Jozef Džurina

Asymptotic properties of third-order differential equations with deviating argument

Czechoslovak Mathematical Journal, Vol. 44 (1994), No. 1, 163–172

ASYMPTOTIC PROPERTIES OF THIRD-ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

JOZEF DŽURINA, Košice

(Received May 18, 1992)

Let us consider the third order linear differential equation

$$y'''(t) + p(t)y[g(t)] = 0. \tag{1}$$

Ohriska [8] has recently shown that using the *v-transformation of an equation* we can deduce oscillatory and asymptotic behavior of the solutions of the equations of the form

$$\Big(r(t)\big(r(t)u'(t)\big)'\Big)' + p(t)u[g(t)] = 0 \tag{2}$$

from that of equation (1).

The aim of this paper is to present a comparison principle which enables us to deduce the asymptotic behavior of the solutions of the equation

$$\Big(r_2(t)\big(r_1(t)u'(t)\big)'\Big)' + p(t)u[g(t)] = 0 \tag{3}$$

from that of equation (2). The desired comparison theorem (cf. Theorem 1) permits us to transfer some asymptotic properties of equation (1) or equation (2) to equation (3).

It is always assumed that functions p, r_1, r_2, r and $g\colon [t_0,\infty) \to (0,\infty)$ are continuous and $g(t) \to \infty$ as $t \to \infty$. We suppose that for $t \geqslant t_0$

$$g(t) \leqslant t, \tag{4}$$

$$R_i(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{r_i(s)} \to \infty \quad \text{as } t \to \infty \text{ for } i = 1,2; \tag{5}$$

$$R(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{r(s)} \to \infty \quad \text{as } t \to \infty. \tag{6}$$

For covenience of notation we put formally $r_0(t) = r_3(t) = 1$, $t \in [t_0, \infty)$ and then we denote:

$$L_0(u; r_0)(t) = u(t),$$
$$L_i(u; r_0, \cdots, r_i)(t) = r_i(t)[L_{i-1}(u; r_0, \cdots, r_{i-1})(t)]' \text{ for } i = 1, 2, 3.$$

We consider only nontrivial solutions of (3). Such a solution is said to be oscillatory if the set of its zeros is unbounded and nonoscillatory otherwise. If $u(t)$ is a nonoscillatory solution of (3) then according to generalization of a lemma of Kiguradze [4, Lemma 3] there is an integer $\ell \in \{0, 2\}$ such that

$$\begin{aligned} u(t)L_i(u; r_0, \cdots, r_i)(t) &> 0, \quad 0 \leqslant i \leqslant \ell, \\ (-1)^{i-\ell} u(t)L_i(u; r_0, \cdots, r_i)(t) &> 0, \quad \ell + 1 \leqslant i \leqslant 3 \end{aligned} \tag{7}$$

for all sufficiently large t. A function $u(t)$ satisfying (7) is said to be a function of degree ℓ. The set of all nonoscillatory solutions of degree ℓ of (3) is denoted by $\mathcal{N}_\ell$. If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of (3), then

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2.$$

The condition (7) with $\ell = 0$ implies that $|L_i(u; r_0, \cdots, r_i)(t)|$ $(0 \leqslant i \leqslant 2)$ are decreasing and $L_i(u; r_0, \cdots, r_i)(t) \to 0$ as $t \to \infty$ for $1 \leqslant i \leqslant 2$. Hartman and Wintner [3] have shown that if in equation (3) $g(t) \equiv t$ then $\mathcal{N}_0 \neq \emptyset$, therefore we are interested in the following extreme situation in which $\mathcal{N} = \mathcal{N}_0$. When this situation occurs, following Kiguradze [5], we say that equation (2) enjoys property (A).

In this paper we have been motivated by the observation that there are very few effective criteria for transfering property (A) from equation (2) to equation (3). Equation (2) has been the object of intensive investigation in recent years and we have many sufficient conditions for equation (2) to have property (A) (see e.g. [8], [9]).

We begin by formulating some preparatory results which are needed in the sequel.

Theorem A. *Let* (5) *hold. Equation* (3) *has property* (A) *if and only if so does the differential inequality*

$$\left\{ \left(r_2(t)(r_1(t)u'(t))' \right)' + p(t)u[g(t)] \right\} \operatorname{sgn} u[g(t)] \leqslant 0. \tag{8}$$

This theorem is a special case of [7, Corollary 1] and exhibits an important relationship between the differential equation (3) and the differential inequality (8).

Theorem B. *Let (6) hold. Further assume that*

$$g \in C^1([t_0, \infty)), \quad g(t) \leqslant t \quad \text{and} \quad g'(t) > 0. \tag{9}$$

Then equation (2) has property (A) if

$$\liminf_{t \to \infty} R^2[g(t)] \int_t^\infty p(s)\,\mathrm{d}s > \frac{1}{3\sqrt{3}}.$$

For the proof see [1, Theorem 11].

Theorem C. *Assume that (6) holds. Then equation (2) has a solution $u(t)$ satisfying*

$$\lim_{t \to \infty} L_2(u, r_0, r, r)(t) = a \in \mathbf{R} - \{0\} \tag{10}$$

if and only if

$$\int^\infty R^2[g(t)]p(t)\,\mathrm{d}t < \infty.$$

The proof is found in Kitamura and Kusano [6].

Lemma 1. *Suppose that (4), (5) and (6) are satisfied. Let $u(t)$ be a positive solution of (3) such that $u \in \mathcal{N}_2$. Assume that*

$$\int^\infty R^2[g(t)]p(t)\,\mathrm{d}t = \infty. \tag{11}$$

Further assume that there exists a real $\lambda > 1$ such that

$$\frac{R_2(t)}{r_1(t)} \geqslant \lambda \frac{R(t)}{r(t)}, \quad \text{for} \quad t \in [t_0, \infty) \tag{12}$$

and

$$\frac{r_1}{r} \text{ is a nonincreasing function.} \tag{13}$$

Then for all $t_1 \geqslant t_0$, and t ($\geqslant t_1$) large enough

$$\begin{aligned} &\frac{1}{r_1(t)} \int_{t_1}^t \frac{1}{r_2(s_2)} \int_{s_2}^\infty p(s_3)u[g(s_3)]\,\mathrm{d}s_3\,\mathrm{d}s_2 \\ &\qquad \geqslant \frac{1}{r(t)} \int_{t_1}^t \frac{1}{r(s_2)} \int_{s_2}^\infty p(s_3)u[g(s_3)]\,\mathrm{d}s_3\,\mathrm{d}s_2 . \end{aligned} \tag{14}$$

Proof. Suppose that $u(t)$ is a positive solution of equation (3) satisfying inequalities (7) with $\ell = 2$ for all $t \geqslant t_1 (\geqslant t_0)$. Integrating twice the inequality $L_2(u; r_0, r_1, r_2)(t) > 0$ yields $u(t) \geqslant c_1 R_1(t)$, where c_1 is a positive constant. Let $t_2 \geqslant t_1$ be chosen so that $u[g(t)] \geqslant c_1 R_1[g(t)]$ for $t \geqslant t_2$. Then

$$\begin{aligned}
&\int_{t_2}^{\infty} \frac{1}{r_2(s_2)} \int_{s_2}^{\infty} p(s_3) u[g(s_3)] \, \mathrm{d}s_3 \, \mathrm{d}s_2 \\
(15) \qquad &\geqslant c_1 \int_{t_2}^{\infty} \frac{1}{r_2(s_2)} \int_{s_2}^{\infty} p(s_3) R_1[g(s_3)] \, \mathrm{d}s_3 \, \mathrm{d}s_2 \\
&\geqslant c_2 \int_{t_2}^{\infty} R_1[g(s_3)] R_2(s_3) p(s_3) \, \mathrm{d}s_3,
\end{aligned}$$

where $c_2 \geqslant c_1$ is a positive constant. Taking (12) and (4) into account we see that

$$R_1[g(t)] R_2(t) \geqslant \int_{t_0}^{g(t)} \frac{R_2(s)}{r_1(s)} \, \mathrm{d}s \geqslant \frac{\lambda}{2} R^2[g(t)],$$

which in view of (11) and (15) implies that

$$(16) \qquad \int_{t_0}^{t} \frac{1}{r_2(s_2)} \int_{s_2}^{\infty} p(s_3) u[g(s_3)] \, \mathrm{d}s_3 \, \mathrm{d}s_2 \to \infty \quad \text{as} \quad t \to \infty.$$

Now assume that $t_1 \geqslant t_0$ is a real number. Denote $P(t) = \int_t^\infty p(s) u[g(s)] \, \mathrm{d}s$, $t \geqslant t_1$. Integration by parts yields

$$\begin{aligned}
\int_{t_1}^{t} \frac{P(s_2)}{r_2(s_2)} \, \mathrm{d}s_2 &= P(t) R_2(t) - P(t_1) R_2(t_1) \\
&\quad + \int_{t_1}^{t} R_2(s) p(s) u[g(s)] \, \mathrm{d}s, \quad t \geqslant t_1.
\end{aligned}$$

From (16) it follows that

$$(17) \qquad \int_{t_1}^{t} R_2(s) p(s) u[g(s)] \, \mathrm{d}s \to \infty \text{ as } \quad t \to \infty.$$

Let $\lambda > 1$ be a real number from (12). Then there exists a $t_2 \geqslant t_1$ such that

$$\begin{aligned}
&\frac{1}{r_1(t)} \left\{ \int_{t_1}^{t} R_2(s) p(s) u[g(s)] \, \mathrm{d}s - P(t_1) R_2(t_1) \right\} \\
&\geqslant \frac{1}{\lambda r_1(t)} \int_{t_1}^{t} R_2(s) p(s) u[g(s)] \, \mathrm{d}s \\
&\geqslant \frac{1}{r(t)} \left\{ \int_{t_1}^{t} R(s) p(s) u[g(s)] \, \mathrm{d}s - P(t_1) R(t_1) \right\}, \quad t \geqslant t_2,
\end{aligned}$$

where we have used (12) and (13). Combining the last inequalities with the fact that

$$\frac{P(t)R_2(t)}{r_1(t)} \geqslant \frac{P(t)R(t)}{r(t)}, \quad t \geqslant t_2,$$

we obtain (14). The proof is complete. □

Now, we are prepared to compare equation (3) with equation (2).

Theorem 1. *Suppose that* (4), (5), (6), (12) *and* (13) *are satisfied. Then equation* (3) *has property* (A) *if so does equation* (2).

P r o o f. Let $u(t)$ be a nonoscillatory solution of (3). Without loss of generality we may assume that $u(t)$ is positive. Suppose that $u(t) \in \mathcal{N}_2$, that is $u(t)$ satisfies inequalities (7) with $\ell = 2$ on $[t_1, \infty)$. Integrating (3) with the aid of (7) we may write

$$u(t) \geqslant u(t_1) + \int_{t_1}^{t} \frac{1}{r_1(s_1)} \int_{t_1}^{s_1} \frac{1}{r_2(s_2)} \int_{s_2}^{\infty} p(s_3)u[g(s_3)] \,\mathrm{d}s_3 \,\mathrm{d}s_2 \,\mathrm{d}s_1, \quad t \geqslant t_1.$$

We have supposed that equation (2) has property (A) and hence equation (2) cannot have any solution $v(t)$ such that $\lim\limits_{t\to\infty} L_2(v; r_0, r, r)(t) = a \in \mathbf{R} - \{0\}$. Therefore by Theorem C the relation (11) is satisfied. Then according to Lemma 1, there exists a $t_2 \geqslant t_1$ such that

$$u(t) \geqslant u(t_1) + \int_{t_2}^{t} \frac{1}{r(s_1)} \int_{t_1}^{s_1} \frac{1}{r(s_2)} \int_{s_2}^{\infty} p(s_3)u[g(s_3)] \,\mathrm{d}s_3 \,\mathrm{d}s_2 \,\mathrm{d}s_1, \quad t \geqslant t_2. \tag{18}$$

Let us denote the right hand side of (18) by $y(t)$. Repeated differentiation of $y(t)$ shows that $L_0(y; r_0)(t) > 0$, $L_1(y; r_0, r)(t) > 0$, $L_2(y; r_0, r, r)(t) > 0$ for $t \geqslant t_2$ and

$$\Big(r(t)\big(r(t)y'(t)\big)'\Big)' + p(t)u[g(t)] = 0, \quad t \geqslant t_2.$$

Since $u[g(t)] \geqslant y[g(t)]$, for all large t, say $t \geqslant t_3$, we obtain

$$\Big(r(t)\big(r(t)y'(t)\big)'\Big)' + p(t)y[g(t)] \leqslant 0, \quad t \geqslant t_3.$$

As y is a function of degree $\ell = 2$, Theorem A ensures that equation (2) cannot enjoy property (A). This is a contradiction, and the proof is complete. □

In the next theorem we illustrate an application of the above-mentioned comparison principle.

Theorem 2. *Suppose that* (5) *and* (9) *are satisfied. Let*

$$\liminf_{t\to\infty} \int_{t_0}^{g(t)} \frac{R_2(s)}{r_1(s)}\,\mathrm{d}s \int_t^{\infty} p(s)\,\mathrm{d}s > \frac{1}{6\sqrt{3}}, \tag{19}$$

and

$$R_2(t)\left(\int_{t_0}^{t} \frac{R_2(s)}{r_1(s)}\,\mathrm{d}s\right)^{-\frac{1}{2}} \quad \textit{be nonincreasing.} \tag{20}$$

Then equation (3) *has property* (A).

Proof. Choose $\lambda > 1$ such that

$$\liminf_{t\to\infty} \int_{t_0}^{g(t)} \frac{R_2(s)}{r_1(s)}\,\mathrm{d}s \int_t^{\infty} p(s)\,\mathrm{d}s > \frac{\lambda}{6\sqrt{3}}. \tag{21}$$

We consider equation (2) with a function $r(t)$ defined by the relation

$$\frac{R(t)}{r(t)} = \frac{1}{\lambda}\frac{R_2(t)}{r_1(t)}, \quad t \geqslant t_0. \tag{22}$$

Integrating (22) and extracting the square root of the resulting equality, we arrive at

$$R(t) = \frac{\sqrt{2}}{\sqrt{\lambda}}\left(\int_{t_0}^{t} \frac{R_2(s)}{r_1(s)}\,\mathrm{d}s\right)^{\frac{1}{2}} \to \infty \text{ as } t \to \infty, \tag{23}$$

where we have used (5). It is easy to see that

$$\frac{1}{r(t)} = R'(t) = \frac{1}{\sqrt{2\lambda}}\frac{R_2(t)}{r_1(t)}\left(\int_{t_0}^{t} \frac{R_2(s)}{r_1(s)}\,\mathrm{d}s\right)^{-\frac{1}{2}}. \tag{24}$$

Hence, function r_1/r is nonincreasing if and only if (20) holds. From (23) we conclude that condition (21) is equivalent to the condition

$$\liminf_{t\to\infty} R^2[g(t)] \int_t^{\infty} p(s)\,\mathrm{d}s > \frac{1}{3\sqrt{3}},$$

which is, as we see from Theorem B, a sufficient condition for equation (2) to have property (A). Our assertion follows from Theorem 1. The proof is complete. $\square$

Corollary 1. *Assume that the hypotheses of Theorem 2 hold except that relation* (20) *is replaced by one of the following conditions:*

$$\frac{2}{r_2(t)}\int_{t_0}^{t}\frac{R_2(s)}{r_1(s)}\,\mathrm{d}s \leqslant \frac{[R_2(t)]^2}{r_1(t)}, \quad t \geqslant t_0, \tag{25}$$

or

$$\frac{r_2}{r_1} \text{ is nondecreasing.} \tag{26}$$

Then equation (3) *has property* (A).

Proof. The function $R_2(t)\left(\int_{t_0}^{t}\frac{R_2(s)}{r_1(s)}\,\mathrm{d}s\right)^{-1/2}$ is nonincreasing if its first derivative is nonpositive, which occurs if (25) holds. Using (26), it is not hard to see that

$$\int_{t_0}^{t}\frac{R_2(s)}{r_1(s)}\,\mathrm{d}s \leqslant \frac{r_2(t)}{r_1(t)}\int_{t_0}^{t}\frac{R_2(s)}{r_2(s)}\,\mathrm{d}s = \frac{r_2(t)}{r_1(t)}\frac{[R_2(t)]^2}{2}, \quad t \geqslant t_0,$$

which is equivalent to (25). The proof is complete. $\square$

Example. Let us consider the equation

$$\left(t^{\frac{1}{2}}\left(t^{\frac{1}{3}}u'(t)\right)'\right)' + \frac{a}{t^{\frac{13}{6}}}u(bt) = 0, \quad t \geqslant 1, \quad b \in (0,1].$$

By Corollary 1, this equation has property (A) if $a > \frac{49}{432\sqrt{3b^{7/6}}}$. Note that we obtain for the equation a better result than e.g. Tanaka's criterion [10] provides.

The technique we have used in the proof of Theorem 2 can be applied to obtain sufficient conditions for equation (3) to have property (A) from those which are known for equation (2) or even for equation (1). The relation (24) shows how to define the function $r(t)$ to obtain equation (2) for comparing with equation (3).

Now we present another application of Theorem 1. For the special case of equation (3), namely for the equation

$$y'''(t) + p(t)y(t) = 0 \tag{27}$$

Chanturia and Kiguradze [2] have obtained the following result.

Theorem D. *Assume that*

$$\liminf_{t\to\infty} t\int_t^{\infty} sp(s)\,\mathrm{d}s > \frac{2\sqrt{3}}{9}$$

or

$$\limsup_{t\to\infty} t\int_t^{\infty} sp(s)\,\mathrm{d}s > 2.$$

Then equation (27) *has property* (A).

We extend the above mentioned result to equation (3).

Theorem 3. *Assume that* (5) *and* (20) *hold and* $g(t) \equiv t$. *Further suppose that*

$$\liminf_{t\to\infty} \left(\int_{t_0}^{t} \frac{R_1(s)}{r_2(s)}\,\mathrm{d}s\right)^{\frac{1}{2}} \int_t^\infty p(s) \left(\int_{t_0}^{s} \frac{R_1(x)}{r_2(x)}\,\mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}s > \frac{\sqrt{3}}{9}$$

or

$$\limsup_{t\to\infty} \left(\int_{t_0}^{t} \frac{R_1(s)}{r_2(s)}\,\mathrm{d}s\right)^{\frac{1}{2}} \int_t^\infty p(s) \left(\int_{t_0}^{s} \frac{R_1(x)}{r_2(x)}\,\mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}s > 1.$$

Then equation (3) *has property* (A).

Proof. Choose $\lambda > 1$ such that

$$\liminf_{t\to\infty} \left(\int_{t_0}^{t} \frac{R_1(s)}{r_2(s)}\,\mathrm{d}s\right)^{\frac{1}{2}} \int_t^\infty p(s) \left(\int_{t_0}^{s} \frac{R_1(x)}{r_2(x)}\,\mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}s > \frac{\lambda\sqrt{3}}{9} \tag{28}$$

and

$$\limsup_{t\to\infty} \left(\int_{t_0}^{t} \frac{R_1(s)}{r_2(s)}\,\mathrm{d}s\right)^{\frac{1}{2}} \int_t^\infty p(s) \left(\int_{t_0}^{s} \frac{R_1(x)}{r_2(x)}\,\mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}s > \lambda. \tag{29}$$

Let us consider equation (2) with the function $r(t)$ given by relation (21). According to the theory of *v-transformation of an equation* (see [8]), equation (2) has property (A) if and only if so does the equation

$$y'''(t) + r[R^{-1}(t)]p[R^{-1}(t)]y(t) = 0, \tag{30}$$

where $R^{-1}(t)$ is the inverse function to $R(t)$. On the other hand, Theorem D ensures that equation (30) has property (A) if

$$\liminf_{t\to\infty} t \int_t^\infty s r[R^{-1}(s)]p[R^{-1}(s)]\,\mathrm{d}s > \frac{2\sqrt{3}}{9}$$

or

$$\limsup_{t\to\infty} t \int_t^\infty s r[R^{-1}(s)]p[R^{-1}(s)]\,\mathrm{d}s > 2,$$

which are in view of (22) equivalent to (28) and (29), respectively. Hence equation (30) as well as equation (2) have property (A). By Theorem 1 we see that the assertion of Theorem 3 holds true. □

The following considerations are intended for extending the previous result to equations with deviating arguments.

Theorem E. *Assume that* (5) *and* (9) *are satisfied. Then equation* (3) *has property* (A) *if so does the equation*

$$\left(r_2(t)\left(r_1(t)u'(t)\right)'\right)' + \frac{p[g^{-1}(t)]}{g'[g^{-1}(t)]}u(t) = 0,$$

where the function $g^{-1}(t)$ *is the inverse to* $g(t)$.

For the proof of Theorem E see e.g. [1] or [7]. From Theorem E and Theorem 3 we have

Theorem 4. *Assume that* (5), (9) *and* (20) *hold. Further suppose that*

$$\liminf_{t\to\infty}\left(\int_{t_0}^{g(t)}\frac{R_1(s)}{r_2(s)}\,\mathrm{d}s\right)^{\frac{1}{2}}\int_t^{\infty}p(s)\left(\int_{t_0}^{g(s)}\frac{R_1(x)}{r_2(x)}\,\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}s > \frac{\sqrt{3}}{9}$$

or

$$\limsup_{t\to\infty}\left(\int_{t_0}^{g(t)}\frac{R_1(s)}{r_2(s)}\,\mathrm{d}s\right)^{\frac{1}{2}}\int_t^{\infty}p(s)\left(\int_{t_0}^{g(s)}\frac{R_1(x)}{r_2(x)}\,\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}s > 1.$$

Then equation (3) *has property* (A).

Taking Corollary 1 into account we see that if we replace condition (20) either by condition (25) or by (26) then the conclusions of Theorems 3 and 4 remain valid.

As a matter of fact we are able to prove a more general comparison theorem. In the sequel we suppose that functions z and $w\colon [t_0,\infty)\to(0,\infty)$ are continuous.

Theorem 5. *Assume that* (5), (6), (9), (12) *and* (13) *are satisfied. Further suppose that*

$$w(t)\geqslant g(t),\quad t\geqslant t_0,$$

$$\int_t^{\infty}z(s)\,\mathrm{d}s\geqslant\int_t^{\infty}p(s)\,\mathrm{d}s,\quad t\geqslant t_0.$$

If equation (2) *has property* (A), *then so does the equation*

$$\left(r_2(t)\left(r_1(t)u'(t)\right)'\right)' + z(t)u[w(t)] = 0. \tag{31}$$

Proof. From Theorem 1 we have that equation (3) has property (A). On the other hand by Theorem 4 in [1] we see that the equation

$$\left(r_2(t)\left(r_1(t)u'(t)\right)'\right)' + z(t)u[g(t)] = 0 \tag{32}$$

has property (A). Applying Theorem 1 in [7] to equations (32) and (31) we get that equation (31) has property (A). The proof is complete. □

References

[1] *J. Džurina*: Comparison theorems for ODEs. Math. Slovaca *42* (1992), 299–315.
[2] *T. A. Chanturija and I. T. Kiguradze*: Asymptotic properties of solutions of nonautonomous ordinary differential equations. Nauka, Moscow, 1990. (In Russian.)
[3] *P. Hartman and A. Wintner*: Linear differential and difference equations with monotone solutions. Amer. J. Math. *75* (1953), 731–743.
[4] *I. T. Kiguradze*: On the oscillation of solutions of the equation $\mathrm{d}^m u/\mathrm{d}t^m + a(t)|u|^n \times \operatorname{sign} u = 0$. Mat. Sb *65* (1964), 172–187. (In Russian.)
[5] *I. T. Kiguradze*: Some singular boundary value problems for ordinary differential equations. University Press, Tbilisy, 1975. (In Russian.)
[6] *Y. Kitamura and T. Kusano*: Nonlinear oscillation of higher-order functional differential equations with deviating arguments. J. Math. Anal. Appl. *77* (1980), 100–119.
[7] *T. Kusano and M. Naito*: Comparison theorems for functional differential equations with deviating arguments. J. Math. Soc. Japan *3* (1981), 509–532.
[8] *J. Ohriska*: Oscillation of differential equations and v-derivatives. Czech. Math. J. *39* (1989), 24–44.
[9] *Ch. G. Philos and Y. G. Sficas*: Oscillatory and asymptotic behavior of second and third order retarded differential equations. Czech. Math. J. *32* (1982), 169–182.
[10] *K. Tanaka*: Asymptotic analysis of odd order ordinary differential equations. Hiroshima Math. J. *10* (1980), 391–408.

Author's address: Department of Mathematical Analysis, Šafárik University, Jesenná 5, 041 54 Košice, Slovakia.