B \to 0$ of the directed system $\{E_\alpha \mid \alpha < \kappa\}$. By [BF; L.1.4] the sequence E is balanced and it is therefore splitting via $\tau\colon B \to G$. Now τ maps B_α into the direct sum of B_α and a set of less than κ many T_β's. Further, let $\omega_\alpha : S_\alpha \oplus B_\alpha \to G$ be the canonical direct limit mapping and denote $\vartheta_\alpha = (1_G - \tau\sigma)\omega_\alpha \big| B_\alpha : B_\alpha \to S$. The regularity of κ leads to the conclusion that the set $F = \{\alpha < \kappa \mid \tau B_\alpha \leqslant S_\alpha \oplus B_\alpha, \vartheta_\alpha B_\alpha \leqslant S_\alpha\}$ is a cub in κ. For $\alpha \in F \cap E$ we now have $\vartheta_\alpha b = (1_G - \tau\sigma)\omega_\alpha b = (1_G - \tau\sigma)\omega_{\alpha+1}\eta_\alpha b = \varphi_\alpha b + \vartheta_{\alpha+1} b$ for each $b \in B_\alpha$, hence $\varphi_\alpha b = -\Pi_\alpha \vartheta_{\alpha+1} b$, which contradicts the choice of φ_α and the proof is complete. $\square$

Let H be a pure subgroup of a torsionfree group G. By a *preseparative chain from H to G* we will mean a smooth increasing union $G = \bigcup_{\alpha<\lambda} H_\alpha$ such that $H_0 = H$, H_α is preseparative in G and $|H_{\alpha+1}/H_\alpha| \leqslant \aleph_1$ for each $\alpha < \lambda$. For $H = 0$ we speak about a *preseparative chain of G*.

3.1. Definition. We say that a torsionfree group G has *enough prebalanced* (EPB), if each subgroup of G of regular cardinality is contained in a prebalanced subgroup of the same cardinality. Further, we say that G has enough preseparatives (EPS), if there is a preseparative chain from H to G whenever H is either prebalanced in G or a countable increasing union of prebalanced subgroups of G. Finally, we say that G has the property (EB_2), if for each prebalanced and TEP subgroup H of G, each prebalanced subgroup of G/H of cardinality less than $|G/H|$ is a B_2-group.

3.2. Lemma. *Let G be a torsionfree group with* (EPB). *If K is a subgroup of G of cardinality $\lambda < |G|$ and $\operatorname{cof}\lambda \neq \omega$, then K is contained in a prebalanced subgroup of G of the same cardinality λ.*

Proof. If λ is regular, there is nothing to prove. If $\operatorname{cof}\lambda = \mu < \lambda$, then $\lambda = \bigcup_{\alpha<\mu} \kappa_\alpha$ smooth, with $\kappa_{\alpha+1}$ regular for each $\alpha < \mu$. Writing $K = \bigcup_{\alpha<\mu} K_\alpha, |K_\alpha| = \kappa_\alpha, \alpha < \mu$, we select $H_0 \supseteq K_0$ prebalanced and of cardinality κ_0. If $H_\beta, \beta < \alpha$, has been defined, then for $\alpha = \gamma + 1$ we take a prebalanced subgroup H_α of G of cardinality κ_α containing $K_\alpha + H_\gamma$ while for α limit we set $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$. Clearly, $H = \bigcup_{\alpha<\mu} H_\alpha$ is of the size λ, contains K and is prebalanced by Lemma 2.2. $\square$

3.3. Lemma. *Let G be a torsionfree group of regular cardinality κ with* (EPB). *Then there is a κ-filtration $G = \bigcup_{\alpha<\kappa} G_\alpha$ such that*

(i) *G_α is preseparative in G for each $\alpha < \kappa$ and prebalanced whenever $\operatorname{cof}\alpha \neq \omega$ provided κ is either limit or of the form $\kappa = \lambda^+$ with $\operatorname{cof}\alpha \neq \omega$;*

(ii) *(CH) if $\kappa = \lambda^+$, $\operatorname{cof}\lambda = \omega$, then G_α is preseparative in G whenever $\operatorname{cof}\alpha \neq \omega_1$ and $G_{\alpha+1} = \bigcup_{i<\omega} G^i_{\alpha+1}$ with $G^i_{\alpha+1}$ prebalanced in G for each $\alpha < \kappa$ and $i < \omega$.*

Proof. (i) Assume first that κ is limit. Then $\kappa = \bigcup_{\alpha<\kappa} \kappa_\alpha$ with $\kappa_{\alpha+1}$ and κ_0 regular. Starting with any smooth union $G = \bigcup_{\alpha<\kappa} H_\alpha$ with $|H_\alpha| = \kappa_\alpha$ take $G_0 \supseteq H_0$ prebalanced and assume that for an ordinal $\alpha < \kappa$ a smooth chain $\{G_\beta \mid \beta < \alpha\}$ has been defined in such a way that G_β is prebalanced for β successor. For α limit we set $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$, while for $\alpha = \gamma + 1$ we select a prebalanced subgroup G_α of G

of cardinality κ_α containing $G_\gamma + H_\alpha$. By Lemma 2.2 G_α is prebalanced whenever $\operatorname{cof}\alpha \neq \omega$ and it is obviously preseparative if $\operatorname{cof}\alpha = \omega$.

So, let $\kappa = \lambda^+$ and $\operatorname{cof}\lambda \neq \omega$. In this case we have a smooth increasing union $G = \bigcup_{\alpha<\kappa} H_\alpha$ with $|H_\alpha| = \lambda$. With respect to Lemma 3.2 we may assume that $H_{\alpha+1}$ is prebalanced for each $\alpha < \kappa$ and consequently Lemma 2.2 gives that H_α is prebalanced whenever $\operatorname{cof}\alpha \neq \omega$ and it is obviously preseparative in the remaining case.

(ii) In this case we have $\lambda = \bigcup_{i<\omega} \kappa_i$, κ_i regular, $i < \omega$, and we start with an auxiliary κ-filtration $G = \bigcup_{\alpha<\kappa} H_\alpha$ with $|H_\alpha| = \lambda$. Express H_0 as a union $H_0 = \bigcup_{i<\omega} H_0^i$ with $|H_0^i| = \kappa_i$. Then we can select $G_0^0 \supseteq H_0^0$, a prebalanced subgroup of cardinality κ_0 and by induction on $i > 0$ we select G_0^i as a prebalanced subgroup of cardinality κ_i containing $H_0^i + G_0^{i-1}$. Setting $G_0 = \bigcup_{i<\omega} G_0^i$, we have G_0 preseparative of the required form and containing H_0. Assume that for some $\alpha < \mu$ we have constructed a smooth increasing chain $\{G_\beta \mid \beta < \alpha\}$ such that $H_\beta \subseteq G_\beta$ and G_β is preseparative in G whenever β is non-limit. For α limit set $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$. Let $\alpha = \gamma + 1$. Expressing $G_\gamma = \bigcup_{i<\omega} G_\gamma^i$ and $H_\alpha = \bigcup_{i<\omega} H_\alpha^i$ we select G_α^0 to be a prebalanced subgroup of G of cardinality κ_0 containing $G_\gamma^0 + H_\alpha^0$. Continuing by induction, let G_α^i be a prebalanced subgroup of the size κ_i containing $G_\gamma^i + H_\alpha^i + G_\alpha^{i-1} (i > 0)$. Setting $G_\alpha = \bigcup_{\alpha<\omega} G_\alpha^i, G_\alpha$ is obviously preseparative and contains $G_\gamma + H_\alpha$ Thus we have constructed a smooth union $G = \bigcup_{\alpha<\kappa} G_\alpha$ with G_α preseparative for each α non-limit. However, if α is limit, then G_α is preseparative for $\operatorname{cof}\alpha = \omega$ by Lemma 2.3, while the same holds by Lemma 2.4 whenever $\operatorname{cof}\alpha \geqslant \omega_2$. Lemma 3.3 is therefore proved. □

3.4. Lemma. *Let A be a pure subgroup of a B_1-group B. Then A is a B_1-group provided one of the following two conditions is satisfied:*

(i) *there is a preseparative chain from A to B;*

(ii) *A is prebalanced in B and B/A has a preseparative chain.*

P r o o f. See [BF2; Th. 4.4]. □

3.5. Lemma. (i) *If A is a corank one preseparative and TEP subgroup of a Butler group B, then A is prebalanced in B;*

(ii) *If A is a preseparative and TEP subgroup of a B_1-group B such that there is a preseparative chain from A to B, then A is prebalanced in B.*

P r o o f. (i) See [BF2; Th. 6.1].

(ii) Let $C \leqslant B$ be any pure subgroup of B containing A as a corank one subgroup. If $B = \bigcup A_\alpha$ is a preseparative chain from A to B, then owing to Lemma 2.3(iii)

$B = \bigcup \langle A_\alpha + C\rangle_*$ is a preseparative chain from C to B and so C is a B_1-group by Lemma 3.4. An application of (i) completes the proof. □

3.6. Lemma. *Let $B = \bigcup\limits_{\alpha<\kappa} B_\alpha$ be a smooth ascending union of* TEP *subgroups where B is a B_1-group with* (EPS). *If $B_{\alpha+1}$ is a smooth ascending union of prebalanced (balanced) subgroups for each $\alpha < \kappa$, then B_α is prebalanced (balanced) for each $\alpha < \kappa$.*

Proof. With respect to Lemmas 2.3 and 2.2 the subgroup $B_{\alpha+1}$ is either prebalanced or a countable union of prebalanced subgroups, so that (EPS) and Lemma 3.5 give the prebalancedness of $B_{\alpha+1}$. Thus B_α is prebalanced in B whenever $\operatorname{cof}\alpha \neq \omega$ by Lemma 2.2 and so using (EPS) and Lemma 3.5 once more we get the prebalancedness of B_α for each $\alpha < \kappa$. To prove the balanced case it suffices to apply Lemma 2.1 in the preceding proof twice. □

3.7. Lemma. *Let H be a prebalanced subgroup of a torsionfree group G such that G/H is a B_2-group. Then $G = \bigcup\limits_{\alpha<\lambda} G_\alpha$ with $G_0 = H$ and $G_{\alpha+1} = G_\alpha + B_\alpha, \alpha < \lambda$, where G_α is pure in G and B_α is a finite rank Butler group.*

Proof. Let $G/H = \bigcup\limits_{\alpha<\lambda} G_\alpha/H$ be a B-filtration of G/H. Then $G_{\alpha+1}/H = G_\alpha/H + B'_\alpha/H, B'_\alpha/H$ a finite rank Butler group. Hence $B'_\alpha = H + B_\alpha$ with B_α a finite rank Butler group, H being prebalanced in G, and the assertion follows easily. □

Recall [AH] that a pure subgroup H of a torsionfree group G is said to be *decent* if for any finite subset S of G there exists a finite number of rank one pure subgroups A_i of G such that $H + \sum A_i$ is pure in G and contains S. Further, a group G is said to have an *axiom-3 family* of decent subgroups if there is a family $\mathscr{F}(G)$ of decent subgroups such that $0, G \in \mathscr{F}(G), \sum_{i\in I} H_i \in \mathscr{F}(G)$ whenever $H_i \in \mathscr{F}(G), i \in I$, and if $H \in \mathscr{F}(G)$ and $X \subseteq G$ is countable, then there is a $K \in \mathscr{F}(G)$ containing $H \cup X$ such that K/H is countable.

3.8. Lemma. (i) *Let B be a B_2-group and let A be any member of a B-filtration of B. Then there are axiom-3 families of decent subgroups $\mathscr{F}(A), \mathscr{F}(B)$ of A, B, respectively, such that $\mathscr{F}(A) \subseteq \mathscr{F}(B)$.*

(ii) *If $B = \bigcup\limits_{\alpha<\lambda} B_\alpha$ smooth is a B_2-group such that all B_α's are members of a B-filtration of B, then there are $\mathscr{F}(B_\alpha)$ and $\mathscr{F}(B)$ such that $\mathscr{F}(B_\beta) \subseteq \mathscr{F}(B_\alpha)$ whenever $\beta \leqslant \alpha$ and $\bigcup\limits_{\beta<\alpha} \mathscr{F}(B_\beta) \subseteq \mathscr{F}(B_\alpha), \alpha$ limit. In particular, $\bigcup\limits_{\alpha<\lambda} \mathscr{F}(B_\alpha) \subseteq \mathscr{F}(B)$.*

Proof. It follows immediately from the proof of [FMa; Th. 7.1] since any subset S closed in μ is closed in each $\lambda > \mu$. $\square$

3.9. Lemma. *Let $\mathscr{F}(B)$ be an axiom-3 family of decent subgroups of a torsionfree group B. Then for each $A \in \mathscr{F}(B)$ there is a B-filtration from A to B.*

Proof. See the proof of [AH; Th. 5.3]. $\square$

κ-Shelah game. Let κ be a regular uncountable cardinal and let G be a torsionfree group of cardinality $|G| > \kappa^+$. We define the *κ-Shelah game* on G in the following way: Player I picks subgroups $G_{2i}, i < \omega$, of cardinality κ and player II picks G_{2i+1} such that $G_i \subseteq G_{i+1}$ for all $i < \omega$. Player II wins if G_{2i+1} is prebalanced in G and TEP in G_{2i+3} for each $i < \omega$.

3.10. Lemma. *Let κ be a regular uncountable cardinal and let B be a B_1-group of cardinality $|B| > \kappa^+$ having the properties* (EPB) *and* (EPS). *Then player* II *has a winning strategy in the κ-Shelah game.*

Proof. Lemma 1.2 in [H] still holds, the κ-Shelah game is determinate and so we are going to show that player I has no winning strategy. By way of contradiction let us assume that I has a winning strategy s and has picked B_0. By transfinite induction we shall construct a smooth chain $\{C_\alpha \mid \alpha < \kappa^+\}$ of subgroups of B of cardinality κ. Let C_0 be a prebalanced subgroup of B of cardinality κ containing B_0. If $0 < \alpha < \kappa^+$ and the $C_\beta, \beta < \alpha$, has been defined, then for α limit we set $C_\alpha = \bigcup\limits_{\beta<\alpha} C_\beta$, while for α non-limit we select C_α as a prebalanced subgroup of B of cardinality κ containing $C_{\alpha-1}$ and all $s(C_{\alpha_1}, C_{\alpha_2}, \ldots, C_{\alpha_{2n}})$, $\alpha_1 < \alpha_2 < \ldots < \alpha_{2n} < \alpha$, $n < \omega$. The union $C = \bigcup\limits_{\alpha<\kappa^+} C_\alpha$ is prebalanced in B by Lemma 2.2 and consequently it is a B_1-group by (EPS) and Lemma 3.4. By Lemma 2.8 there is a cub $U \subseteq \kappa^+$ such that C_α has TEP in C for all $\alpha \in U$ and consequently all these C_α's are prebalanced by Lemma 3.6.

Now when player I has chosen B_{2i} in the κ-Shelah game, then player II picks B_{2i+1} to be C_α, where α is the least non-limit element of U such that $B_{2i} \subseteq C_\alpha$. $\square$

4.1. Theorem. *Let B be a B_1-group the having properties* (EPB) *and* (EPS). *Then $B = \bigcup_{\alpha<\mu} B_\alpha$ is an increasing smooth union of prebalanced and* TEP *subgroups B_α, $|B_\alpha| < |B|$ provided one of the following two conditions holds:*

(i) *$|B|$ is regular;*

(ii) *$|B|$ is singular and B has* (EB_2).

Proof. Assume (i). For $|B| = \kappa = \aleph_0$, B is a B_2-group by [BS; Th. 3.4] and if $B = \bigcup_{n<\lambda} B_n$, $\lambda \leqslant \omega$, is a B-filtration of B, then all B_n's are TEP in B by Corollary 2.6.

Assume now that κ is a regular uncountable cardinal. It follows from Lemma 3.3 and its proof that there is a subgroup B_0 of B of cardinality less than κ which is either prebalanced or an increasing countable union of prebalanced subgroups. By (EPS) there is a preseparative chain $B = \bigcup_{\alpha<\kappa} C_\alpha$ from B_0 to B. By Lemma 2.8 we have a subchain $B = \bigcup_{\alpha<\kappa} B_\alpha$ with B_α preseparative and TEP in B for each $\alpha < \kappa$ and it suffices to use Lemma 3.5.

Assume now (ii). There is a smooth union $\kappa = \bigcup_{\alpha<\mu} \kappa_\alpha$ with $\kappa_0 > \mu = \operatorname{cof} \kappa$ and κ_α regular whenever α is non-limit. Further, let $B = \bigcup_{\alpha<\mu} C_\alpha$ be a smooth union such that $|C_\alpha| = \kappa_\alpha$. With respect to Lemmas 2.2 and 3.2 we may assume that C_α is prebalanced in B whenever $\operatorname{cof} \alpha \neq \omega$.

Set $C_\alpha^0 = C_\alpha$ for each $\alpha < \mu$ and assume that C_α^k has been defined for all $0 \leqslant k \leqslant n$ and $\alpha < \mu$. For α limit or 0 set $B_\alpha^n = C_\alpha^n$ and for α successor take B_α^n according to the κ_α- Shelah game C_α^0, B_α^0, C_α^1, B_α^1, ... For each $\alpha < \mu$ list the elements of each B_α^n as $\{b_\alpha^j \mid j < \kappa_\alpha\}$. Now let A_α^n be the subgroup of B generated by B_α^n and all b_γ^j, $\gamma < \mu$, $j < \kappa_\alpha$. By hypothesis, B_α^n is a B_2-group and consequently by [FMa; Th. 7.2] it has an axiom-3 family $\mathcal{F}(B_\alpha^n)$ of decent subgroups. The routine set-theoretical arguments lead to the conclusion that C_α^{n+1} can be selected such that it has cardinality κ_α, contains $\bigcup_{\beta\leqslant\alpha} A_\beta^n$ and $C_\alpha^{n+1} \cap B_{\alpha+1}^n \in \mathcal{F}(B_{\alpha+1}^n)$.

Thus $C_\alpha^n \leqslant B_\alpha^n \leqslant A_\alpha^n \leqslant C_\alpha^{n+1}$ for all $\alpha < \mu$ and $n < \omega$ and consequently we can set $B_\alpha = \bigcup_{n<\omega} A_\alpha^n = \bigcup_{n<\omega} B_\alpha^n = \bigcup_{n<\omega} C_\alpha^n$ for each $\alpha < \mu$. Now $\bigcup_{\alpha<\mu} B_\alpha = \bigcup_{\alpha<\mu} \bigcup_{n<\omega} C_\alpha^n \geqslant \bigcup_{\alpha<\mu} C_\alpha = B$, $|B_\alpha| = \kappa_\alpha < \kappa$. Further, for α non-limit the condition (EB_2) gives that $B_\alpha^{n+1}/B_\alpha^n$ is a B_2-group. Consequently, by Lemma 3.7, the B-filtration of B_α^n extends to that of B_α^{n+1} and so by induction we get that B_α is a B_2-group for each α non-limit.

It follows now from Lemma 3.8 that we can assume $\mathscr{F}(B_\alpha^n) \subseteq \mathscr{F}(B_\alpha^{n+1})$ and consequently $\bigcup_{n<\omega} \mathscr{F}(B_\alpha^n) \subseteq \mathscr{F}(B_\alpha)$.

Let $\alpha < \mu$ be arbitrary. We have $B_\alpha = B_\alpha \cap B_{\alpha+1} = B_\alpha \cap (\bigcup_{n<\omega} B_{\alpha+1}^n) = \bigcup_{n<\omega} (B_\alpha \cap B_{\alpha+1}^n) = \bigcup_{n<\omega} ((\bigcup_{k<\omega} C_\alpha^k) \cap B_{\alpha+1}^n) = \bigcup_{n<\omega} \bigcup_{k<\omega} (C_\alpha^k \cap B_{\alpha+1}^n) = \bigcup_{n<\omega} (C_\alpha^{n+1} \cap B_{\alpha+1}^n) \in \bigcup_{n<\omega} \mathscr{F}(B_{\alpha+1}^n) \subseteq \mathscr{F}(B_{\alpha+1})$. Lemma 3.9 shows that the B-filtration of B_α extends to that of $B_{\alpha+1}$ and consequently B_α is TEP in $B_{\alpha+1}$ by Corollary 2.6. Obviously, B_α is TEP in B for each $\alpha < \mu$.

Let $\alpha < \mu$ be a limit ordinal and take $b \in B_\alpha$ arbitrarily. Then $b \in B_\alpha^n$ for some $n < \omega$ and consequently $b = b_\alpha^j$ for some $j < \kappa_\alpha$. Thus $j < \kappa_\beta$ for some $\beta < \alpha$, the chain $\{\kappa_\alpha \mid \alpha < \mu\}$ being assumed smooth. This yields $b \in A_\beta^n \leqslant B_\beta$ showing the smoothness of $\bigcup_{\alpha<\mu} B_\alpha$.

Finally, each $B_{\alpha+1}$ is an increasing countable union of prebalanced subgroups and consequently each $B_\alpha, \alpha < \kappa$, is prebalanced by Lemma 3.6.

The proof is now complete. □

4.2. Theorem. *Let B be a B_1-group having the properties* (EPB), (EPS) *and* (EB$_2$). *Then B is a B_2-group.*

P r o o f. By Theorem 4.1 we can write $B = \bigcup_{\alpha<\mu} B_\alpha, \mu = \operatorname{cof} \kappa, \kappa = |B|$, where $|B_\alpha| < \kappa$ and B_α is prebalanced and TEP in B. By hypothesis, B_0 is a B_2-group. Assume that for some $\alpha < \mu$ all $B_\beta, \beta < \alpha$, are B_2-groups such that for $\gamma < \beta$ the B-filtration of B_γ extends to that of B_β. Now for α limit we continue by smoothness, while for α successor it suffices to use Lemma 3.7. □

4.3. Corollary. *($V = L$) Any B_1-group B is a B_2-group.*

P r o o f. Assume there are B_1-groups without being B_2-groups and let B be one of them of the smallest possible cardinality. By [FMa; L. 2.3] (cf. [DHR; L. 5.2, 5.4]) B has (EPB). Let A be a prebalanced and TEP subgroup of B and let C/A be prebalanced in B/A with $|C/A| < |B/A| \leqslant |B|$. By Lemma 2.7 B/A is a B_1-group and so is C/A by Lemma 3.4, B/C having a preseparative chain by [FMa; Th. 4.3] and Lemma 2.3. Thus B has the property (EB$_2$). If A is prebalanced in B then B/A has a separative chain and consequently there is a preseparative chain from A to B by Lemma 2.3. If $A = \bigcup_{m<\omega} A_m$ is an ascending union with A_m prebalanced in B then by [FMa; L. 4.1] we have $B = \bigcup_{\alpha<\kappa} B_\alpha$ with $A_m + B_\alpha$ separative in B and $|B_{\alpha+1}/B_\alpha| \leqslant \aleph_1$. Now $A + B_\alpha = \bigcup_{m<\omega} (A_m + B_\alpha)$ is preseparative by Lemma 2.3 and $(A + B_{\alpha+1})/(A + B_\alpha)$ is of cardinality at most $\aleph_1$ as a homomorphic image of

$B_{\alpha+1}/B_{\alpha}$. Thus B has (EPS) and Theorem 4.2 gives a contradiction completing the proof. □

4.4. Theorem. *Any pure subgroup of a B_1-group B with countable typeset is a B_1-group.*

Proof. Any pure subgroup of B is obviously preseparative and it suffices to use Lemma 3.4. □

Recall [FV] that a torsionfree group is said to be *locally Butler* if any of its pure subgroups of finite rank is Butler.

4.5. Corollary [B4]. *Any B_1-group with countable typeset is locally Butler.*

4.6. Corollary [N]. *Any pure homogeneous subgroup of a completely decomposable group with countable typeset is completely decomposable.*

Proof. Any such subgroup is a B_1-group by Theorem 4.4 and it suffices to use [B3]. □

Now we are ready to present the following direct generalization of [DR; Th. 7].

4.7. Theorem. *If B is a B_1-group with countable typeset and has TEP over its pure subgroup A, then A is decent in B and B/A is a B_1-group with countable typeset.*

Proof. The subgroup A is prebalanced in B by Lemma 3.5 and consequently B/A has countable typeset, each type being a finite union of types of elements of B. Further, B/A is a B_1-group by Lemma 2.7, hence it is locally Butler by Corollary 4.5 and A is decent in B by [FV; Th. 8]. □

4.8. Theorem. *Within the class of torsionfree groups with countable typesets the classes of B_1 and B_2-groups coincide.*

Proof. Assume the existence of a B_1-group with countable typeset which is not a B_2-group and has the smallest possible cardinality κ. Clearly, B has (EPS) and by Theorems 4.7 and 4.4 also (EB_2). With respect to Theorem 4.2 it remains now to show that B has (EPB). So let C be a pure subgroup of B of regular cardinality $\lambda < \kappa$ and let D be a pure subgroup of B containing C and having the cardinality λ^+. If $D = \bigcup D_\alpha$ is a λ^+-filtration of D starting from C, then Lemma 2.8 gives the existence of some D_α which is TEP in D, D being a B_1-group by Theorem 4.5. An application of Lemma 3.5 completes the proof. □

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Author's address: Dept. of Mathematics, Charles University, Sokolovská 83, 186 00 Praha, Czech Republic.